SOME FURTHER REMARKS ON THE ITERATES OF THE $\varphi$ AND THE $\sigma$-FUNCTIONS

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1. Introduction

**Notations.** $\mathbb{N}$ = set of positive integers, $\mathcal{P}$ = set of primes; $\omega(n)$ = the number of distinct prime factors of $n$, $\varphi(n)$ = Euler’s totient function, $\sigma(n)$ = the sum of positive divisors of $n$. For some multiplicative function $f : \mathbb{N} \to \mathbb{N}$, let $f_0(n) = n$, $f_1(n) = f(n)$, $f_{j+1}(n) = f(f_j(n))$ ($j = 1, 2, \ldots$). For the variable $x$ let $x_1 = \log x$, $x_2 = \log x_1$,... The letters $p, q$ with or without suffixes always denote prime numbers. The largest prime factor of $n$ is denoted by $P(n)$, the smallest prime factor of $n$ is $p(n)$.

As usual let

\begin{equation}
\Psi(x, y) = \#\{n \leq x : P(n) \leq y\} \quad (x \geq y \geq 2),
\end{equation}

\begin{equation}
\Phi(x, y) = \#\{n \leq x : p(n) > y\}.
\end{equation}

It is known (see Tenenbaum [1], Theorem I.4.2) that

\begin{equation}
\Phi(x, y) = x \prod_{p \leq y} \left( 1 - \frac{1}{p} \right) \left( 1 + O \left( \frac{1}{(\log y)^2} \right) \right)
\end{equation}

if

\begin{equation*}
2 \leq y \leq \exp \left( \frac{x_1}{10x_2} \right).
\end{equation*}
Several questions on the prime factors of $\varphi_k(n)$ (k-fold iterate of $\varphi$), and on $\sigma_k(n)$, furthermore the size of the set of $n \leq x$ satisfying $(n, \varphi_k(n)) = 1$ were investigated. A non-complete list of the relevant paper is: [2]-[14].

2. On the function $E_k(n) := (n, \varphi_k(n))$

Let $k \geq 1$ be fixed,

\begin{equation}
E_k(n) := (n, \varphi_k(n)).
\end{equation}

\begin{equation}
K_k(n) := (n, \sigma_k(n)).
\end{equation}

Let

\begin{equation}
A(n, y) := \prod_{\substack{p \leq y \\ p \nmid n}} p^\alpha.
\end{equation}

**Theorem 1.** We have

\begin{equation}
\frac{1}{x} \# \{ n \leq x : E_k(n) \neq A(n, x^k) \} = o_x(1),
\end{equation}

\begin{equation}
\frac{1}{x} \# \{ n \leq x : K_k(n) \neq A(n, x^k) \} = o_x(1).
\end{equation}

**Proof.** We shall prove only (2.4). The proof of (2.5) is similar, so we omit it. Let $\varepsilon_x$ be a sequence tending to zero (slowly). Let $A_1$ be the set of those $n \leq x$ for which $p|n$ holds for some $p \in L_x = \left[ x^{k-\varepsilon_x}, x^{k+\varepsilon_x} \right]$. Then

\begin{equation}
\#(A_1) \leq \sum_{p \in L_x} \left[ \frac{x}{p} \right] \leq x \log \frac{k + \varepsilon_x}{k - \varepsilon_x} + O \left( \frac{x}{x^3} \right) = O \left( \left( \varepsilon_x + \frac{1}{x^3} \right) x \right).
\end{equation}

Let $A_2$ be the set of those $n \leq x$ for which $q|E(n)$ for some $q \neq x^{k/2}$. We shall say that $p_0, p_1, \ldots, p_k$ is a chain of primes, if

\[ p_{j+1} - 1 \equiv 0 \pmod{p_j} \quad (j = 0, \ldots, k - 1) \]
Some further remarks on the iterates of the $\varphi$ and the $\sigma$-functions holds. Assume that $p_0 | \varphi_k(n)$. Then either $p_0^2 | \varphi_{k-1}(n)$, or not, and in the second case $p_1 | \varphi_{k-1}(n)$, where $p_1 - 1 \equiv 0 \pmod{p_0}$. We can proceed as we did in [13] and deduce that

\[(2.7) \quad \#(A_2) \ll \sum_{q > x_k^2} \frac{x}{qp_k},\]

where we sum over all chains $q = p_0, p_1, \ldots, p_k \leq x$.

The following assertion is proved in [4].

**Lemma 1.** Let

\[\delta(x, k, l) := \sum_{p \leq x} \frac{1}{p}.\]

For $l = 1$ or $-1$ and $k \leq x$, $x \geq 3$ we have

\[\delta(x, k, l) \leq \frac{c_1 x}{\varphi(k)},\]

where $c_1$ is an absolute constant.

By using Lemma 1, we have

\[\sum_{x_k^2 < q} \frac{1}{q} \sum_{p_1} \sum_{p_2} \ldots \sum_{p_k} \frac{1}{p_k} \leq c_1 x_2 \sum_{q} \sum_{p_1} \ldots \sum_{p_{k-1}} \frac{1}{p_{k-1}} \leq \ldots \leq c_1^k x_2^k \sum_{q > x_k^2} 1/q^2 = O(1/x_3).\]

Thus

\[(2.8) \quad \#(A_2) = O\left(\frac{x}{x_3}\right).\]

Let $Y_x$ be a sequence tending to infinity slowly. We assume that $Y_x = O(x_4)$. Let $\kappa(n)$ be the largest prime power divisor of $n$, with exponent at least 2, i.e.

\[\kappa(n) := \max_{p^a \mid n} p^a,\]

Let $A_3 := \{n \leq x : \kappa(n) \geq Y_x\}$. It is obvious that

\[(2.9) \quad \#(A_3) \ll \sum_{p^a \geq Y_x} \frac{x}{p^a} \ll \frac{x}{\sqrt{Y_x}}.\]
Let \( \pi \in \mathcal{P} \), \( \pi < x^{1/2} - \varepsilon \), and \( \mathcal{A}_\pi \) be the set of those \( n = \pi \nu \leq x \), \( \nu \in \mathbb{N} \), for which \( (\pi, \varphi_k(\pi \nu)) = 1 \). This holds only if \( (\pi, \varphi_k(\nu)) = 1 \).

Let \( \mathcal{B}_k(\pi) \) be the set of all those primes \( p_k \) for which there exists a chain the starting element of which is \( p_0 = \pi \) (thus \( p_0, p_1, \ldots, p_k \) is a suitable chain). It is clear that \( (\pi, \varphi_k(\nu)) = 1 \) implies that \( (\pi, \mathcal{B}_k(\pi)) = 1 \).

Thus, by Brun’s sieve we have

\[
\#(\mathcal{A}_\pi) \leq \frac{c \pi}{x} \prod_{\substack{p \in \mathcal{B}_k(\pi) \\ p < x}} (1 - 1/p).
\]  

By using the method given in [12], [13] we obtain that

\[
\sum_{p \in \mathcal{B}_k \atop p < x} 1/p \geq \frac{1}{2} x^{\varepsilon/2} \quad \text{(say)},
\]

whence

\[
\#(\mathcal{A}_\pi) \ll \frac{x}{\pi} \exp \left( -\frac{1}{2} x^{\varepsilon/2} \right),
\]

and so

\[
\sum_{\pi < x^{1/2} - \varepsilon x} \#(\mathcal{A}_\pi) \ll x x^{\varepsilon/2}.
\]

If \( n \leq x \) is such a number which does not belong to \( \mathcal{A}_1 \cup \mathcal{A}_2 \cup (\cup \mathcal{A}_\pi) \), then \( E_k(n) = A(n, x^{1/2}) \). (2.4) is proved.

By similar method one can prove

**Theorem 2.** We have

\[
\#\{p \leq x : E_k(p + a) \neq A(p + a, x^{1/2})\} = o(\pi(x)),
\]

\[
\#\{p \leq x : K_k(p + a) \neq A(p + a, x^{1/2})\} = o(\pi(x)),
\]

as \( x \to \infty \). Here \( a \neq 0 \) is an arbitrary integer.
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3. Some lemmas

3.1. Let \( \pi_r(x) = \# \{ n \leq x : \omega(n) = r \} \). Hardy and Ramanujan [14] proved that

\[
\pi_r(x) \leq \frac{x (x_2 + c)^{r-1}}{x_1 (r-1)!}
\]

holds for every \( r \in \mathbb{N} \), \( x \geq 3 \), with a suitable absolute constant \( c \).

Hence one can prove immediately

**Lemma 3.** For every \( c_1 \in \mathbb{R} \) there exists \( c_2 \in \mathbb{R} \) such that

\[
\# \{ n \leq x : \omega(n) \geq c_2 x_2 \} \ll \frac{x}{x_1^{c_1}}.
\]

3.2. Let \( Q \) be an arbitrary prime in the interval \( x_2^k \leq Q \leq x_1^{1/3} \). Let \( \kappa_0, \kappa_1, \ldots \) be a sequence of completely additive functions defined for primes \( p \) as follows:

\[
\kappa_0(p) = \begin{cases} 1, & \text{if } p = Q, \\ 0, & \text{if } p \neq 0, \end{cases}
\]

\[
\kappa_{j+1}(p) = \sum_{q \in \mathbb{P}, q | p} \kappa_j(q).
\]

Let

\[
S_j(y) := \sum_{p \leq y} \kappa_j(p).
\]

**Lemma 4.** Let \( x^{1/4} \leq y \leq x \). Then, for \( j \geq 2 \),

\[
S_j(y) = (\text{li} y)(\log \log y)^{j-1} \left( \frac{(\text{li} y)(\log \log y)^{j-2}}{Q^{(m-1)/m}} \right) + O \left( \frac{(\text{li} y)(\log \log y)^{j-2}}{Q^{(m-1)/m}} \right)
\]

and

\[
S_1(y) = \pi(y, Q, 1) = \frac{\text{li} y}{Q - 1} + O \left( \frac{\text{li} y}{Q} e^{-e\sqrt{\log y}} \right).
\]
Here $m$ is an arbitrary positive integer, the constants implied by the error terms may depend on $j$ and $m$.

Lemma 4 is proved in [3] (Lemma 7).

Let

\[(3.8) \quad T^{(k)}_Y := \{\nu \leq Y : p(\nu) > x_2^{k}\},\]

\[(3.9) \quad T^{(k)}_Y (s) := \# \{\nu \in T^{(k)}_Y : \omega(E_k(\nu)) = s\}.\]

**Lemma 5.** Let $\sqrt{x} \leq Y \leq x$, $x \geq 100$. Then, with a suitable constant $c$ which may depend only on $k$, we have

\[(3.10) \quad T^{(k)}_Y (s) \leq \frac{1}{s!} \left( \frac{c}{x_3} \right)^s Y\]

for every $s \geq 1$, and for every fixed $s_0$

\[(3.11) \quad T^{(k)}_Y (s) \ll \frac{1}{s!} \left( \frac{c}{x_3} \right)^s \Phi(y, x_2^{k}),\]

if $s = 1, \ldots, s_0$. Here $\Phi$ is defined by (1.2).

**Proof.** Let $(x_2^{k} \leq) Q_1 < \ldots < Q_s$ be primes for which $Q_1 \ldots Q_s | E_k(\nu)$. Repeating the argument used in [13] one can see that the number of these $\nu \in T^{(k)}_Y$ is less than

\[(3.12) \quad \sum_{p_1^{(k)}, \ldots, p_s^{(k)}} \Phi \left( \frac{Y}{Q_1 \ldots Q_s p_1^{(k)} \ldots p_s^{(k)}}, x_2^{k} \right),\]

where $p_j^{(k)}$ is the final element of the chain of primes $Q_j (= p_j^{(0)}), p_j^{(1)}, \ldots, p_j^{(k)}$.

The contribution of the extraordinary cases when $p_j^{(1)} = p_j^{(2)}$ ($j_1 \neq j_2$), or if $p_j^{(1)} | \varphi_{k-1}(\nu)$ is smaller than (3.13) (which is an upper bound of (3.12)).

Since $\Phi(Y, x_2^{k}) \leq Y$, we obtain from Lemma 1 that (3.12) is less than

\[(3.13) \quad \frac{Y x_2^{ks}}{Q_1^{k} \ldots Q_s^{k}}.\]
whence by summing over all sets of primes \((x_2^k \leq Q_1 < \ldots < Q_s < x)\) we obtain that

\[
\sum Q_1^2 \ldots Q_s^2 \leq \frac{1}{s!} \left\{ \sum_{x_2^k < Q < x} \frac{1}{Q^2} \right\}^s \leq \frac{1}{s!} \left( \frac{c}{x_2^k x_3} \right)^s.
\]

Hence (3.10) is immediate.

To prove (3.11), we estimate \(T^{(k)}(s)\) by

\[
\sum \Phi (Q_1 \ldots Q_s p_1^{(k)} \ldots p_s^{(k)}, x_2^k) + 
\sum_{Q_s > x_1^{1/3}} \sum_{p_i^{(k)} \leq x_1^{1/3}} \left[ \frac{Y}{Q_1 \ldots Q_s p_1^{(k)} \ldots p_s^{(k)}} \right] = \sum_1 + \sum_2.
\]

As earlier, we obtain that

\[
\sum_2 \leq \sum_{Q_s > x_1^{1/3}} \frac{1}{Q_s^2} \left( \sum_{Q > x_2^k} \frac{1}{Q^2} \right)^{s-1} (c_1 x^k)^s \ll \frac{Y \cdot x_3^{-1}}{x_1}.
\]

say. In \(\sum_1\) first we sum over those \(p_1^{(k)} \ldots , p_s^{(k)}\) for which \(\max_{l=1,\ldots,k} p_l^{(k)} < x_1^{1/4}\).

For such collection of \(p_1^{(k)} \ldots , p_s^{(k)}\):

\[
(3.14) \quad \Phi \left( \frac{Y}{Q_1 \ldots Q_s p_1^{(k)} \ldots p_s^{(k)}}, x_2^k \right) \ll \frac{Y \cdot x_3^{-1}}{Q_1 \ldots Q_s p_1^{(k)} \ldots p_s^{(k)}},
\]

and for the others we use the trivial inequality

\[
(3.15) \quad \Phi \left( \frac{y}{Q_1 \ldots Q_s p_1^{(k)} \ldots p_s^{(k)}}, x_2^k \right) \leq \left[ \frac{Y}{Q_1 \ldots Q_s p_1^{(k)} \ldots p_s^{(k)}} \right].
\]

Summing up the right hand side of (3.14) over \(Q_1, \ldots, Q_s\) we obtain the bound

\[
Y \cdot x_3^{-1} \cdot \frac{1}{s!} \left( \frac{c}{x_3} \right)^s.
\]
It remains to estimate the cases when \( \max p_1^{(k)} \geq x^{1/4s} \). We shall estimate

\[
\sum_{x > p_1^{(k)} > x^{1/4s}} 1/p_1^{(k)},
\]

where \( p_1^{(k)} \) is the final element of the chain \( Q_l = (p_1^{(0)}), p_1^{(1)}, \ldots, p_1^{(k)} \).

Let \( M := x^{1/4s}, T \) be the smallest integer for which \( 2^T M \geq x \). Thus \( T = O(x_1) \). Let us define the sequence of the completely additive functions \( \kappa_j \) by the rules (3.3), (3.4) with the choice \( Q = Q_l \).

Applying Lemma 4 we get immediately that

\[
\sum_{r=0}^{T} \frac{1}{2^r M} S_k (2^r M) \ll \begin{cases} 
\frac{x^{k-1}}{Q_1} + \frac{x^{k-2}}{Q_1} / (m-1)/m & \text{if } k \geq 2, \\
1/Q_1 & \text{if } k = 1.
\end{cases}
\]

The proof of Lemma 4 can be completed easily.

4. Some theorems

Let \( z \geq 1 \) be a constant, \( h(n) := z^{\omega(n)} \). Let

\[
M_k(x) := \sum_{n \leq x} h(E_k(n)),
\]

\[
T_k(x) := \sum_{n \leq x} h(K_k(n)).
\]

**Theorem 3.** We have

\[
\frac{M_k(x)}{x} = (1 + o_x(1)) C(kx_3)^{z-1},
\]

\[
\frac{T_k(x)}{x} = (1 + o_x(1)) C(kx_3)^{z-1},
\]

where \( C = C(z, k) \) is a nonzero constant.
Proof. We shall prove only (4.1). The proof of (4.2) is similar, so we omit it.

By using Lemma 3, we can find a constant $c_3$ such that

\[
\sum_{\omega(n) > c_3 \sqrt{x}} h(E_k(n)) \ll x/x_1^2,
\]

say.

Let

\[
\mathcal{R}_1 := \{ n \leq x \mid \omega(n) \geq c_3 \sqrt{x} \}.
\]

Let

\[
\mathcal{R}_2(V) := \{ n \leq x \mid A(n, x_2^k) \geq V \},
\]

where $V \in [x_1, x^{1/4}]$.

By using the known inequality

\[
\Psi(x, y) \ll x \exp \left( -\frac{1}{2} \log x \log y \right) \quad \text{as} \quad 2 \leq y \leq x
\]

(see e.g. [1] Chapter III. 5. Theorem 1), we can deduce that

\[
\# \mathcal{R}_2(V) \ll x \cdot k x_3 \exp \left( -\frac{1}{2} \frac{\log V}{k x_3} \right).
\]

Indeed,

\[
\# \mathcal{R}_2(V) \leq x \sum_{\substack{V \leq D \leq x \\text{\scriptsize{prime}}} \atop \rho(D) \leq \sqrt{x}} \frac{1}{D} \leq x \cdot \sum_{j=0}^{j_0} \frac{1}{2j+1} \Psi \left( 2^{j+1} V, x_2^k \right),
\]

where $j_0$ is the smallest integer for which $2^{j_0} V \geq x$. From (4.8), (4.10) the inequality (4.9) follows.

Let now $V = \exp(x_2^3)$. We have

\[
\# \mathcal{R}_2(\exp(x_2^3)) \ll x/x_1^B,
\]

where $B$ is an arbitrary large positive constant.
Let $D \leq \exp(x^2)$ be fixed and let

$$U_1(D) := \sum h(E_k(n))$$

for those $n = D\nu$, for which $\nu \in T_{x/D}^k$ and $\omega(E_k(\nu)) \neq 0$. If $\omega(E_k(\nu)) = s$, then $h(E_k(n)) \leq \omega(D) \cdot z^s$, consequently, by Lemma 5,

$$U_1(D) \ll \frac{\omega(D)x}{D^x} \cdot \frac{1}{x} \sum_{s=1}^{\infty} \frac{1}{s!} \left( \frac{cz}{x^3} \right)^s + \frac{1}{s!} \sum_{s \geq s_0 + 1} \left( \frac{cz}{x^3} \right)^s \ll \frac{\omega(D)x}{D^x}.$$

(4.12)

Collecting our inequalities we obtain that

$$M_k(x) \leq \sum_{D \leq \exp(x^2)} \frac{\omega(D)\phi\left(\frac{x}{D}, x^k/2\right)}{P(D) \leq x^2} + O\left(\frac{x}{x^2} \sum_{D \leq \exp(y^2)} \frac{\omega(D)}{D} \right) + O(\frac{x}{x^1}) \quad \text{say.}$$

(4.13)

Let $\varepsilon_x$ be a sequence tending to zero slowly. Let us count for a fixed $D$ those $\nu \in T_{x/D}$ for which there exists at least one prime $\pi < x^{k-\varepsilon_x}$, such that $\pi | D$ and $(\pi, \phi_k(D\nu)) = 1$.

By using the Brun sieve and repeating the argument for getting the inequality (2.10), (2.11) we can deduce that the size of these $\nu$ is less than

$$\ll \frac{x}{D} \frac{x^2}{\sqrt{x^1}}.$$

(4.14)

Thus

$$M_k(x) \geq \sum_{D_1 \leq \exp(x^2)} \frac{\omega(D_1)\phi\left(\frac{x}{D_1}, x^{k(1-\varepsilon_x)}/2\right)}{P(D_1) \leq x^2} + O\left(\frac{xx^2}{\sqrt{x}^1} \sum_{D_1} \frac{z}{D_1} \right).$$

(4.15)
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Hence, by using (1.3) we deduce that

\[(4.16) \quad \frac{M_k(x)}{x} \geq (1 + o_x(1)) \prod_{p < x^{1/(1-\varepsilon)}} (1 - 1/p) \prod_{p < x^{1/(1-\varepsilon)}} \left(1 + \frac{z}{p-1}\right).\]

On the other hand, from (4.13) we obtain that

\[(4.17) \quad \frac{M_k(x)}{x} \leq (1 + o_x(1)) \prod_{p < x^{1/(1-\varepsilon)}} (1 - 1/p) \left(1 + \frac{z}{p-1}\right).\]

Since

\[\prod_{x^{1/(1-\varepsilon)} < p < x^k} \left(1 - \frac{1}{p}\right) \left(1 + \frac{z}{p-1}\right) \to 1 \quad \text{as} \quad \varepsilon_x \to 0,\]

therefore

\[\frac{M_k(x)}{x} = (1 + o_x(1)) \exp \left((z - 1) \sum_{p < x^k} 1/p + B_1 + O \left(\frac{1}{x^3}\right)\right) = (1 + o_x(1))(kx_3)^{z-1}C \]

with a suitable constant \(C = C(k, z) \neq 0\). Thus (4.3) is true.

By a somewhat complicated argument we would be able to prove the following

**Theorem 4.** Let \(a \neq 0, \ z \geq 1, \ h(n) := n^\omega(n),\)

\[U_k(x) := \sum_{p \leq x} h(E_k(p + a)), \quad V_k(x) := \sum_{p \leq x} h(K_k(p + a)).\]

Then

\[\frac{U_k(x)}{\pi(x)} = (1 + o_x(1))C^*(kx_3)^{z-1}, \quad \frac{V_k(x)}{\pi(x)} = (1 + o_x(1))C^*(kx_3)^{z-1},\]

where \(C^* = C^*(z, k)\) is a nonzero constant.
References


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