

ADDITIVE UNIQUENESS SETS FOR MULTIPLICATIVE FUNCTIONS

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Abstract. We proved that if a multiplicative function F and a positive integer k satisfy $F(2) \neq 0$, $F(5) \neq 1$ and

$$F(n^2 + m^2 + k + 1) = F(n^2 + k) + F(m^2 + 1) \quad \text{for all } n, m \in \mathbb{N},$$

then $F(n) = n$ for all positive integers n , $(n, 2) = 1$.

1. Introduction

In this paper, let \mathbb{N} and \mathcal{P} stand for the set of positive integers and prime numbers, respectively. We denote by \mathcal{M} the set of all multiplicative functions f such that $f(1) = 1$. Furthermore, we deal with the set \mathcal{B} of non-negative integers which can be represented as a sum of two squares of integers and with \mathcal{S} the set of all squares of positive integers.

In the following subsets A and B of \mathbb{N} are called additive uniqueness sets (AU-sets) for \mathcal{M} if $f \in \mathcal{M}$ satisfies

$$f(a + b) = f(a) + f(b) \quad \text{for all } a \in A \quad \text{and} \quad b \in B,$$

then $f(n) = n$ for all $n \in \mathbb{N}$. In 1992 C.Spiro [6] showed that $A = B = \mathcal{P}$ are AU-sets for \mathcal{M} . In [1] it is proved that $A = \mathcal{S}$ and $B = \mathcal{P}$ are also AU-sets for

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\mathcal{M} . Recently in [5] (see also [4]) the second author showed that if $k \in \mathbb{N}$ and $f \in \mathcal{M}$ satisfy the conditions $f(4)f(9) \neq 0$ and

$$f(n^2 + m^2 + k) = f(n^2) + f(m^2 + k) \quad \text{for all } n, m \in \mathbb{N},$$

then $f(n) = n$ for all $n \in \mathbb{N}$, $(n, 2k) = 1$. The proof of this result is used on the following theorem of K.-H. Indlekofer and N. M. Timofeev [3]:

Theorem (IT). *Let C be a non-zero integer and $A, B \in \mathbb{N}$ such that $(A, B) = 1$, $(AB, 2C) = 1$. Then there exists a positive constant $\theta = \theta(A, B, C)$ such that*

$$|\{n \leq x : A(n + C) = B(m + C), (A, n + C) = 1, n, m \in \mathcal{B}\}| > \theta \frac{x}{\log x}$$

holds for all $x \geq x_0(A, B, C)$.

We are interested in characterizing all multiplicative functions F satisfying the condition

$$(1) \quad F(n^2 + m^2 + a + b) = F(n^2 + a) + F(m^2 + b) \quad \text{for all } n, m \in \mathbb{N},$$

where a, b are given positive integers. Presently we are unable to determine all such solutions. Our purpose in this paper is to solve (1) for the case $\min(a, b) = 1$.

Theorem. *Assume that $k \in \mathbb{N}$ and $F \in \mathcal{M}$ satisfy the condition*

$$F(n^2 + m^2 + k + 1) = F(n^2 + 1) + F(m^2 + k) \quad \text{for all } n, m \in \mathbb{N}.$$

(a) *If $F(2) = 0$, then*

$$F(n^2 + m^2 + k + 1) = F(n^2 + 1) = F(m^2 + k) = 0 \quad \text{for all } n, m \in \mathbb{N}.$$

(b) *If $F(2) \neq 0$ and $F(5) = 1$, then $k \equiv 3 \pmod{4}$ and*

$$\begin{cases} F(n^2 + 1) & = \chi_2(n) + 1, \\ F(n^2 + k) & = \chi_2(n) - 1, \\ F(n^2 + m^2 + k + 1) & = \chi_2(n) + \chi_2(m) \end{cases}$$

for all $n, m \in \mathbb{N}$, where χ_2 denotes the principal character (mod 2).

(c) *If $F(2) \neq 0$ and $F(5) \neq 1$, then $F(n) = n$ for all $n \in \mathbb{N}$, $(n, 2) = 1$.*

2. Lemmas concerning an arithmetical function satisfying (1)

In this section we assume that $a, b \in \mathbb{N}$ and F is an arithmetical function with condition (1). Let

$$S_j := F(j^2 + a) \quad \text{for all } j \in \mathbb{N}.$$

First we note from (1) that

$$F(n^2 + a) + F(m^2 + b) = F(m^2 + a) + F(n^2 + b),$$

consequently

$$(2) \quad F(n^2 + b) = S_n + D \quad \text{and} \quad F(n^2 + m^2 + a + b) = S_n + S_m + D$$

hold for all $n, m \in \mathbb{N}$, where $D := F(b + 1) - F(a + 1)$.

In the proof of our theorem, we shall use the following two lemmas.

Lemma 1. *We have*

$$(3) \quad S_k + S_l = S_u + S_v \quad \text{if} \quad k^2 + l^2 = u^2 + v^2$$

and

$$(4) \quad S_x + S_y = S_z \quad \text{if} \quad x^2 + y^2 + a = z^2.$$

Proof. Assume that positive integers k, l, u, v satisfy $k^2 + l^2 = u^2 + v^2$. Then by (2) we have

$$F(k^2 + l^2 + a + b) = S_k + S_l + D \quad \text{and} \quad F(u^2 + v^2 + a + b) = S_u + S_v + D,$$

therefore (3) is true.

Now assume that positive integers x, y, z satisfy the equation $x^2 + y^2 + a = z^2$. Then we infer from (2) that

$$F(x^2 + y^2 + a + b) = F(z^2 + b)$$

and

$$F(x^2 + y^2 + a + b) = S_x + S_y + D, \quad F(z^2 + b) = S_z + D.$$

Thus (4) is true.

Lemma 2. *We have*

$$(5) \quad S_{n+12} = S_{n+9} + S_{n+8} + S_{n+7} - S_{n+5} - S_{n+4} - S_{n+3} + S_n$$

for all $n \in \mathbb{N}$ and

$$(6) \quad \begin{cases} S_7 &= 2S_5 - S_1, \\ S_8 &= 2S_5 + S_4 - 2S_1, \\ S_9 &= S_6 + 2S_5 - S_2 - S_1, \\ S_{10} &= S_6 + 3S_5 - S_3 - 2S_1, \\ S_{11} &= S_6 + 4S_5 - S_3 - S_2 - 2S_1, \\ S_{12} &= S_6 + 4S_5 + S_4 - S_2 - 4S_1. \end{cases}$$

Proof. This is Lemma 1 of [5].

3. The proof of the theorem

In this section we assume that $k \in \mathbb{N}$ and $F \in \mathcal{M}$ satisfy the condition

$$F(n^2 + m^2 + k + 1) = F(n^2 + 1) + F(m^2 + k) \quad \text{for all } n, m \in \mathbb{N}.$$

We shall use the notations and results of Lemma 1-2 with $a = 1$ and $b = k$. Let $S_j := f(j^2 + 1)$. Then, by (2), we have

$$(7) \quad F(n^2 + m^2 + k + 1) = S_n + S_m + D \quad \text{and} \quad F(n^2 + k) = S_n + D$$

for all $n, m \in \mathbb{N}$, where $D = F(k + 1) - F(2)$.

First we note from Lemma 1 that if $x^2 + y^2 + 1 = z^2$, then $S_x + S_y = S_z$. Since $2^2 + 2^2 + 1 = 3^2$ and $4^2 + 8^2 + 1 = 9^2$, we have

$$(8) \quad \begin{cases} S_3 &= 2S_2, \\ S_9 &= S_4 + S_8. \end{cases}$$

On the other hand, by using the facts

$$F(10) = F(2)F(5) \quad \text{and} \quad F(2)F(65) = F(5)F(26),$$

we have

$$(9) \quad S_3 = S_1S_2 \quad \text{and} \quad S_1S_8 = S_2S_5.$$

Therefore, we infer from (6), (8) and (9) that $S_4 = S_9 - S_8 = S_6 - S_4 - S_2 + S_1$, consequently

$$(10) \quad \begin{cases} S_6 &= 2S_4 + S_2 - S_1, \\ S_7 &= 2S_5 - S_1, \\ S_8 &= 2S_5 + S_4 - 2S_1, \\ S_9 &= 2S_5 + 2S_4 - 2S_1, \\ S_{10} &= 3S_5 + 2S_4 - S_2 - 3S_1, \\ S_{11} &= 4S_5 + 2S_4 - 2S_2 - 3S_1, \\ S_{12} &= 4S_5 + 3S_4 - 5S_1. \end{cases}$$

Lemma 3. *If $S_2 = 0$, then $S_1 = 0$.*

Proof. In the following we assume that $S_2 = F(5) = 0$. Hence we obtain from (8)-(10) that

$$S_3 = 0 \quad \text{and} \quad S_8 = S_{12} = 0,$$

because $S_8 = F(65) = F(5)F(13) = 0$ and $S_{12} = F(145) = F(5)F(29) = 0$. An application of the formula of S_8 and S_{12} in (10) shows that $S_4 = S_1$ and $S_5 = \frac{1}{2}S_1$. Consequently, one can check from (10) the following results:

$$\begin{cases} S_2 = S_3 = S_7 = S_8 = S_{12} = 0, \\ S_1 = S_4 = S_6 = S_9 = S_{11}, \\ S_5 = S_{10} = \frac{1}{2}S_1. \end{cases}$$

Hence, it follows from (5) that the sequence $\{S_n\}_{n=1}^{\infty}$ is also periodic, namely

$$(11) \quad S_n = S_m \quad \text{if} \quad n \equiv m \pmod{5} \quad \text{and} \quad S_j \in \left\{ S_1, 0, 0, S_1, \frac{1}{2}S_1 \right\}$$

for all $j \in \mathbb{N}$.

We shall prove that $S_1 = 0$. Assume that $S_1 \neq 0$. Then, by using (11), we have

$$\frac{1}{2}S_1 = S_5 = F(2)F(13) = S_1F(13),$$

$$S_1 = S_{21} = F(21^2 + 1) = F(2)F(13)F(17) = F(13)S_1S_4 = F(13)S_1^2$$

and

$$S_1 = S_4 = S_{34} = F(34^2 + 1) = F(13)F(89),$$

which imply that

$$F(13) = \frac{1}{2}, \quad S_1 = 2 \quad \text{and} \quad F(89) = 4.$$

Hence

$$1 = \frac{1}{2}S_1 = S_{55} = F(55^2 + 1) = F(2)F(17)F(89) = S_1^2 F(89) = 2^2 \cdot 4 = 16,$$

which is impossible.

Thus, $S_1 = 0$ and the proof of Lemma 3 is complete.

Lemma 4. *If $S_1 = 0$, then*

$$F(n^2 + m^2 + k + 1) = F(n^2 + 1) = F(m^2 + k) = 0 \quad \text{for all } n, m \in \mathbb{N}.$$

Proof. We assume that $S_1 = F(2) = 0$. In this case, we have

$$S_{2t+1} = F((2t+1)^2 + 1) = F(2)F(2t^2 + 2t + 1) = 0$$

for all $t \in \mathbb{N}$. Hence

$$S_1 = S_3 = S_5 = S_7 = S_9 = S_{11} = 0.$$

It is easy to check from the formula of S_3 and S_9 in (8)-(10) that

$$S_2 = S_4 = 0.$$

Hence, (10) gives

$$S_6 = S_8 = S_{10} = S_{12} = 0.$$

Thus we have proved that $S_i = 0$ for all $i \in \mathbb{N}$, $1 \leq i \leq 12$, which with (5) show that $S_n = F(n^2 + 1) = 0$ for all $n \in \mathbb{N}$.

Now we use Theorem (IT) to show that $D = F(k+1) - F(2) = 0$. Assume that $D \neq 0$. Then, from (7) and from our assumptions, we have

$$(12) \quad F(\nu + C) = D \quad \text{for all } \nu \in \mathcal{B},$$

where $C = k + 1$. Let $A := (2C)^2 + 1$, $B = 1$. Then $(A, B) = (AB, 2C) = 1$, therefore Theorem (IT) implies that there are positive integers $\nu, \mu \in \mathcal{B}$ such that

$$A(\nu + C) = \mu + C \quad \text{and} \quad (A, \nu + C) = 1.$$

From (12) we obtain

$$F(A)D = F(A)F(\nu + C) = F(A(\nu + C)) = F(\mu + C) = D,$$

which with $D \neq 0$ implies $F(A) = 1$. This is impossible, because

$$F(A) = F((2C)^2 + 1) = S_{2C} = 0.$$

The part (a) of our theorem is proved.

Lemma 5. *If $F(2) \neq 0$, then $S_1 = F(2) = 2$ and $S_2 = F(5) \in \{1, 5\}$.*

Proof. Assume that $S_1 = F(2) \neq 0$. First we note from Lemma 3 and Lemma 4 that $S_1 \neq 0$ and $S_2 \neq 0$ are satisfied. It follows from (8) and (9) that $S_3 = S_1 S_2 = 2S_2$, consequently $S_1 = 2$.

Next we prove that

$$(13) \quad S_5 = S_4 + 2S_2 - 1.$$

Indeed, using the following relations

$$17^2 + 1 = 2 \cdot (12^2 + 1), \quad 1^2 + 13^2 = 7^2 + 11^2 \quad \text{and} \quad 1^2 + 17^2 = 11^2 + 13^2,$$

which with (3), using the fact $S_1 = F(2) = 2$ and the multiplicativity of F , imply

$$(14) \quad S_{17} = 2S_{12}, \quad 2 + S_{13} = S_7 + S_{11} \quad \text{and} \quad 2 + S_{17} = S_{11} + S_{13}.$$

Thus, we have established, in view of (10), that

$$S_7 = 2 + S_{13} - S_{11} = 4 + S_{17} - 2S_{11} = 4 + 2S_{12} - 2S_{11}$$

and so

$$S_7 - 4 = 2(S_{12} - S_{11}) = 2(S_4 + 2S_2 - 4).$$

Therefore (13) is true, because it is known from (10) that $S_7 - 4 = 2(S_5 - 3)$.

Now we prove that

$$(15) \quad S_2 \in \{1, 5\}.$$

We obtain from (10) and (13)-(14) that

$$\begin{aligned} S_{13} &= S_7 + S_{11} - 2 = (2S_5 - 2) + (4S_5 + 2S_4 - 2S_2 - 6) - 2 = \\ &= 6S_5 + 2S_4 - 2S_2 - 10 = 6(S_4 + 2S_2 - 1) + 2S_4 - 2S_2 - 10 = \\ &= 8S_4 + 10S_2 - 16 \end{aligned}$$

and

$$S_{13} = F(13^2 + 1) = F(2)F(5)F(17) = 2S_2S_4,$$

which show that

$$(16) \quad (S_2 - 4)S_4 = 5S_2 - 8.$$

Let $x := S_2$. It is obvious from the above relation that $x - 4 \neq 0$, therefore $S_4 = (5x - 8)/(x - 4)$.

On the other hand, we infer from (9), (10) and (13) that

$$2S_8 = S_2S_5 = S_2(S_4 + 2S_2 - 1) = x \left(\frac{5x - 8}{x - 4} + 2x - 1 \right)$$

and

$$2S_8 = 2(2S_5 + S_4 - 2S_1) = 4(S_4 + 2S_2 - 1) + 2S_4 - 8 = 6 \left(\frac{5x - 8}{x - 4} \right) + 8x - 12.$$

After simplifications of these relations of $2S_8$, we obtain the equation of the form

$$x(x - 1)(x - 5) = 0,$$

consequently $x = S_2 \in \{0, 1, 5\}$. Thus (15) is proved, since $x = S_2 \neq 0$. The Lemma 5 is proved.

Lemma 6. *If $F(2) \neq 0$ and $F(5) = 1$, then $k \equiv 3 \pmod{4}$, $D = -2$ and $S_n = \chi_2(n) + 1$ for all $n \in \mathbb{N}$.*

Proof. Assume that $F(2) \neq 0$ and $F(5) = 1$. Then it follows from (8), (13) and (16) that $S_3 = S_5 = 2$ and $S_4 = 1$. Consequently, we infer from (10) that

$$S_n = S_{(n,2)} \quad \text{for } n \in \mathbb{N}, \quad 1 \leq n \leq 12,$$

which by (5) implies that the sequence $\{S_n\}_{n=1}^{\infty}$ is periodic, namely

$$(17) \quad S_n = S_{(n,2)} = \chi_2(n) + 1 \quad \text{for all } n \in \mathbb{N},$$

where χ_2 denotes the principal character (mod 2).

In order to prove that k is odd, we note from (7) and (17) that

$$F(k + 25 + 1) = F(3^2 + 4^2 + k + 1) = S_3 + S_4 + D = 3 + D,$$

$$F(k + 26 + 1) = F(1^2 + 5^2 + k + 1) = S_1 + S_5 + D = 4 + D$$

and

$$F(k + 26)F(k + 27) = F((k + 26)^2 + 5^2 + k + 1) = S_{k+26} + 2 + D,$$

which imply

$$(18) \quad S_k = \chi_2(k) + 1 = D^2 + 6D + 10.$$

Hence $\chi_2(k) = (D + 3)^2 \in \{0, 1\}$, consequently

$$(19) \quad D \in \{-4, -3, -2\}.$$

Assume that k is even. Then (18) implies that $D = -3$. By

$$\left(\frac{k}{2}\right)^2 + 1 + k + 1 = \left(\frac{k}{2} + 1\right)^2 + 1,$$

we have

$$S_{\frac{k}{2}+1} = S_{\frac{k}{2}} + S_1 + D = S_{\frac{k}{2}} - 1,$$

which gives

$$1 = \chi_2\left(\frac{k}{2}\right) - \chi_2\left(\frac{k}{2} + 1\right).$$

This is true for the case when $k \equiv 2 \pmod{4}$.

Since $F(n^2 + k) = S_n + D = \chi_2(n) - 2$ and $F(2) = S_1 = 2$, we have

$$F\left(2\ell^2 + \frac{k}{2}\right) = -1 \text{ for all } \ell \in \mathcal{N}.$$

Let $p = 4Q + 1 \in \mathcal{P}$ and $p > k$. Then

$$\begin{aligned} \left(2Q^2 + \frac{k}{2}, (2Q + 1)^2 + k\right) &= \left(2Q^2 + \frac{k}{2}, 4Q + 1\right) = \\ &= (4Q^2 + k, p) = (k - Q, p) = 1 \end{aligned}$$

and

$$\left(2Q^2 + \frac{k}{2}\right) [(2Q + 1)^2 + k] = 2 \left(Q(2Q + 1) + \frac{k}{2}\right)^2 + \frac{k}{2}$$

imply that

$$F\left(2Q^2 + \frac{k}{2}\right) F[(2Q + 1)^2 + k] = F\left[2 \left(Q(2Q + 1) + \frac{k}{2}\right)^2 + \frac{k}{2}\right].$$

Hence we infer that

$$S_{2Q+1} + D = S_{2Q+1} - 3 = 1, \text{ i.e. } S_{2Q+1} = 4.$$

This is impossible. Therefore k is odd.

Assume that k is odd. Then the relation

$$F\left[\left(\frac{k+3}{2}\right)^2 + 2^2 + k + 1\right] = F\left[\left(\frac{k+5}{2}\right)^2 + 1\right],$$

with (7) and (18)-(19) implies

$$D = S_{\frac{k+5}{2}} - S_{\frac{k+3}{2}} - S_2 = S_{\frac{k+5}{2}} - S_{\frac{k+3}{2}} - 1 \in \{-4, -3, -2\}.$$

This is true for the case when $k \equiv 3 \pmod{4}$ and $D = -2$.

Lemma 6 and the part (b) of our theorem is proved.

Lemma 7. *If $F(2) \neq 0$ and $F(5) \neq 1$, then*

$$(20) \quad \begin{cases} F(n^2 + 1) & = n^2 + 1, \\ F(n^2 + k) & = n^2 + k, \\ F(n^2 + m^2 + k + 1) & = n^2 + m^2 + k + 1 \end{cases}$$

are satisfied for all $n, m \in \mathbb{N}$.

Proof. First we note from Lemma 5 that $F(2) \neq 0$ and $F(5) \neq 1$ imply that $S_1 = F(2) = 2$ and $S_2 = F(5) = 5$. Hence we infer from (8) and (16) that $S_3 = 2S_2 = 10$ and

$$S_4 = \frac{(5S_2 - 8)}{S_2 - 4} = 17.$$

These with (10) and (13) imply that $S_n = F(n^2 + 1) = n^2 + 1$ for $1 \leq n \leq 12$. Hence we infer from (5) that the first relation of (20) is true.

Next we prove that

$$(21) \quad D = k - 1.$$

Indeed, if k is odd, then

$$F\left[\left(\frac{k+3}{2}\right)^2 + 2^2 + k + 1\right] = F\left[\left(\frac{k+5}{2}\right)^2 + 1\right],$$

with the first relation of (20) implies

$$D = S_{\frac{k+5}{2}} - S_{\frac{k+3}{2}} - S_2 = \left(\frac{k+5}{2}\right)^2 + 1 - \left(\frac{k+3}{2}\right)^2 - 1 - 5 = k - 1.$$

If k is even, then by

$$\left(\frac{k}{2}\right)^2 + 1 + k + 1 = \left(\frac{k}{2} + 1\right)^2 + 1,$$

we also obtain from the first relation of (20) that

$$D = S_{\frac{k}{2}+1} - S_{\frac{k}{2}} - S_1 = \left(\frac{k}{2} + 1\right)^2 + 1 - \left(\frac{k}{2}\right)^2 - 1 - 2 = k - 1.$$

Thus (21) is proved, therefore all relations of (20) follow from (7). The proof of Lemma 7 is complete.

In order to complete the proof of our theorem, it remains to deduce from (20) the following result.

Lemma 8. *If (20) holds, then*

$$(22) \quad F(n) = n \quad \text{for all } (n, 2) = 1.$$

Proof. First we prove (22) for $n \in \mathbb{N}$ with the condition $(n, 2(k+1)) = 1$. Assume that $n \in \mathbb{N}$, $(n, 2(k+1)) = 1$. Then the Theorem (IT) implies that there are positive integers μ and ν such that

$$n(\mu + k + 1) = \nu + k + 1 \quad \text{and} \quad (n, \mu + k + 1) = 1.$$

Thus, from (20) we have

$$\begin{aligned} n(\mu + k + 1) &= \nu + k + 1 = F(\nu + k + 1) = F[n(\mu + k + 1)] = \\ &= F(n)F(\mu + k + 1) = F(n)(\mu + k + 1), \end{aligned}$$

which proves (22) under the condition $(n, 2(k+1)) = 1$.

Next we prove that

$$(23) \quad F(p^\alpha) = p^\alpha \quad \text{for } p \in \mathcal{P}, p > 2 \quad \text{and} \quad p \mid k + 1.$$

Let $2, p_1, \dots, p_r$ be all distinct prime divisors of $2(k+1)$. Let $p \in \{p_1, \dots, p_r\}$ and $\alpha \in \mathbb{N}$. We consider the equation

$$(24) \quad x^2 + k = p^\alpha y.$$

Since $k \equiv -1 \pmod{p}$, therefore there are $x_\alpha, y_\alpha \in \mathbb{N}$ such that

$$x_\alpha^2 + k = y_\alpha p^\alpha \quad \text{and} \quad (p^\alpha - x_\alpha)^2 + k = (p^\alpha - 2x_\alpha + y_\alpha) p^\alpha.$$

It is obvious that one of y_α and $p^\alpha - 2x_\alpha + y_\alpha$ is coprime to p . Assume that $x_\alpha, y_\alpha \in \mathbb{N}$ satisfy (24) and $(y_\alpha, p) = 1$. Let $x = p^\alpha t + x_\alpha$ and $y = p^\alpha t^2 + 2x_\alpha t + y_\alpha$. Then (x, y) is also a solution of (24).

Since $p^\alpha t^2 + 2x_\alpha t + y_\alpha \equiv 0 \pmod{p_i}$ has at most two solutions and $p_i \geq 3$ for all $1 \leq i \leq r$, consequently there is $t_i \in \mathbb{N}$ such that

$$(p^\alpha t_i^2 + 2x_\alpha t_i + y_\alpha, p_i) = 1 \quad \text{for } i = 1, \dots, r.$$

On the other hand, it is easy to see that

$$(p^\alpha (y_\alpha + 1)^2 + 2x_\alpha (y_\alpha + 1) + y_\alpha, 2) = (y_\alpha + 1 + y_\alpha, 2) = 1.$$

Hence an application of the Chinese Remainder Theorem shows that there is $t_0 \in \mathbb{N}$ for which

$$(p^\alpha t_0^2 + 2x_\alpha t_0 + y_\alpha, 2(k+1)) = 1.$$

Thus we have proved that

$$(x_0, y_0) = (p^\alpha t_0 + x_\alpha, p^\alpha t_0^2 + 2x_\alpha t_0 + y_\alpha)$$

is a solution of (24) with the condition $(y_0, 2(k+1)) = 1$.

Finally, we infer from (20) and (22) that

$$p^\alpha y_0 = x_0^2 + k = F(x_0^2 + k) = F(p^\alpha y_0) = F(p^\alpha) F(y_0) = F(p^\alpha) y_0,$$

which proves (23). The proof of Lemma 8 is complete.

The theorem is proved.

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