

**A REMARK
ON THE PRODUCT PARTITION
OF INTEGERS INTO k PARTS**

I. Kátai (Budapest, Hungary)

M.V. Subbarao†

Abstract. Let $f_k(n)$ be the number of solutions of the equation $n = m_1 m_2 \dots m_k$ in integers ($2 \leq m_1 < m_2 < \dots < m_k$). The authors analyze the question how $\sum_{n \leq x} f_k(n)$ can be estimated with good remainder terms by using known result.

1. Let $k \in \mathbb{Z}$ be an integer, $f_k(n)$ be the number of solutions of the equation

$$n = m_1 m_2 \dots m_k,$$

in integers ($2 \leq m_1 < m_2 < \dots < m_k$).

Let

$$F(t, s) = \prod_{n=2}^{\infty} \left(1 + \frac{t}{n^s} \right).$$

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Then

$$\begin{aligned} \log F(t, s) &= \sum_{k=1}^{\infty} \frac{t^k}{k} \cdot (-1)^{k-1} \left(\sum_{n=2}^{\infty} \frac{1}{n^{ks}} \right) = \\ &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1} t^k}{k} (\zeta(ks) - 1) = \\ &= t\zeta(s) - t + a(t, s), \\ a(t, s) &= \sum_{k=2}^{\infty} \frac{(-1)^{k-1} t^k}{k} (\zeta(ks) - 1). \end{aligned}$$

It is clear that for $|t| < \sqrt{2}$, the function $a(t, s)$ as a function of s is regular and bounded in $\operatorname{Re} s > \frac{1}{2} + \delta$ where δ is an arbitrary positive constant.

Let

$$F_k(s) = \sum_{n=1}^{\infty} \frac{f_k(n)}{n^s}.$$

We have $e^{-t+a(t,s)} = b_0(s) + b_1(s)t + b_2(s)t^2 + \dots$, $b_0(s) = 1$, where $b_\nu(s)$ are bounded in $\sigma > \frac{1}{2} + \delta$, they are polynomials of $\zeta(2s), \dots, \zeta(\nu s)$ for every ν . The explicit form of them can be computed.

Let $x_1 = \log x$, $x_2 = \log x_1, \dots$.

Since

$$\begin{aligned} F(t, s) &= 1 + \sum_{k=1}^{\infty} t^k F_k(s), \\ e^{t\zeta(s)} &= \sum_{k=0}^{\infty} \frac{\zeta(s)^k}{k!} t^k, \end{aligned}$$

therefore

$$(1.1) \quad F_k(s) = \sum_{\nu=0}^k \frac{b_\nu(s)}{(k-\nu)!} \zeta^{k-\nu}(s).$$

We would like to estimate

$$S_k(x) = \sum_{n \leq x} f_k(n).$$

Since

$$e^{-t+a(t,s)} = \left(\sum \frac{(-t)^{\nu_0}}{\nu_0!} \right) \prod_{k=2}^{\infty} \left(\sum_{m=0}^{\infty} \frac{(-1)^{(k-1)m} t^{km}}{k^m} (\zeta(k,s) - 1)^m \right),$$

therefore

$$(1.2) \quad \frac{b_\nu(s)}{(k-\nu)!} = E_1 \cdot G_1(s) + \dots + E_p \cdot G_p(s),$$

where the general form of $G_l(s)$ can be written as

$$(1.3) \quad G(s) = \zeta^{m_1}(a_1 s) \dots \zeta^{m_q}(a_q s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s},$$

where $(2 \leq) a_1 < \dots < a_q$, m_1, \dots, m_s are positive integers, and

$$m_1 a_1 + \dots + m_q a_q \leq \nu.$$

Let

$$(1.4) \quad \begin{aligned} D(s) &= D(s|G) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}, \\ B(y) &= (B(y|G) =) \sum_{n \leq y} g(n). \end{aligned}$$

From (1.3) we have

$$(1.5) \quad B(y) = \sum_{n_1^{a_1} \dots n_q^{a_q} \leq x} d_{m_1}(n_1) \dots d_{m_q}(n_q).$$

Lemma 1. *We have*

$$(1.6) \quad B(y) \leq C x^{1/a_1} x_1^{m_1-1} \zeta\left(\frac{a_2}{a_1}\right) \dots \zeta\left(\frac{a_q}{a_1}\right)$$

for $q \geq 2$, and

$$(1.7) \quad B(y) \leq C x^{1/a_1} x_1^{m_1-1} \quad \text{if } q = 1.$$

Proof. Since

$$\begin{aligned} \sum_{n^a \leq x} d_m(n) &= \sum_{u_1 \dots u_m \leq x^{1/a}} 1 = \sum_{u_1 \dots u_{m-1} \leq x^{1/a}} \frac{x^{1/a}}{u_1 \dots u_{m-1}} \leq \\ &\leq x^{1/a} \left(\sum_{u < x^{1/a}} \frac{1}{u} \right)^{m-1} \leq \\ &\leq x^{1/a} \left(\frac{1}{a} x^{1/a} + c \right)^{m-1}, \end{aligned}$$

therefore the assertion is true for $q = 1$.

Let $q \geq 2$. Assume that the assertion is true for $q - 1$.

Then

$$\begin{aligned} \sum_{n_1^{a_1} \dots n_q^{a_q} \leq x} d_{m_1}(n_1) \dots d_{m_q}(n_q) &= \sum_{n_2^{a_2} \dots n_q^{a_q} \leq x} d_{m_2}(n_2) \dots d_{m_q}(n_q) \sum_{n_2, \dots, n_q}, \\ \sum_{n_2, \dots, n_q} &= \sum_{n_1 \leq \left(\frac{x}{n_2^{a_2} \dots n_q^{a_q}} \right)^{1/a_1}} d_{m_1}(n_1) \leq cx^{1/a_1} x_1^{m_1-1} \cdot \frac{1}{n_2^{a_2/a_1} \dots n_q^{a_q/a_1}}, \end{aligned}$$

and so

$$(1.8) \quad B(y) \leq cx^{1/a_1} x_1^{m_1-1} \zeta \left(\frac{a_2}{a_1} \right)^{m_2-1} \dots \zeta \left(\frac{a_q}{a_1} \right)^{m_q-1}.$$

Lemma 2. *Let*

$$\begin{aligned} D_k(x) &:= \sum_{n \leq x} d_k(n) = xQ_{k-1}(\log x) + \Delta_k(x), \\ Q_{k-1}(\log x) &:= \operatorname{Res}_{s=1} x^{s-1} \zeta^k(s) s^{-1}. \end{aligned}$$

Let $\alpha_2 \leq \alpha_3 \leq \dots$ be such a sequence for which $\alpha_2 > \frac{1}{2}$, and for each $\varepsilon > 0$,

$$\Delta_k(x) = O(x^{\alpha_k + \varepsilon})$$

holds. Possible value of α_k can be found in [1], Theorem 13.2.

2. Let us estimate

$$(2.1) \quad E(x \mid G, k, \nu) = \sum_{mn \leq x} g(m) d_{k-\nu}(n),$$

where G is a function of form (1.3).

Then

$$\begin{aligned} E(x \mid G, k, \nu) &= \sum_{m \leq x} g(m) D_{k-\nu}(x/m) = \\ &= \sum_{m \leq x} g(m) \left\{ \frac{x}{m} Q_{k-\nu-1} \left(\log \frac{x}{m} \right) + O \left(\left(\frac{x}{m} \right)^{\alpha_k + \varepsilon} \right) \right\} = \\ &= \sum_{m \leq x} \frac{g(m)}{m} Q_{k-\nu-1} \left(\log \frac{x}{m} \right) + O \left(x^{\alpha_k + \varepsilon} \sum_{m \leq x} \frac{g(m)}{m^{\alpha_k + \varepsilon}} \right). \end{aligned}$$

Since $\alpha_{k-\nu} > 1/2$, therefore

$$\sum \frac{g(m)}{m^{\alpha_{k-\nu} + \varepsilon}} < \infty.$$

Let

$$Q_{k-\nu-1}(y) = \sum_{\mu=0}^{k-\nu-1} e_{\mu} y^{\mu}.$$

Therefore

$$\begin{aligned} \sum_{m \leq x} \frac{g(m)}{m} \sum_{\mu=0}^{k-\nu-1} e_{\mu} (x_1 - (\log m))^{\mu} &= \sum_{h=0}^{k-\nu-1} x_1^h U_h(x), \\ U_h(x) &= \sum_{l=0}^{k-\nu-h-1} d_{h,l} \sum_{m \leq x} \frac{g(m)}{m} (\log m)^l. \end{aligned}$$

Let

$$(2.2) \quad \eta_l := \sum \frac{g(m)}{m} (\log m)^l.$$

Since

$$\begin{aligned} \sum_{m \geq x} g(m) \frac{(\log m)^l}{m} &\ll \sum_{t=0}^{\infty} \frac{[\log(2^t x)]^l}{2^t x} \sum_{2^t x < m < 2^{t+1} x} g(m) \ll \\ &\ll \sum_{t=0}^{\infty} \frac{(\log 2^t x)^l}{\sqrt{2^t x}} (\log x)^{m_1-1} \zeta(2)^{\nu} \ll \frac{(\log x)^K}{\sqrt{x}} \end{aligned}$$

holds with a suitable large K , therefore

$$U_h(x) = \sum_{h=0}^{k-\nu-l-1} d_{h,l}\eta_l + O\left(x^{-1/2+\varepsilon}\right),$$

$\varepsilon > 0$ is an arbitrary small positive integer. Let us observe furthermore that in (1.1) $b_0(s) = 1$.

We proved the following

Lemma 3. *For (2.1) we have*

$$E(x \mid G, k, \nu) = x\tilde{Q}_{k-\nu-1}(\log x) + O\left(x^{\alpha_k+\varepsilon}\right).$$

From Lemma 3 the following assertion follows.

Theorem 1. *Let $k \geq 2$ be an arbitrary integer. Then*

$$S_k(x) = x\tilde{P}_{k-1}(\log x) + O\left(x^{\alpha_k+\varepsilon}\right),$$

where $\tilde{P}_{k-1}(y) = \pi_{k-1}y^{k-1} + \dots + \pi_0$ is a polynomial, the leading coefficient π_{k-1} satisfies $\pi_{k-1} = \frac{1}{k!}$.

Remarks.

1. A.F. Lavrik [2] counted the coefficients of the polynomials Q_{k-1} in Lemma 2. By his method and by counting the coefficients of the expansions $b_\nu(s) = b_\nu^{(0)} + b_\nu^{(1)}(s-1) + \dots$, one can determine the coefficients of \tilde{P}_{k-1} in Theorem 1.

2. A.A. Karacuba [3] proved, by using the method of I.M. Vinogradov, that

$$\sum_{n \leq x} d_k(n) = xP_{k-1}(x_1) + O\left(x^{1-\frac{c}{k^{2/3}}+\varepsilon}\right)$$

uniformly as $k/x_2 \leq \varepsilon_x$, where $\varepsilon_x \rightarrow 0$ arbitrarily, $\varepsilon > 0$, $c > 0$, the constant implied by the error term is absolute, P_{k-1} is a polynomial of degree $k-1$, the leading term of which is 1.

By using his theorem we can deduce that Theorem 1 remains valid uniformly as $\frac{k}{x_2} \leq \varepsilon_x$, with $\alpha_k = 1 - \frac{c}{k^{2/3}}$.

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I. Kátai

Department of Computer Algebra
Eötvös Loránd University
and Research Group of Applied
Number Theory of the
Hungarian Academy of Sciences
Pázmány Péter sét. 1/C.
H-1117 Budapest, Hungary
katai@compalg.inf.elte.hu

M.V. Subbarao

University of Alberta
Edmonton, Alberta, T6G 2G1
Canada
m.v.subbarao@ualberta.ca