THE MAXIMAL OPERATOR OF THE (C,α) MEANS OF THE WALSH–FOURIER SERIES

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Abstract. The main aim of this paper is to prove that for the boundedness of the maximal operator σ_*^α from the Hardy space $H_p(I)$ to space $L_p(I)$ the assumption $p > 1/(\alpha+1)$ is essential.

We denote the set of non-negative integers by $\mathbf{N}$. By a dyadic interval in $I : [0,1)$ we mean one of the form $\left[\frac{l}{2^k}, \frac{l+1}{2^k}\right)$ for some $k \in \mathbf{N}$, $0 \le l < 2^k$. Given $k \in \mathbf{N}$ and $x \in [0,1)$, let $I_k(x)$ denote the dyadic interval of length 2^{-k} which contains the point x.

We also use the notation $\text{mes}(A)$ for the Lebesgue measure of any measurable set A.

Let $r_0(x)$ be a function defined by

$$r_0(x) = \begin{cases} 1, & \text{if } x \in [0,1/2) \\ -1, & \text{if } x \in [1/2,1) \end{cases}, \qquad r_0(x+1) = r_0(x).$$

The Rademacher system is defined by

$$r_n(x) = r_0(2^n x), \quad n \ge 1 \quad \text{and} \quad x \in [0,1).$$

Let $w_0, w_1, \dots$ represent the Walsh functions, i.e. $w_0(x) = 1$ and if $n = 2^{(n_1)} + \dots + 2^{(n_r)}$ is a positive integer with $n_1 > n_2 > \dots > n_r$, then

$$w_n(x) = r_{n_1}(x) \dots r_{n_r}(x).$$

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The Walsh-Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

Recall that

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in [0, 1/2^n), \\ 0, & \text{if } x \in [1/2^n, 1). \end{cases} \tag{1}$$

The partial sums of the Walsh-Fourier series are defined as follows:

$$S_m(f, x) = \sum_{j=0}^{m-1} \hat{f}(j) w_j(x),$$

where the number

$$\hat{f}(j) = \int_I f(x) w_j(x) dx$$

is said to be j-th Walsh-Fourier coefficient of the function f.

The norm (or quasinorm) of the space $L_p(I)$ is defined by

$$||f||_p := \left(\int_I |f(x)|^p dx \right)^{1/p} \qquad (0 < p < +\infty).$$

The space weak-$L_p(I)$ consists of all measurable functions f for which

$$||f||_{\text{weak}-L_p(I)} := \sup_{\lambda > 0} \lambda \operatorname{mes}(|f| > \lambda)^{1/p} < +\infty.$$

The σ-algebra generated by the dyadic I_k interval of length 2^{-k} will be denoted by F_k $(k \in \mathbf{N})$.

Denote by $f = (f^{(n)}, n \in \mathbf{N}$ martingale with respect to $(F_n, n \in \mathbf{N}$ (for details see, e.g. [7, 10]). The maximal function of martingale f is defined by

$$f^* = \sup_{n \in \mathbf{N}} |f^{(n)}|.$$

In case $f \in L_1(I)$, the maximal function can also be given by

$$f^*(x) = \sup_{n\geq 1} \frac{1}{\mathrm{mes}(I_n(x))} \left| \int\limits_{I_n(x)} f(u)du \right|, \qquad x \in I.$$

For $0 < p < \infty$ the Hardy martingale space $\mathbf{H}_p(I)$ consist all martingales for which

$$||f||_{H_p} := ||f^*||_p < \infty.$$

If $f \in L_1(I)$ then it is easy to show that the sequence $(S_{2^n}(f) : n \in \mathbf{N})$ is a martingale. If f is a martingale, that is $f = (f^{(0)}, f^{(1)}, \dots)$ then the Walsh-Fourier coefficients must be defined in a little bit different way:

$$\hat{f}(j) = \lim_{k\to\infty} \int\limits_{I^d} f^{(k)}(x) w_j(x) dx.$$

The Walsh-Fourier coefficients of the function $f \in L_1(I)$ are the same as the ones of the martingale $(S_{2^n}(f) : n \in \mathbf{N}$ obtained from the function f.

The (C,α) means of the Walsh-Fourier series of the martingale f is given by

$$\sigma_n^\alpha(f,x) = \frac{1}{A_{n-1}^\alpha} \sum_{j=1}^{n} A_{n-j}^{\alpha-1} S_j(f,x),$$

where

$$A_n^\alpha := \frac{(1+\alpha)\dots(n+\alpha)}{n!}$$

for any $n \in \mathbf{N}, \alpha \neq -1, -2, \dots$ It is known that [11] $A_n^\alpha \sim n^\alpha$.

For the martingale f we consider the maximal operator

$$\sigma_*^\alpha f = \sup_n |\sigma_n^\alpha(f,x)|.$$

The (C,α) kernel is defined by

$$K_n^\alpha(x) := \frac{1}{A_{n-1}^\alpha} \sum_{k=1}^{n} A_{n-j}^{\alpha-1} D_k(x).$$

The first result with respect to the a.e. convergence of the Walsh-Fejér means $\sigma_n^1 f$ is due to Fine [1]. Later, Schipp [4] showed that the maximal operator $\sigma_*^1 f$ is of weak type $(1,1)$, from which the a.e. convergence follows by standard argument [3]. Schipp's result implies by interpolation also the boundedness of $\sigma_*^1 : L_p \to L_p$ $(1 < p \leq \infty)$. This fails to hold for $p = 1$, but Fujii [2] proved that σ_*^1 is bounded from the dyadic Hardy space H_1 to the space L_1 (see also Simon [5]). Fujii's theorem was extended by Weisz [8]. Namely, he proved that the maximal operator of the Fejér means of the one-dimensional Walsh-Fourier series is bounded from the martingale Hardy space $H_p(I)$ to the space $L_p(I)$ for $p > 1/2$. Simon [6] gave a counterexample, which shows that this boundedness does not hold for $0 < p < 1/2$.

The maximal operator σ_*^α $(0 < \alpha < 1)$ of the (C, α) means of the Walsh-Paley Fourier series was investigated by Weisz [9]. In his paper Weisz proved the boundedness of $\sigma_*^\alpha : H_p \to L_p$ when $p > 1/(1+\alpha)$. In [9] Weisz conjectured that for the boundedness of the maximal operator σ_*^α from the Hardy space $H_p(I)$ to the space $L_p(I)$ the assumption $p > 1/(\alpha+1)$ is essential. We give answer to the question and prove that the maximal operator σ_*^α of the (C, α) means of the Walsh-Paley Fourier series is not bounded from the Hardy space $H_{1/\alpha+1}(I)$ to the space $L_{1/\alpha+1}(I)$. The following is true.

Theorem 1. *Let $\alpha \in (0,1)$. Then the maximal operator σ_*^α of the (C, α) means of the Walsh-Fourier series is not bounded from the Hardy space $H_{1/(\alpha+1)}(I)$ to the space $L_{1/(\alpha+1)}(I)$.*

In order to prove Theorem 1 we need the following lemma.

Lemma 1. *Let $1 < n \in \mathbf{N}$. Then*

$$\int\limits_I \max_{1 \leq N \leq 2^n} \left(A_{N-1}^\alpha |K_N^\alpha(x)|\right)^{1/(\alpha+1)} dx \geq c(\alpha) \frac{n}{\log n}.$$

Proof. It is evident that

$$\int\limits_I D_j(x) D_i(x) dx = \min\{i, j\}.$$

Then we can write

$$\int\limits_I \left(\sum_{j=1}^M A_{M-j}^{\alpha-1} D_j(x) \right)^2 dx =$$

$$= \sum_{j=1}^M \sum_{i=1}^M A_{M-j}^{\alpha-1} A_{M-i}^{\alpha-1} \int\limits_I D_j(x) D_i(x) dx = \tag{2}$$

$$= \sum_{j=1}^M \sum_{i=1}^M A_{M-j}^{\alpha-1} A_{M-i}^{\alpha-1} \min\{i, j\} \geq c_1(\alpha) M^{2\alpha+1}.$$

It is well-known that [9]

$$(3)\qquad \int_I |K_M^\alpha(x)|dx \le c_2(\alpha) < \infty, \quad M = 1,2...$$

Denote

$$E_{N_i} := \{x \in I : |K_{N_i}^\alpha(x)| \le c(\alpha)N_i\}$$

and

$$G_{N_i} := I \setminus E_{N_i},$$

where

$$N_i := \frac{2^n}{N^i}, \quad i = 1,2,..., \left[\frac{n}{\log_2 n}\right], \quad n \ge 2$$

and $c(\alpha)$ is some positive constant discussed later.

From (2) and (3) we can write

$$(4)\qquad c_1(\alpha)N_i^{2\alpha+1} \le$$

$$\le \int_I (A_{N_i-1}^\alpha |K_{N_i}^\alpha(x)|)^2 dx =$$

$$= \int_{E_{N_i}} (A_{N_i-1}^\alpha |K_{N_i}^\alpha(x)|)^2 dx + \int_{G_{N_i}} (A_{N_i-1}^\alpha |K_{N_i}^\alpha(x)|)^2 dx \le$$

$$\le c(\alpha)A_{N_i-1}^\alpha N_i \int_{E_{N_i}} (A_{N_i-1}^\alpha |K_{N_i}^\alpha(x)|dx +$$

$$+ \int_{G_{N_i}} (A_{N_i-1}^\alpha |K_{N_i}^\alpha(x)|)^{(2\alpha+1)/(\alpha+1)} (A_{N_i-1}^\alpha |K_{N_i}^\alpha(x)|)^{1/(\alpha+1)} dx \le$$

$$\le c(\alpha)c_3(\alpha)N_i^{2\alpha+1} + c_4(\alpha)N_i^{2\alpha+1} \int_{G_{N_i}} (A_{N_i-1}^\alpha |K_{N_i}^\alpha|(x)|)^{1/(\alpha+1)} dx.$$

Define

$$c(\alpha) = \frac{c_1(\alpha)}{2c_3(\alpha)}.$$

Then from (4) we get

$$(5)\qquad \int_{G_{N_i}} (A_{N_i-1}^\alpha |K_{N_i}^\alpha|(x)|)^{1/(\alpha+1)} dx \ge c_5(\alpha) > 0.$$

Denote

$$\Omega_{N_i} := G_{N_i} \setminus \bigcup_{j=1}^{i-1} G_{N_j}.$$

From the definition of the set G_{N_i} we obtain

$$c(\alpha)N_i \mathrm{mes}(G_{N_i}) < \int_{G_{N_i}} |K_{N_i}^{\alpha}(x)|dx \leq c_6(\alpha).$$

Hence

$$\mathrm{mes}(G_{N_i}) \leq \frac{c_7(\alpha)}{N_i}. \tag{6}$$

Combining (5) and (6) we get

$$\begin{aligned}
&\int_{\Omega_{N_i}} (A_{N_i-1}^{\alpha}|K_{N_i}^{\alpha}(x)|)^{1/(\alpha+1)}dx \geq \\
&\geq \int_{G_{N_i}} (A_{N_i-1}^{\alpha}|K_{N_i}^{\alpha}(x)|)^{1/(\alpha+1)}dx - \sum_{j=1}^{i-1} \int_{G_{N_i}} (A_{N_i-1}^{\alpha}|K_{N_i}^{\alpha}(x)|)^{1/(\alpha+1)}dx \geq \\
&\geq c_8(\alpha) - c_9(\alpha)N_i \sum_{j=1}^{i-1} \mathrm{mes}(G_{N_j}) \geq \\
&\geq c_8(\alpha) - c_{10}(\alpha)N_i \sum_{j=1}^{i-1} \frac{1}{N_j} \geq \\
&\geq c_8(\alpha) - \frac{c_{11}(\alpha)}{n} \geq c_{12}(\alpha), \quad \text{for} \quad n \geq n_0.
\end{aligned}$$

Consequently we can write

$$\begin{aligned}
&\int_I \max_{1\leq N\leq 2^n} (A_{N-1}^{\alpha}|K_N^{\alpha}(x)|)^{1/(\alpha+1)}dx \geq \\
&\geq \sum_{i=1}^{[n/(\log n)]} \int_{\Omega_{N_i}} \max_{1\leq N\leq 2^n} (A_{N-1}^{\alpha}|K_N^{\alpha}(x)|)^{1/(\alpha+1)}dx \geq \\
&\geq \sum_{i=1}^{[n/(\log n)]} \int_{\Omega_{N_i}} (A_{N_i-1}^{\alpha}|K_{N_i}^{\alpha}(x)|)^{1/(\alpha+1)}dx \geq \\
&\geq c_{13}(\alpha)\frac{n}{\log n}.
\end{aligned}$$

Lemma 1 is proved.

Proof of Theorem 1. Let $1 < n \in \mathbf{N}$ and

$$f_n(x) := D_{2^{n+1}}(x) - D_{2^n}(x).$$

Then we can write that

$$S_k(f_n;x) = \begin{cases} 0, & \text{if } k = 0,\dots,2^n, \\ D_k(x) - D_{2^n}(x), & \text{if } k = 2^n+1,\dots,2^{n+1}-1, \\ f_n(x), & \text{if } k \geq 2^{n+1}. \end{cases} \tag{7}$$

We have

$$f_n^*(x) = \sup_k |S_{2^k}(f_n;x)| = f_n(x)|,$$

$$\|f_n\|_{H_p} = \|f_n^*\|_p = \|D_{2^n}(x)\|_p = 2^{n(1-1/p)}. \tag{8}$$

Since

$$D_{k+2^n} - D_{2^n} = w_{2^n} D_k, \qquad k = 1,2,\dots,2^n,$$

from (7) we obtain

$$\sigma_*^\alpha f_n(x) \geq \max_{1\leq M\leq 2^n} \left|\sigma_{2^n+M}^\alpha(f_n;x)\right| =$$

$$= \max_{1\leq M\leq 2^n} \frac{1}{A_{2^n+N}^\alpha} \left|\sum_{k=2^n+1}^{2^n+M} A_{2^n+M-k}^{\alpha-1} S_k(f_n;x)\right| \geq$$

$$\geq \frac{1}{A_{2^{n+1}}^\alpha} \max_{1\leq M\leq 2^n} \left|\sum_{k=1}^{M} A_M^{\alpha-1} - k(D_{k+2^n}(x) - D_{2^n}(x))\right| \geq$$

$$\geq \frac{c_{13}(\alpha)}{2^{n\alpha}} \max_{1\leq M\leq 2^n} \left|\sum_{k=1}^{M} A_{M-k}^{\alpha-1} D_k(x)\right|.$$

Then from Lemma 1 we get

$$\frac{\|\sigma_*^\alpha f_n\|_{1/(\alpha+1)}}{\|f_n\|_{1/(\alpha+1)}} \geq \frac{c_{15}(\alpha)}{2^{n\alpha 2}2^{-n\alpha}} \left(\int_I \max_{1\leq M\leq 2^n} (A_{M-1}^\alpha |K_M^\alpha(x)|)^{1/(\alpha+1)} dx\right)^{\alpha+1} \geq$$

$$\geq c_{16}(\alpha) \left(\frac{n}{\log n}\right)^{\alpha+1} \to \infty \quad \text{as} \quad n \to \infty.$$

Theorem 1 is proved.

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