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DISTRIBUTION OF 2–ADDITIVE FUNCTIONS UNDER SOME CONDITIONS

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Abstract. Distribution of 2-additive functions under the condition $\alpha(n) = k$ is investigated, where $\alpha(n)$ is the sum of digits in the binary expansion of n.

1. Introduction and formulation of the theorems

Let $\varepsilon_j(n)$ be the j'th digit in the binary expansion of n,

$$n = \sum_{j=0}^{\infty} \varepsilon_j(n) \cdot 2^j, \quad \varepsilon_j(n) \in \{0,1\}. \tag{1.1}$$

Let $\mathcal{A}_2$ be the class of 2-additive and $\mathcal{M}_2$ be the class of 2-multiplicative functions.

A function $f : \mathbb{N}_0 (= \mathbb{N} \cup \{0\}) \to \mathbb{R}$ belongs to $\mathcal{A}_2$, if

$$f(0) = 0, \text{ and } f(n) := \sum_{j=0}^{\infty} \varepsilon_j(n) f(2^j),$$

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Mathematics Subject Classification: 11K65, 11P99, 11N37 and $g : \mathbb{N}_0 \to \mathbb{C}$ belongs to $\mathcal{M}_2$, if

$$g(0) = 1, \text{ and } g(n) := \prod_{j=0}^{\infty} g(\varepsilon_j(n) \cdot 2^j).$$

Let $\overline{\mathcal{M}_2}$ be the set of those $g \in \mathcal{M}_2$ for which additionally $|g(n)| = 1 \quad (n \in \in \mathbb{N}_0)$ holds.

Let $\alpha(n) = \sum_{j=0}^{\infty} \varepsilon_j(n)$ be the so called "sum of digits" function.

Let

$$\mathcal{E}_{N,k} = \{n < 2^N \mid \alpha(n) = k\}, \text{ and}$$
$$\eta = \eta_{N,k} = \frac{k}{N}.$$

Here we continue our work [1].

Theorem 1. *Let $g \in \overline{\mathcal{M}_2}$ be such a function for which*

$$\sum_{j=0}^{\infty}(1 - g(2^j)) \tag{1.2}$$

is convergent. Let

$$M_\eta := \prod_{j=0}^{\infty} \left((1-\eta) + g(2^j)\eta\right). \tag{1.3}$$

Let $\delta > 0$ be a constant. Then

$$\max_{\delta \le \frac{k}{N} \le 1-\delta} \left| \frac{1}{\binom{N}{k}} \sum_{n \in \mathcal{E}_{N,k}} g(n) - M_{\eta_{N,k}} \right| \to 0 \quad (N \to \infty).$$

Theorem 2. *Let $f \in \mathcal{A}_2$ such that $\sum f(2^j)$, $\sum f^2(2^j)$ are convergent. Let $\varphi_\eta(\tau)$ be the characteristic function of $\Theta = \xi_0 + \xi_1 + \ldots$, where $\xi_0, \xi_1, \ldots$ are independent random variables,*

$$P(\xi_\nu = 0) = 1 - \eta, \quad P(\xi_\nu = f(2^\nu)) = \eta.$$

Thus

$$\varphi_\eta(\tau) = \prod_{j=0}^{\infty} \left((1-\eta) + \eta \cdot e^{i\tau f(2^j)} \right).$$

Let $F_\eta(y)$ *be the distribution function of* Θ.

Then

$$\max_{\delta \le \frac{k}{N} \le 1-\delta} \sup_{y\in\mathbb{R}} \left| \frac{1}{\binom{N}{k}} \# \{ n \in \mathcal{E}_{N,k},\ f(n) < y \} - F_\eta(y) \right| \to 0 \quad (N \to \infty).$$

Here $\delta > 0$ *is an arbitrary small constant.*

Theorem 3. *Let* $f \in \mathcal{A}_2,\ f(2^j) = O(1)$. *Let* $A_N = \sum_{j=0}^{N-1} f(2^j),\ m_N(\eta) :=$ $:= \eta A_N$,

$$\sigma_N^2(\eta) = (1-\eta)\eta \sum_{j=0}^{N-1} \left(f(2^j) - \frac{A_N}{N} \right)^2$$

$\eta \in [\delta,\ (1-\delta)],\ \delta > 0$ *be a constant.*

Assume that $\sigma_N^2\left(\frac{1}{2}\right) \to \infty \quad (N \to \infty)$. *Then*

$$\lim_{N\to\infty} \sup_{\frac{k}{N} \in [\delta,\ 1-\delta]} \sup_{y\in\mathbb{R}} \left| \frac{1}{\binom{N}{k}} \# \left\{ n \in \mathcal{E}_{N,k} \ \left| \frac{f(n) - m_N\left(\frac{k}{N}\right)}{\sigma_N\left(\frac{k}{N}\right)} < y \right. \right\} - \Phi(y) \right| = 0.$$

The proof of this last theorem is very similar to the proof of Theorem 3 in [1], so we omit it.

2. Proof of Theorem 1 and 2

It is enough to prove Theorem 1. Theorem 2 follows hence, if we consider $g_\tau(n) = e^{i\tau f(n)}$ and apply Theorem 1.

The proof is almost the same as that of Theorem 2 in [1].

Let M be a large fixed integer, $\arg g(2^j) = h(2^j)$, $h(2^j) \in [-\pi, \pi]$. From (1.2) we obtain that

$$\sum |1 - g(2^j)|^2 \asymp \sum h^2(2^j) < \infty,$$

and that $\sum h(2^j)$ is convergent. Thus $g(2^j) \to 1 \quad (j \to \infty)$. Let h be defined on $\mathbb{N}_0$ as a 2-additive function. Then $g(n) = e^{ih(n)}$.

Let

$$g_M(n) = \prod_{j=0}^{M-1} g(\varepsilon_j(n) \cdot 2^j), \quad h_M(n) = \sum_{j=0}^{M-1} h(\varepsilon_j(n) \cdot 2^j),$$

$$h_M^*(n) = \sum_{j=M}^{N-1} h(\varepsilon_j(n) \cdot 2^j).$$

We have

$$\frac{1}{\binom{N}{k}} \sum_{n \in \mathcal{E}_{N,k}} h_M^*(n) = \sum_{j=M}^{N-1} h(2^j) \cdot \frac{\binom{N-1}{k-1}}{\binom{N}{k}},$$

$$\frac{1}{\binom{N}{k}} \sum_{n \in \mathcal{E}_{N,k}} h_M^{*2}(n) = \sum_{j=M}^{N-1} h^2(2^j) \frac{\binom{N-1}{k-1}}{\binom{N}{k}} +$$

$$+ \sum_{\substack{i_1 \neq i_2 \\ M \le i_1, i_2 \le N-1}} \frac{\binom{N-2}{k-2}}{\binom{N}{k}} h(2^{i_1}) h(2^{i_2}).$$

Furthermore

$$\frac{\binom{N-1}{k-1}}{\binom{N}{k}} = \frac{k}{N} = \eta, \quad \frac{\binom{N-2}{k-2}}{\binom{N}{k}} = \frac{k(k-1)}{N(N-1)} = \eta^2 \left(1 + O\left(\frac{1}{N}\right)\right).$$

Hence we obtain that

$$\frac{1}{\binom{N}{k}}\sum_{n\in\mathcal{E}_{N,k}}\left(h_M^*(n)-\eta\sum_{j=M}^{N-1}h(2^j)\right)^2\ll_\eta\sum_{j=M}^{N-1}h^2(2^j)+$$
$$+\frac{1}{N}\sum_{M\le i,\ j\le N-1}|h(2^i)|\cdot|h(2^j)|.$$

The right hand side tends to zero as $M\to\infty$. It implies that

$$\limsup_{N\to\infty}\max_{\delta\le\frac{k}{N}\le 1-\delta}\frac{1}{\binom{N}{k}}\left|\sum_{n\in V_{N,k}}(g(n)-g_M(n))\right|=\Delta(M)\to 0\quad\text{as } M\to\infty.$$

To estimate $\frac{1}{\binom{N}{k}}\sum_{n<2^N}g_M(n)$, we write each $n\in\mathcal{E}_{N,k}$ as $n=t+q^Mm$. For a fixed t, $n\in\mathcal{E}_{N,k}$ if and only if $m\in\mathcal{E}_{N-M,\ k-\alpha(t)}$, thus

$$\frac{1}{\binom{N}{k}}\sum_{n<2^N}g_M(n)=\sum_{t=0}^{2^M-1}g(t)\cdot\frac{\binom{N-M}{k-\alpha(t)}}{\binom{N}{k}}=$$
$$=\sum_{t=0}^{2^M-1}g(t)\left(\frac{\eta}{1-\eta}\right)^{\alpha(t)}\cdot(1-\eta)^M(1+o_N(1))=$$
$$=(1+o_N(1))(1-\eta)^M\sum_{t=0}^{2^M-1}g(t)\left(\frac{\eta}{1-\eta}\right)^{\alpha(t)}=$$
$$=(1+o_N(1))(1-\eta)^M\prod_{j=0}^{M-1}\left(1+g(2^j)\frac{\eta}{1-\eta}\right)=$$
$$=(1+o_N(1))\prod_{j=0}^{M-1}((1-\eta)+g(2^j)\eta).$$

The relation is uniform as $\frac{k}{N}\in[\delta,\ 1-\delta]$. Hence the theorem is immediate.

3. Final remarks

We can prove the following assertions.

Theorem 4. *Let* $g \in \overline{\mathcal{M}}_2,\ \delta > 0$ *and assume that there is a sequence* $k_N = k$ *such that*

$$\frac{1}{\binom{N}{k}} \sum_{n \in \mathcal{E}_{N,k}} g(n) - M_{\eta_{N,k}} \to 0 \quad as \quad N \to \infty,\ k = k_N.$$

Then (1.2) *is convergent.*

Theorem 5. *Let* $f \in \mathcal{A}_2,\ \delta > 0$*, and assume that for a suitable sequence* $k = k_N$ *such that* $\eta \in (\delta, 1-\delta)$ *we have*

$$\sup_{y \in \mathbb{R}} \left| \frac{1}{\binom{N}{k}} \# \{n \in \mathcal{E}_{N,k}, f(n) < y\} - F_{\eta_{N,k}}(y) \right| \to 0$$

as $N \to \infty,\ k = k_N$. *Then the series'* $\sum f(2^j),\ \sum f^2(2^j)$ *are convergent.*

We shall prove these assertions in more general form in a subsequent paper.

Reference

[1] **Kátai I. and Subbarao M.V.,** Distribution of additive and q-additive functions under some conditions, *Publ. Math. Debrecen,* **64** (2004), 167-187.

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