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OPTIMIZATION METHODS MODELED BY SECOND ORDER DIFFERENTIAL EQUATION

T. Hajba (Győr, Hungary)

Abstract. In this paper we investigate the continuous version of the Fletcher-Reeves algorithm described by a system of second order differential equations. We define the connections between the functions of the coefficients under which the minimum point of the function will be an asymptotically stable limit point of the trajectories.

1. Preliminaries

Let $f : I\!R^n \to I\!R$ be a continuous differentiable function and let us consider the minimization problem

$$f(x) \to \inf_{x \in I\!R^n}, \tag{1}$$

assuming that there exists a unique $x_* \in I\!R^n$ such that

$$f(x_*) = f_* = \inf_{x \in I\!R^n} f(x).$$

Lots of methods are developed for the solution of this problem. A family of the methods consists of the so called methods of conjugate directions. As a prototype of this family can be considered the method of Fletcher and Reeves [5], namely, starting from x_0 and $p_0 = -f'(x_0)$ compute the pair of points

$$\begin{aligned} x_{k+1} &= x_k + \alpha_k p_k, \\ p_{k+1} &= -f'(x_{k+1}) + \beta_k p_k, \end{aligned} \qquad k = 1, 2, \dots, \tag{2}$$

where the parameter α_k usually is chosen as the local minimizer of the function f along to the direction p_k and for β_k there are different choices, for example

$$\beta_k = \frac{||f'(x_{k+1})||^2}{||f'(x_k)||^2}\,.$$

If we set aside the particular choices of α_k and β_k, the iteration (2) can be considered as a numerical integration with stepsize $h_k = 1$ by the Euler-method of the system of differential equations

$$\begin{aligned} \dot{x} &= \alpha(t)p, \\ \dot{p} &= -f'(x + \alpha(t)p) + (\beta(t) - 1)p \end{aligned} \tag{3}$$

with the initial values

$$x(t_0) = x_0, \quad p(t_0) = -f'(x_0). \tag{4}$$

It is obvious, that (3)-(4) is equivalent with the system of second order differential equations

$$\ddot{x} - \frac{\dot{\alpha}(t) + \beta(t) - 1}{\alpha(t)}\dot{x} + \alpha(t)f'(x + \dot{x}) = 0 \tag{5}$$

with the initial values

$$x(t_0) = x_0, \qquad \dot{x}(t_0) = -\alpha(t_0)f'(x_0). \tag{6}$$

Our aim is to determine the functions $\alpha(t)$ and $\beta(t)$ such that the trajectories will be asymptotically stable in the point x_* of the minimum.

Modeling the iterative numerical methods of optimization with differential equations has been executed in several papers. However, almost all papers deal only with either the gradient or the Newton method and they are modeled by a system of first order differential equations (see e.g. [1], [3], [4], [6], [7], [9], etc.).

It is worthy to remark that the first order continuous minimization models can be divided into two classes. To the first class belong those models described by a system of first order differential equations for which the point x_0 is a stationary point of the system. In this case the convergence of the trajectories to x_0 is equivalent with the asymptotic stability of x_0, therefore the Lyapunov function methods are useful to prove the convergence with an appropriately chosen Lyapunov function (see e.g. [3], [4], [9]). To the second class of the models belong those continuous first order models, for which the minimum point is not stationary, but along the trajectories the right hand side vector of the differential equation system tends to null-vector if $t \to \infty$. In this case the Lyapunov type methods are also useful, but under more rigorous conditions (see [7], [8]).

Since the n-dimensional second order differential equation system can be written into a $2n$-dimensional first order system, we can also speak about stationary and non-stationary second order models.

In our investigation the following lemma will play basic role to obtain the conditions of the convergence. It can be obtained immediately from the Gronwall lemma.

Lemma 1. [8] *Suppose that the non-negative scalar function $x(t)$ is defined for $t_0 \le t < \infty$ and satisfies the following differential inequality*

$$\frac{d}{dt}x(t) \le -a(t)x(t) + b(t)$$

for $t_0 \le t < \infty$, where the functions $a(t) > 0$ and $b(t)$ are integrable on any finite interval and they have the following properties

$$\int_{t_0}^{\infty} a(t)dt = \infty, \qquad \lim_{t\to\infty} \frac{b(t)}{a(t)} = 0.$$

Then $\lim_{t\to\infty} x(t) = 0.$

In the following we will assume about the examined model, that all trajectories of the system of differential equations are defined on the whole $[t_0, \infty)$. This is not a very strict assumption, since if we assume, for example, the Lipschitz continuity of the gradient vector $f'(x)$, then this assumption satisfies.

2. A simplified model

Let $x(t)$ and $p(t)$ be continuously differentiable vector-functions $I\!R \to I\!R^n$. Let us consider the following system of differential equations

$$\begin{aligned} \dot{x} &= \alpha(t)p, \\ \dot{p} &= -f'(x) + \beta(t)p \end{aligned} \tag{7}$$

with the initial values (4). Here the function $f(x)$ is taken from the optimization problem (1), and $f'(x)$ is its gradient.

In the case when in the system (7) $\alpha(t) \equiv 1$ and $\beta(t) \equiv \beta_0$, the convergence of the trajectories and the appropriate choice has been analyzed in [2].

Here we consider the system (7) with function parameters.

Proposition 1. *Assume that*

1. *f is defined and continuously differentiable strongly convex function on $I\!R^n$ with the modulus of convexity $\kappa > 0$;*
2. *$\alpha(t)$ is a positive, continuously differentiable function;*
3. *$\beta(t)$ is non-positive, continuous function;*
4. *between the parameter-functions α and β the following connections hold:*

   4.a. $\dfrac{-\dot{\alpha}(t)}{\alpha(t)} - \alpha(t) \leq 2\alpha(t) + \beta(t) < 0$ *for all* $t_0 < t$*;*

   4.b. $-\kappa \leq 2\alpha(t) + \beta(t)$ *for every* $t_0 < t$*;*

   4.c. $\int\limits_{t_0}^{\infty} (2\alpha(t) + \beta(t))dt = -\infty$*;*

Then there exists a unique $x_* \in I\!R^n$ such that

$$\inf_{x \in I\!R^n} f(x) = f(x_*)$$

and for any trajectories of (7) we have

$$\lim_{t\to\infty} f(x(t)) = f(x_*), \qquad \lim_{t\to\infty} ||x(t) - x_*|| = 0, \qquad \lim_{t\to\infty} ||p(t)|| = 0.$$

Proof. From the strong convexity of the function follows the unique existence of x_*.

Let us introduce the function

$$g(t) = \frac{1}{\alpha(t)}(f(x(t)) - f(x_*)) + \frac{1}{2}||x(t) - x_*||^2 + \frac{1}{2}||x(t) - x_* + p(t)||^2.$$

For the derivative of $g(t)$ we have

$$\begin{aligned} \frac{d}{dt}g(t) = {} & \frac{-\dot{\alpha}(t)}{\alpha^2(t)}(f(x(t)) - f(x_*)) + \langle f'(x(t)), p(t)\rangle + \\ & + \alpha(t)\langle x(t) - x_*, p(t)\rangle + \\ & + \langle \alpha(t)p(t) - f'(x(t)) + \beta(t)p(t), x(t) - x_* + p(t)\rangle = \\ = {} & \frac{-\dot{\alpha}(t)}{\alpha^2(t)}(f(x(t)) - f(x_*)) + (\alpha(t) + \beta(t))||p(t)||^2 + \\ & + (2\alpha(t) + \beta(t))\langle x(t) - x_*, p(t)\rangle - \langle f'(x(t)), x(t) - x_*\rangle. \end{aligned} \tag{8}$$

Since $f(x)$ is strongly convex function with the modulus of convexity $\kappa > 0$ if and only if

$$\langle f'(x(t)), x(t) - x_* \rangle \geq f(x(t)) - f(x_*) + \kappa ||x(t) - x_*||^2,$$

the equality (8) turns into the inequality

$$(9) \qquad \frac{d}{dt} g(t) \leq \left(-\frac{\dot{\alpha}(t)}{\alpha^2(t)} - 1 \right) (f(x(t)) - f(x_*)) + (\alpha(t) + \beta(t)) ||p(t)||^2 + \\ + (2\alpha(t) + \beta(t)) \langle x(t) - x_*, p(t) \rangle - \kappa ||x(t) - x_*||^2.$$

From (9) by using the conditions 4.a and 4.b we get that

$$\frac{d}{dt} g(t) \leq (2\alpha(t) + \beta(t)) \left[\frac{1}{\alpha(t)} (f(x(t)) - f(x_*)) + \frac{1}{2} ||x(t) - x_*||^2 + \right. \\ \left. + \frac{1}{2} ||x(t) - x_* + p(t)||^2 \right] = \\ = (2\alpha(t) + \beta(t)) g(t)$$

for $t_0 \leq t$.

Taking into consideration the condition 4.c it follows from the inequality (9) that the function g satisfies the conditions of Lemma 1, therefore

$$(10) \qquad \lim_{t \to \infty} g(t) = 0.$$

Since $g(t)$ is the sum of non-negative functions, from (10) follows that every member of the sum tends to 0. It proves the validity of

$$\lim_{t \to \infty} ||x(t) - x_*|| = 0$$

and using the continuity of the function f the validity of

$$\lim_{t \to \infty} f(x(t)) = f(x_*).$$

The last statement follows from the inequality

$$||p(t)|| = ||x(t) - x_* + p(t) + x_* - x(t)|| \leq ||x(t) - x_* + p(t)|| + ||x_* - x(t)||$$

and from the fact that the terms in the right hand side tend to 0. ♢

Remark 1. The conditions of the theorem can be satisfied. Specifically, if $\alpha(t) = \frac{\alpha_0}{t+a}$, $\beta(t) = -\frac{\beta_0}{t+a}$, where $a > -t_0$ and the parameters α_0, β_0 belong to the polyhedron

$$P = \left\{(\alpha_0, \beta_0) \in I\!R^2 : 3\alpha_0 - \beta_0 \geq 1,\ 2\alpha_0 \leq \beta \leq 2\alpha_0 + \kappa(t_0 + a),\ \alpha_0, \beta_0 > 0\right\},$$

then taking into consideration, that $\kappa > 0$ and $t_0 + a > 0$, we have that P is not empty. For example, with $\alpha_0 = 3\kappa$, $\beta_0 = 7\kappa$ the functions $\alpha(t)$ and $\beta(t)$ will be appropriate if $a = 1$ and $t_0 = 0$.

Remark 2. The system (7) is stationary with the stationary point $(x_*, 0) \in I\!R^n \times I\!R^n$.

In the practice the computation of the gradient $f'(x)$ can be solved only with some error, moreover in a lot of cases we know only its approximation. In the following proposition we will prove, that the convergence of trajectories to the minimum point can be preserved under some condition for the approximating function.

Proposition 2. *Let us suppose that the conditions of the Proposition 1 are fulfilled. Let $\phi(x, t)$ be an approximation of the gradient function $f'(x)$, satisfying the inequality*

$$||f'(x(t)) - \phi(x(t), t)||^2 \leq \delta(t) \quad \forall t \in I\!R, x \in I\!R^n$$

and assume that the accuracy function $\delta(t)$ in the approximation fulfills the following connection with the parameter functions $\alpha(t)$ and $\beta(t)$

$$\lim_{t\to\infty} \frac{\delta(t)}{\alpha(t)(2\alpha(t) + \beta(t))} = 0. \tag{11}$$

Then there exists a unique solution x_* of the problem (1) and for any trajectories of system of differential equations

$$\begin{aligned} \dot{x} &= \alpha(t)p, \\ \dot{p} &= -\phi(x, t) + \beta(t)p, \end{aligned} \tag{12}$$

we have

$$\lim_{t\to\infty} f(x(t)) = f(x_*), \qquad \lim_{t\to\infty} ||x(t) - x_*|| = 0, \qquad \lim_{t\to\infty} ||p(t)|| = 0.$$

Proof. The existence and uniqness of x_* follows from the strong convexity of the function $f(x)$.

Let $x(t)$ be a trajectory of (12) and along this trajectory execute the function

$$g(t) = \frac{1}{\alpha(t)}(f(x(t)) - f(x_*)) + \frac{1}{2}||x(t) - x_*||^2 + \frac{1}{2}||x(t) - x_* + p(t)||^2.$$

The derivative of this function is

$$\begin{aligned}\frac{d}{dt}g(t) = &\frac{-\dot\alpha(t)}{\alpha^2(t)}(f(x(t)) - f(x_*)) + \langle f'(x(t)), p(t)\rangle + \alpha(t)\langle x(t) - x_*, p(t)\rangle + \\ &+ \langle \alpha(t)p(t) - \phi(x,t) + \beta(t)p(t), x(t) - x_* + p(t)\rangle = \\ = &\frac{-\dot\alpha(t)}{\alpha^2(t)}(f(x(t)) - f(x_*)) + \langle f'(x(t)), p(t)\rangle + (\alpha(t) + \beta(t))\langle p(t), p(t)\rangle + \\ &+ (2\alpha(t) + \beta(t))\langle x(t) - x_*, p(t)\rangle + \\ &+ \langle f'(x(t)) - \phi(x(t),t), x(t) - x_* + p(t)\rangle - \\ &- \langle f'(x(t)), x(t) - x_* + p(t)\rangle.\end{aligned}$$

Taking into consideration the strong convexity of the function $f(x)$ we have

$$\begin{aligned}\frac{d}{dt}g(t) \le &\left(\frac{-\dot\alpha(t)}{\alpha^2(t)} - 1\right)(f(x(t)) - f(x_*)) + (\alpha(t) + \beta(t))||p(t)||^2 + \\ &+ (2\alpha(t) + \beta(t))\langle x(t) - x_*, p(t)\rangle - \kappa||x(t) - x_*||^2 + \\ &+ \langle f'(x(t)) - \phi(x,t), x(t) - x_* + p(t)\rangle.\end{aligned} \tag{13}$$

Since $\alpha(t) > 0$, hence $0 \le ||\frac{1}{\sqrt{\alpha(t)}}u + \sqrt{\alpha(t)}v||^2$ for all $u, v \in I\!R^n$, and from this we get the inequality

$$\langle u, v\rangle \le \frac{1}{2\alpha(t)}||u||^2 + \frac{\alpha(t)}{2}||v||^2.$$

Applying this inequality to $\langle f'(x(t)) - \phi(x,t), x(t) - x_* + p(t)\rangle$ in (13) and using the second assumption of the proposition we have

$$\begin{aligned}\frac{d}{dt}g(t) \le &\left(\frac{-\dot\alpha(t)}{\alpha^2(t)} - 1\right)(f(x(t)) - f(x_*)) + (\alpha(t) + \beta(t))||p(t)||^2 + \\ &+ (2\alpha(t) + \beta(t))\langle x(t) - x_*, p(t)\rangle - \kappa||x(t) - x_*||^2 + \\ &+ \frac{\delta(t)}{2\alpha(t)} + \frac{\alpha(t)}{2}||x(t) - x_*||^2 + \frac{\delta(t)}{2\alpha(t)} + \frac{\alpha(t)}{2}||p(t)||^2.\end{aligned}$$

Using the assumptions 4.a and 4.b from the Proposition 1, we get that

$$\frac{d}{dt}g(t) \le (2\alpha(t)+\beta(t))\left(\frac{1}{\alpha(t)}(f(x(t))-f(x_*))+\frac{1}{2}||x(t)-x_*||^2+\right.$$

$$\left.+\frac{1}{2}||x(t)-x_*+p(t)||^2\right)+\frac{\delta(t)}{\alpha(t)}=$$

$$=(2\alpha(t)+\beta(t))g(t)+\frac{\delta(t)}{\alpha(t)} \tag{14}$$

for $t_0 \le t$.

Taking into consideration the condition 4.c from the Proposition 1 and the connection (11), it follows from the inequality (14) that the function g satisfies the conditions of Lemma 1, therefore

$$\lim_{t\to\infty} g(t) = 0$$

and hence analogously to the end of the proof of the Proposition 2 we obtain all statements of the proposition. ♢

Remark 3. If the parameter functions $\alpha(t)$ and $\beta(t)$ are taken from the Remark 1. then the condition for the accuracy will be satisfied choosing $\delta(t) = \frac{1}{(t+a)^{2+\varepsilon}}$, $\varepsilon > 0$.

Remark 4. Depending on the approximating function $\phi(x,t)$ the system (11) may be either stationary or non-stationary, but in both cases the convergence of the trajectories to the minimum point can be guaranteed with appropriated δ-depending choice of the function $\alpha(t)$ and $\beta(t)$.

3. The continuous analog of the Fletcher-Reeves algorithm

Let us return to the continuous model (3). It seems to be natural that the shifting the argument of the gradient function needs a distinguished discussion, and it is expected that additional assumptions will be desired.

Proposition 3. *Assume that*

1. *f is defined and continuously differentiable strongly convex function on $I\!R^n$ with the modulus of convexity $\kappa > 0$;*

2. *the gradient of the function f is Lipschitz continuous with the Lipschitz constant $L > 0$;*
3. *$\alpha(t)$ is a positive, continuously differentiable function;*
4. *$\beta(t)$ is a non-positive, continuous function;*
5. *between the parameter-functions $\alpha(t)$ and $\beta(t)$ the following connections hold:*

   5.a. $\sup\limits_{t \geq t_0}(\alpha(t) + \beta(t)) < \infty$;

   5.b. $0 \leq \frac{\dot{\alpha}(t)}{\alpha(t)} + \beta(t) \leq \min(2\kappa\alpha(t), 1 - \kappa\alpha(t) - \frac{L^2\alpha}{4\kappa})$ *for all* $t_0 < t$;

   5.c. $\int\limits_{t_0}^{\infty}(\frac{\dot{\alpha}(t)}{\alpha(t)} + \beta(t) - \kappa\alpha(t))dt = -\infty.$

Then there exists a unique $x_* \in \mathbb{R}^n$ such that

$$\inf_{x \in \mathbb{R}^n} f(x) = f(x_*)$$

and for any trajectories of (3)-(4) we have

$$\lim_{t \to \infty} f(x(t)) = f(x_*),$$

$$\lim_{t \to \infty} ||x(t) + \alpha(t)p(t) - x_*|| = 0, \qquad \lim_{t \to \infty} ||x(t) - x_*|| = 0.$$

Proof. From the strong convexity of the function follows the unique existence of x_*.

The assumptions 2 and 5.a guarantee that every solution of the system (3)-(4) exists on $[t_0, \infty)$.

Let us introduce the function

$$g(t) = \frac{1}{2}||x(t) + \alpha(t)p(t) - x_*||^2 + \frac{1}{2}\alpha^2(t)||p(t)||^2.$$

For the derivative of $g(t)$ along the trajectories we have

$$\begin{aligned}\frac{d}{dt}g(t) = &\langle x(t) + \alpha(t)p(t) - x_*, -f'(x(t) + \alpha(t)p(t))\rangle + \\ &+ \alpha(t)\left(\frac{\dot{\alpha}(t)}{\alpha(t)} + \beta(t)\right)\langle x(t) + \alpha(t)p(t) - x_*, p(t)\rangle + \\ &+ \dot{\alpha}(t)\alpha(t)||p(t)||^2 + \\ &+ \alpha^2(t)\langle p(t), -f'(x(t) + \alpha(t)p(t)) + (\beta(t) - 1)p(t)\rangle.\end{aligned}$$

Using the strong convexity of f and the Lipschitz continuity of f' we obtain

$$\begin{aligned}
\frac{d}{dt}g(t) &= -\kappa\alpha(t)||x(t)+\alpha(t)p(t)-x_*||^2+ \\
&\quad +\alpha(t)\left(\frac{\dot{\alpha}(t)}{\alpha(t)}+\beta(t)\right)\langle x(t)-x_*, p(t)\rangle + \\
&\quad +\alpha^2(t)\left(2\left(\frac{\dot{\alpha}(t)}{\alpha(t)}+\beta(t)\right)-1+\frac{L^2\alpha(t)}{4\kappa}\right)||p(t)||^2 = \\
&= -\kappa\alpha(t)||x(t)-x_*||^2+ \\
&\quad +\alpha\left(\frac{\dot{\alpha}(t)}{\alpha(t)}-2\kappa\alpha(t)\right)\langle x(t)-x_*, p\rangle + \\
&\quad +\alpha^2(t)\left(2\left(\frac{\dot{\alpha}(t)}{\alpha(t)}+\beta(t)\right)-1+\frac{L^2\alpha(t)}{4\kappa}-\kappa\alpha(t)\right).
\end{aligned}$$

Taking into consideration the assumption 5.b from the last inequality follows the inequality

$$\frac{d}{dt}g(t) \le \left(\frac{\dot{\alpha}(t)}{\alpha(t)}-\kappa\alpha(t)\right)g(t)$$

which satisfies the conditions of the Lemma 1. Consequently, $\lim_{t\to\infty} g(t) = 0$.

Since $g(t)$ is a sum of nonnegative functions, therefore

$$\lim_{t\to\infty} ||x(t)+\alpha(t)p(t)-x_*|| = 0, \qquad \lim_{t\to\infty} \alpha(t)||p(t)|| = 0.$$

From these limits by the triangular inequality we obtain

$$\lim_{t\to\infty} ||x-x_*|| = 0.$$

The first statement of the proposition follows from the continuity of the function f. ♢

Remark 5. It can be shown that the families of functions $\alpha(t)$ and $\beta(t)$ are non-empty, but the reciprocal functions are usually not adequate.

The special case, where we can use the continuous analogue of the Fletcher-Reeves method is the minimization of quadratic function. Namely, let us execute the minimization problem

$$\frac{1}{2}\langle x, Ax\rangle + \langle c, x\rangle \to \min_{x\in \mathbb{R}^n}, \tag{15}$$

where A is a $n \times n$ positive definite symmetrical matrix and $c \in \mathbb{R}^n$. With these assumptions the problem (15) has a unique solution x_*.

For this problem the modeling system of differential equations (3) turns into the following system

$$\begin{aligned} \dot{x} &= \alpha(t)p, \\ \dot{p} &= -(Ax + c) + ((\beta(t) - 1)I - \alpha(t)A)\, p. \end{aligned} \tag{16}$$

Proposition 4. *Assume that*

1. *$\alpha(t)$ is a positive, continuously differentiable decreasing function;*
2. *$\beta(t)$ is a continuous function;*
3. *between the parameter functions $\alpha(t)$ and $\beta(t)$ the following connections hold:*
   - *3.a.* $\dfrac{-\dot{\alpha}(t)}{\alpha(t)(\alpha(t)+1)} - \dfrac{2\alpha(t)}{\alpha(t)+1} \le \alpha(t) + \beta(t) < 0$ *for every* $t_0 < t$;
   - *3.b.* $-\frac{1}{2}d \le \alpha(t) + \beta(t)$ *for every* $t_0 < t$; *where d is the smallest eigenvalue of A;*
   - *3.c.* $\int_{t_0}^{\infty} (\alpha(t) + \beta(t))dt = -\infty$.

Then for all trajectories of (16) we have $\lim_{t\to\infty} ||x(t) - x_*|| = 0$.

Proof. Let

$$g(t) = \frac{1}{2}\left(1 + \frac{1}{\alpha(t)}\right)\langle A(x(t) - x_*, x(t) - x_*\rangle + \frac{1}{2}||x(t) - x_* + p(t)||^2.$$

Then

$$\begin{aligned} \frac{d}{dt}g(t) = & \frac{-\dot{\alpha}(t)}{2\alpha^2(t)}\langle A(x(t) - x_*), x(t) - x_*\rangle + \\ & + (\alpha(t) + 1)\langle A(x(t) - x_*), p(t)\rangle + \\ & + \langle \alpha(t)p(t) - (Ax(t) + c) + (\beta(t)I - \alpha(t)A)p(t), x(t) - x_* + p(t)\rangle. \end{aligned}$$

Since $Ax_* + c = 0$, we have that

$$\begin{aligned}\frac{d}{dt}g(t) = &\frac{-\dot\alpha(t)}{2\alpha^2(t)}\langle A(x(t)-x_*), x(t)-x_*\rangle +\\ &+(\alpha(t)+1)\langle A(x(t)-x_*), p(t)\rangle +\\ &+(\alpha(t)+\beta(t))||p(t)||^2+\\ &+(\alpha(t)+\beta(t))\langle x(t)-x_*, p(t)\rangle -\\ &-\langle Ax(t)+c-(Ax_*+c), x(t)-x_*+p(t)\rangle -\\ &-\alpha(t)\langle A(x(t)-x_*), p(t)\rangle - \alpha(t)\langle Ap(t), p(t)\rangle =\\ = &\frac{-\dot\alpha(t)}{2\alpha^2(t)}\langle A(x(t)-x_*), x(t)-x_*\rangle + (\alpha(t)+\beta(t))||p(t)||^2+\\ &+(\alpha(t)+\beta(t))\langle x(t)-x_*, p(t)\rangle -\\ &-\alpha(t)\langle Ap(t), p(t)\rangle - \langle A(x(t)-x_*), x(t)-x_*\rangle.\end{aligned}$$

Since A is positive definite we have that

$$\begin{aligned}\frac{d}{dt}g(t) \leq &(\frac{-\dot\alpha(t)}{2\alpha^2(t)} - \frac{1}{2})\langle A(x(t)-x_*), x(t)-x_*\rangle +\\ &+(\alpha(t)+\beta(t))||p(t)||^2+\\ &+(\alpha(t)+\beta(t))\langle x(t)-x_*, p(t)\rangle - \frac{1}{2}d\langle x(t)-x_*, x(t)-x_*\rangle.\end{aligned}$$

Under the assumptions of 3.a and 3.b we have that

$$\begin{aligned}\frac{d}{dt}g(t) \leq (\alpha(t)+\beta(t))\Bigg((1+\frac{1}{\alpha(t)})\langle A(x(t)-x_*), x(t)-x_*\rangle +\\ +\frac{1}{2}||x(t)-x_*+p(t)||^2\Bigg) =\\ = (\alpha(t)+\beta(t))g(t)\end{aligned}$$

for $t_0 \leq t$.

Taking into consideration the assumption 3.c it follows that the function g satisfies the conditions of Lemma 1, therefore

$$\lim_{t\to\infty} g(t) = 0$$

and hence

$$\lim_{t\to\infty} ||x(t) - x_*|| = 0.$$

$\diamondsuit$

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T. Hajba
Department of Mathematics
Faculty of Engineering Sciences
Széchenyi István University
Egyetem tér 1.
H-9026 Győr, Hungary