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SOME FURTHER REMARKS ON THE ITERATES OF THE φ AND THE σ-FUNCTIONS

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1. Introduction

Notations. $\mathbb{N}$ = set of positive integers, $\mathcal{P}$ = set of primes; $\omega(n)$ = the number of distinct prime factors of n, $\varphi(n)$ = Euler's totient function, $\sigma(n)$ = the sum of positive divisors of n. For some multiplicative function $f:\mathbb{N}\to\mathbb{N}$, let $f_0(n)=n,\ f_1(n)=f(n),\ f_{j+1}(n)=f(f_j(n))\quad (j=1,2,\dots)$. For the variable x let $x_1=\log x,\ x_2=\log x_1,\dots$. The letters p,q with or without suffixes always denote prime numbers. The largest prime factor of n is denoted by $P(n)$, the smallest prime factor of n is $p(n)$.

As usual let

$$\Psi(x,y)=\#\{n\le x\ :\ P(n)\le y\}\qquad (x\ge y\ge 2), \tag{1.1}$$

$$\Phi(x,y)=\#\{n\le x\ :\ p(n)>y\}. \tag{1.2}$$

It is known (see Tenenbaum [1], Theorem I.4.2) that

$$\Phi(x,y)=x\prod_{p\le y}\left(1-\frac{1}{p}\right)\left\{1+O\left(\frac{1}{(\log y)^2}\right)\right\} \tag{1.3}$$

if

$$2\le y\le \exp\left(\frac{x_1}{10x_2}\right).$$

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Several questions on the prime factors of $\varphi_k(n)$ (k-fold iterate of φ), and on $\sigma_k(n)$, furthermore the size of the set of $n \le x$ satisfying $(n, \varphi_k(n)) = 1$ were investigated. A non-complete list of the relevant paper is: [2]-[14].

2. On the function $E_k(n) := (n, \varphi_k(n))$

Let $k \ge 1$ be fixed,

$$E_k(n) := (n, \varphi_k(n)). \tag{2.1}$$

$$K_k(n) := (n, \sigma_k(n)). \tag{2.2}$$

Let

$$A(n, y) := \prod_{\substack{p^\alpha \| n \\ p < y}} p^\alpha. \tag{2.3}$$

Theorem 1. *We have*

$$\frac{1}{x} \# \left\{ n \le x \ : \ E_k(n) \ne A(n, x_2^k) \right\} = o_x(1), \tag{2.4}$$

$$\frac{1}{x} \# \left\{ n \le x \ : \ K_k(n) \ne A(n, x_2^k) \right\} = o_x(1). \tag{2.5}$$

Proof. We shall prove only (2.4). The proof of (2.5) is similar, so we omit it. Let ε_x be a sequence tending to zero (slowly). Let $\mathcal{A}_1$ be the set of those $n \le x$ for which $p|n$ holds for some $p \in L_x = \left[x_2^{k-\varepsilon_x}, x_2^{k+\varepsilon_x} \right]$. Then

$$\#(\mathcal{A}_1) \le \sum_{p \in L_x} \left[\frac{x}{p} \right] \le x \log \frac{k + \varepsilon_x}{k - \varepsilon_x} + O\left(\frac{x}{x_3} \right) = O\left(\left(\varepsilon_x + \frac{1}{x_3} \right) x \right). \tag{2.6}$$

Let $\mathcal{A}_2$ be the set of those $n \le x$ for which $q|E(n)$ for some $q > x_2^k$.

We shall say that $p_0, p_1, \dots, p_k$ is a chain of primes, if

$$p_{j+1} - 1 \equiv 0 \pmod{p_j} \quad (j = 0, \dots, k-1)$$

holds. Assume that $p_0|\varphi_k(n)$. Then either $p_0^2|\varphi_{k-1}(n)$, or not, and in the second case $p_1|\varphi_{k-1}(n)$, where $p_1-1\equiv 0 \pmod{p_0}$. We can proceed as we did in [13] and deduce that

$$\#(\mathcal{A}_2) \ll \sum_{q>x_2^k} \frac{x}{qp_k}, \tag{2.7}$$

where we sum over all chains $q=p_0,p_1,\dots,p_k(\le x)$.

The following assertion is proved in [4].

Lemma 1. *Let*

$$\delta(x,k,l) := \sum_{\substack{p\le x\\ p\equiv l \pmod k}} 1/p.$$

For $l=1$ or -1 and $k\le x$, $x\ge 3$ we have

$$\delta(x,k,l) \le \frac{c_1 x}{\varphi(k)},$$

where c_1 is an absolute constant.

By using Lemma 1, we have

$$\sum_{(x_2^k<q)} \frac{1}{q} \sum_{p_1}\sum_{p_2}\cdots\sum_{p_k}\frac{1}{p_k} \le c_1x_2 \sum_{q}\sum_{p_1}\cdots\sum_{p_{k-1}}\frac{1}{p_{k-1}} \le$$
$$\le \dots \le c_1^k x_2^k \sum_{q>x_2^k} 1/q^2 = O\left(1/x_3\right).$$

Thus

$$\#(\mathcal{A}_2) = O\left(\frac{x}{x_3}\right). \tag{2.8}$$

Let Y_x be a sequence tending to infinity slowly. We assume that $Y_x = O(x_4)$. Let $\kappa(n)$ be the largest prime power divisor of n, with exponent at least 2, i.e.

$$\kappa(n) := \max_{\substack{p^\alpha|n\\ \alpha\ge 2}} p^\alpha.$$

Let $\mathcal{A}_3 := \{n\le x \;:\; \kappa(n)\ge Y_x\}$. It is obvious that

$$\#(\mathcal{A}_3) \ll \sum_{\substack{p^a\ge Y_x\\ a\ge 2}} \frac{x}{p^a} \ll \frac{x}{\sqrt{Y_x}}. \tag{2.9}$$

Let $\pi \in \mathcal{P},\ \pi < x_2^{k-\varepsilon_x}$, and $\mathcal{A}_\pi$ be the set of those $n = \pi\nu \le x,\ \nu \in \mathbb{N}$, for which $(\pi, \varphi_k(\pi\nu)) = 1$. This holds only if $(\pi, \varphi_k(\nu)) = 1$.

Let $\mathcal{B}_k(\pi)$ be the set of all those primes p_k for which there exists a chain the starting element of which is $p_0 = \pi$ (thus $p_0, p_1, \ldots, p_k$ is a suitable chain). It is clear that $(\pi, \varphi_k(\nu)) = 1$ implies that $(\pi, \mathcal{B}_k(\pi)) = 1$.

Thus, by Brun's sieve we have

$$\#(\mathcal{A}_\pi) \le \frac{cx}{\pi} \prod_{\substack{p \in \mathcal{B}_k(\pi) \\ p < x}} (1 - 1/p). \tag{2.10}$$

By using the method given in [12], [13] we obtain that

$$\sum_{\substack{p \in \mathcal{B}_k \\ p < x}} 1/p \ge \frac{1}{2} x_2 \quad \text{(say)},$$

whence

$$\#(\mathcal{A}_\pi) \ll \frac{x}{\pi} \exp\left(-\frac{1}{2} x_2\right), \tag{2.11}$$

and so

$$\sum_{\pi < x_2^{k-\varepsilon_x}} \#(\mathcal{A}_\pi) \ll x x_2 \sqrt{x_1}. \tag{2.12}$$

If $n \le x$ is such a number which does not belong to $\mathcal{A}_1 \cup \mathcal{A}_2 \cup (\cup \mathcal{A}_\pi)$, then $E_k(n) = A(n, x_2^k)$. (2.4) is proved.

By similar method one can prove

Theorem 2. *We have*

$$\#\{p \le x \ :\ E_k(p+a) \ne A(p+a, x_2^k)\} = o(\pi(x)), \tag{2.13}$$

$$\#\{p \le x \ :\ K_k(p+a) \ne A(p+a, x_2^k)\} = o(\pi(x)), \tag{2.14}$$

as $x \to \infty$. Here $a \ne 0$ is an arbitrary integer.

3. Some lemmas

3.1. Let $\pi_r(x) = \#\{n \le x \ : \ \omega(n) = r\}$. Hardy and Ramanujan [14] proved that

$$\pi_r(x) \le \frac{x}{x_1} \frac{(x_2 + c)^{r-1}}{(r-1)!} \tag{3.1}$$

holds for every $r \in \mathbb{N}$, $x \ge 3$, with a suitable absolute constant c.

Hence one can prove immediately

Lemma 3. *For every $c_1 \in \mathbb{R}$ there exists $c_2 \in \mathbb{R}$ such that*

$$\#\{n \le x \ : \ \omega(n) \ge c_2 x_2\} \ll \frac{x}{x_1^{c_1}}. \tag{3.2}$$

3.2. Let Q be an arbitrary prime in the interval $x_2^k \le Q \le x_1^{1/3}$. Let $\kappa_0, \kappa_1, \dots$ be a sequence of completely additive functions defined for primes p as follows:

$$\kappa_0(p) = \begin{cases} 1, & \text{if } p = Q, \\ 0, & \text{if } p \ne 0, \end{cases} \tag{3.3}$$

$$\kappa_{j+1}(p) = \sum_{\substack{q \in \mathcal{P} \\ q \mid p-1}} \kappa_j(q). \tag{3.4}$$

Let

$$S_j(y) := \sum_{p \le y} \kappa_j(p). \tag{3.5}$$

Lemma 4. *Let $x^{1/4} \le y \le x$. Then, for $j \ge 2$,*

$$S_j(y) = \frac{(\operatorname{li} y)(\log\log y)^{j-1}}{(j-1)!(Q-1)} + O\left(\frac{(\operatorname{li} y)(\log\log y)^{j-2}}{Q^{(m-1)/m}}\right) \tag{3.6}$$

and

$$S_1(y) = \pi(y, Q, 1) = \frac{\operatorname{li} y}{Q-1} + O\left(\frac{\operatorname{li} y}{Q} e^{-c\sqrt{\log y}}\right). \tag{3.7}$$

Here m is an arbitrary positive integer, the constants implied by the error terms may depend on j and m.

Lemma 4 is proved in [3] (Lemma 7).

Let

$$\mathcal{T}_Y^{(k)} := \{\nu \le Y \;:\; p(\nu) > x_2^k\}, \tag{3.8}$$

$$T_Y^{(k)}(s) := \#\left\{\nu \in \mathcal{T}_Y^{(k)} \;:\; \omega(E_k(\nu)) = s\right\}. \tag{3.9}$$

Lemma 5. *Let $\sqrt{x} \le Y \le x$, $x \ge 100$. Then, with a suitable constant c which may depend only on k, we have*

$$T_Y^{(k)}(s) \le \frac{1}{s!}\left(\frac{c}{x_3}\right)^s Y \tag{3.10}$$

for every $s \ge 1$, and for every fixed s_0

$$T_Y^{(k)}(s) \ll \frac{1}{s!}\left(\frac{c}{x_3}\right)^s \Phi(y, x_2^k), \tag{3.11}$$

if $s = 1, \dots, s_0$. Here Φ is defined by (1.2).

Proof. Let $(x_2^k \le)\ Q_1 < \ldots < Q_s$ be primes for which $Q_1 \ldots Q_s | E_k(\nu)$. Repeating the argument used in [13] one can see that the number of these $\nu \in \mathcal{T}_y^{(k)}$ is less than

$$\sum_{p_1^{(k)}, \dots, p_s^{(k)}} \Phi\left(\frac{Y}{Q_1 \ldots Q_s\, p_1^{(k)} \ldots p_s^{(k)}},\ x_2^k\right), \tag{3.12}$$

where $p_j^{(k)}$ is the final element of the chain of primes $Q_j (= p_j^{(0)}),\ p_j^{(1)}, \dots, p_j^{(k)}$.

The contribution of the extraordinary cases when $p_{j_1}^{(l_1)} = p_{j_2}^{(l_2)}$ $(j_1 \ne j_2)$, or if $p_l^{(t)2} \mid \varphi_{k-t}(\nu)$ is smaller than (3.13) (which is an upper bound of (3.12)).

Since $\Phi(Y, x_2^k) \le Y$, we obtain from Lemma 1 that (3.12) is less than

$$\frac{Y x_2^{ks}}{Q_1^2 \ldots Q_s^2}, \tag{3.13}$$

whence by summing over all sets of primes $(x_2^k \le)\ Q_1 < \ldots < Q_s\ (< x)$ we obtain that

$$\sum \frac{1}{Q_1^2 \ldots Q_s^2} \le \frac{1}{s!} \left\{ \sum_{x_2^k < Q < x} 1/Q^2 \right\}^s \le \frac{1}{s!} \left(\frac{c}{x_2^k x_3} \right)^s.$$

Hence (3.10) is immediate.

To prove (3.11), we estimate $T_Y^{(k)}(s)$ by

$$\sum_{(x_2^k <) Q_1 < \ldots < Q_s \le x_1^{1/3}} \ \sum_{p_1^{(k)}, \ldots, p_s^{(k)}} \Phi\left(\frac{Y}{Q_1 \ldots Q_s\ p_1^{(k)} \ldots p_s^{(k)}},\ x_2^k \right) +$$
$$+ \sum_{\substack{Q_1 < \ldots < Q_s \\ Q_s > x_1^{1/3}}} \ \sum_{p_1^{(k)}, \ldots, p_s^{(k)}} \left[\frac{Y}{Q_1 \ldots Q_s\ p_1^{(k)} \ldots p_s^{(k)}} \right] = \sum\nolimits_1 + \sum\nolimits_2 .$$

As earlier, we obtain that

$$\sum\nolimits_2 \le \sum_{Q_s > x_1^{1/3}} \frac{1}{Q_s^2} \cdot \frac{1}{(s-1)!} \left\{ \sum_{Q > x_2^k} 1/Q^2 \right\}^{s-1} (c_1 x_2^k)^s \ll \frac{Y}{x_1},$$

say. In $\sum_1$ first we sum over those $p_1^{(k)}, \ldots, p_s^{(k)}$ for which $\max\limits_{l=1,\ldots,k} p_l^{(k)} < x^{1/4s}$.

For such collection of $p_1^{(k)}, \ldots, p_s^{(k)}$:

$$\Phi\left(\frac{Y}{Q_1 \ldots Q_s\ p_1^{(k)} \ldots p_s^{(k)}},\ x_2^k \right) \ll \frac{Y \cdot x_3^{-1}}{Q_1 \ldots Q_s\ p_1^{(k)} \ldots p_s^{(k)}}, \tag{3.14}$$

and for the others we use the trivial inequality

$$\Phi\left(\frac{y}{Q_1 \ldots Q_s\ p_1^{(k)} \ldots p_s^{(k)}},\ x_2^k \right) \le \left[\frac{Y}{Q_1 \ldots Q_s\ p_1^{(k)} \ldots p_s^{(k)}} \right]. \tag{3.15}$$

Summing up the right hand side of (3.14) over $Q_1, \ldots, Q_s$ we obtain the bound

$$Y \cdot x_3^{-1} \cdot \frac{1}{s!} \left(\frac{c}{x_3} \right)^s.$$

It remains to estimate the cases when $\max p_l^{(k)} \geq x^{1/4s}$. We shall estimate

$$\sum_{x>p_l^{(k)}>x^{1/4s}} 1/p_l^{(k)}, \tag{3.16}$$

where $p_l^{(k)}$ is the final element of the chain $Q_l(=p_l^{(0)}),\ p_l^{(1)},\dots,p_l^{(k)}$.

Let $M := x^{1/4s}$, T be the smallest integer for which $2^T M \geq x$. Thus $T = O(x_1)$. Let us define the sequence of the completely additive functions κ_j by the rules (3.3), (3.4) with the choice $Q = Q_l$.

Applying Lemma 4 we get immediately that

$$\sum_{r=0}^{T} \frac{1}{2^r M} S_k\left(2^{r+1}M\right) \ll \begin{cases} \frac{x_2^{k-1}}{Q_l} + x_2^{k-2}/Q_l^{(m-1)/m} & \text{if} \quad k \geq 2, \\ 1/Q_l & \text{if} \quad k = 1. \end{cases}$$

The proof of Lemma 4 can be completed easily.

4. Some theorems

Let $z \geq 1$ be a constant, $h(n) := z^{\omega(n)}$. Let

$$M_k(x) := \sum_{n\leq x} h(E_k(n)), \tag{4.1}$$

$$T_k(x) := \sum_{n\leq x} h(K_k(n)). \tag{4.2}$$

Theorem 3. *We have*

$$\frac{M_k(x)}{x} = (1+o_x(1))C(kx_3)^{z-1}, \tag{4.3}$$

$$\frac{T_k(x)}{x} = (1+o_x(1))C(kx_3)^{z-1}, \tag{4.4}$$

where $C = C(z,k)$ is a nonzero constant.

Proof. We shall prove only (4.1). The proof of (4.2) is similar, so we omit it.

By using Lemma 3, we can find a constant c_3 such that

$$\sum_{\omega(n)>c_3x_2} h(E_k(n)) \ll x/x_1^2, \tag{4.5}$$

say.

Let

$$\mathcal{R}_1 := \{n \le x \mid \omega(n) \ge c_3x_2\}. \tag{4.6}$$

Let

$$\mathcal{R}_2(V) := \{n \le x \mid A(n, x_2^k) \ge V\}, \tag{4.7}$$

where $V \in [x_1, x^{1/4}]$.

By using the known inequality

$$\Psi(x,y) \ll x \exp\left(-\frac{1}{2}\frac{\log x}{\log y}\right) \quad \text{as} \quad 2 \le y \le x \tag{4.8}$$

(see e.g. [1] Chapter III. 5. Theorem 1), we can deduce that

$$\#\mathcal{R}_2(V) \ll x \cdot kx_3 \exp\left(-\frac{1}{2}\frac{\log V}{kx_3}\right). \tag{4.9}$$

Indeed,

$$\#\mathcal{R}_2(V) \le x \sum_{\substack{V \le D \le x \\ P(D) \le x_2^k}} \frac{1}{D} \le x \cdot \sum_{j=0}^{j_0} \frac{1}{2^jV} \Psi\left(2^{j+1}V,\ x_2^k\right), \tag{4.10}$$

where j_0 is the smallest integer for which $2^{j_0}V \ge x$. From (4.8), (4.10) the inequality (4.9) follows.

Let now $V = \exp(x_2^2)$. We have

$$\#\mathcal{R}_2(\exp(x_2^2)) \ll x/x_1^B, \tag{4.11}$$

where B is an arbitrary large positive constant.

Let $D \le \exp(x_2^2)$ be fixed and let

$$U_1(D) := \sum\nolimits^* h(E_k(n))$$

for those $n = D\nu$, for which $\nu \in \mathcal{T}_{x/D}^k$ and $\omega(E_k(\nu)) \neq 0$. If $\omega(E_k(\nu)) = s$, then $h(E_k(n)) \le z^{\omega(D)} \cdot z^s$, consequently, by Lemma 5,

$$\begin{aligned} U_1(D) &\ll \frac{z^{\omega(D)}x}{Dx_3}\left\{\sum_{s=1}^{s_0}\frac{1}{s!}\left(\frac{cz}{x_3}\right)^s + x_3\sum_{s\ge s_0+1}\frac{1}{s!}\left(\frac{cz}{x_3}\right)^s\right\} \ll \\ &\ll \frac{z^{\omega(D)}x}{Dx_3^2}. \end{aligned} \tag{4.12}$$

Collecting our inequalities we obtain that

$$\begin{aligned} M_k(x) \le &\sum_{\substack{D\le\exp\left(x_2^k\right)\\ P(D)\le x_2^k}} z^{\omega(D)}\phi\left(\frac{x}{D}, x_2^k\right) + \\ + \quad &O\left(\frac{x}{x_3^2}\sum_{\substack{D\le\exp\left(x_2^k\right)\\ P(D)\le x_2^k}}\frac{z^{\omega(D)}}{D}\right) + O\left(x/x_1\right), \quad \text{say.} \end{aligned} \tag{4.13}$$

Let ε_x be a sequence tending to zero slowly. Let us count for a fixed D those $\nu \in \mathcal{T}_{x/D}$ for which there exists at least one prime $\pi < x_2^{k-\varepsilon_x}$, such that $\pi | D$ and $(\pi, \varphi_k(D\nu)) = 1$.

By using the Brun sieve and repeating the argument for getting the inequality (2.10), (2.11) we can deduce that the size of these ν is less than

$$\ll \frac{x}{D}\frac{x_2}{\sqrt{x_1}}. \tag{4.14}$$

Thus

$$\begin{aligned} M_k(x) \ge &\sum_{\substack{D_1\le\exp\left(x_2^k\right)\\ P(D_1)\le x_2^{k(1-\varepsilon_x)}}} z^{\omega(D_1)}\phi\left(\frac{x}{D_1},\ x_2^{k(1-\varepsilon_x)}\right) + \\ &+ O\left(\frac{xx_2}{\sqrt{x_1}}\sum\frac{z}{D_1}\right). \end{aligned} \tag{4.15}$$

Hence, by using (1.3) we deduce that

$$\frac{M_k(x)}{x} \geq (1+o_x(1)) \prod_{p<x_2^{k(1-\varepsilon_x)}} (1-1/p) \prod_{p<x_2^{k(1-\varepsilon_x)}} \left(1+\frac{z}{p-1}\right). \tag{4.16}$$

On the other hand, from (4.13) we obtain that

$$\frac{M_k(x)}{x} \leq (1+o_x(1)) \prod_{p<x_2^k} (1-1/p) \left(1+\frac{z}{p-1}\right). \tag{4.17}$$

Since

$$\prod_{x_2^{k(1-\varepsilon_x)}<p<x_2^k} \left(1-\frac{1}{p}\right)\left(1+\frac{z}{p-1}\right) \to 1 \quad \text{as} \quad \varepsilon_x \to 0,$$

therefore

$$\begin{aligned}\frac{M_k(x)}{x} &= (1+o_x(1)) \exp\left((z-1)\sum_{p<x_2^k} 1/p + B_1 + O\left(\frac{1}{x_3}\right)\right) = \\ &= (1+o_x(1))(kx_3)^{z-1}C\end{aligned}$$

with a suitable constant $C = C(k,z)$ $(\neq 0)$. Thus (4.3) is true.

By a somewhat complicated argument we would be able to prove the following

Theorem 4. *Let $a \neq 0$, $z \geq 1$, $h(n) := z^{\omega(n)}$,*

$$\begin{aligned}U_k(x) &:= \sum_{p\leq x} h(E_k(p+a)),\\ V_k(x) &:= \sum_{p\leq x} h(K_k(p+a)).\end{aligned}$$

Then

$$\begin{aligned}\frac{U_k(x)}{\pi(x)} &= (1+o_x(1))C^*(kx_3)^{z-1},\\ \frac{V_k(x)}{\pi(x)} &= (1+o_x(1))C^*(kx_3)^{z-1},\end{aligned}$$

where $C^* = C^*(z,k)$ is a nonzero constant.

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