MEAN BEHAVIOUR OF UNIFORMLY SUMMABLE q–MULTIPLICATIVE FUNCTIONS

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Abstract. In this paper a complete characterization of q-multiplicative functions $f \in \mathcal{L}^*$ is given.

1. Introduction

In 1968 G. Halász proved the following mean-value theorem for multiplicative functions.

Theorem A. (Halász [5]). *Let f be a multiplicative function, $|f(n)| \le 1$, $(n = 1, 2, \ldots)$. If there is a real number a such that the series*

$$\sum_p \frac{(1 - \mathrm{Re} f(p) p^{-ia})}{p}$$

converges, then as $x \to \infty$

$$\sum_{n \le x} f(n) = \frac{x^{1+ia}}{1+ia} \prod_{p \le x} \left(1 - \frac{1}{p}\right) \left(1 + \sum_{m=1}^{\infty} p^{-m(1+ia)} f(p^m)\right) + o(x).$$

On the other hand, if there is no such number a, then

$$x^{-1} \sum_{n \le x} f(n) \to 0 \qquad (x \to \infty).$$

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In either case there are constants D, α, and a slowly-oscillating function $L(u)$ with $|L(u)| = 1$, so that as $x \to \infty$

$$\sum_{n \le x} f(n) = Dx^{1+i\alpha} L(\log x) + o(x).$$

The function L and the constants α, D may be given explicitely (see for example Halász [5] and K.-H. Indlekofer [7]).

For $f : \mathbb{N}_0 \to \mathbb{C}$ define, for any real number $\alpha \ge 1$,

$$\|f\|_\alpha := \left(\limsup_{N \to \infty} \frac{1}{N} \sum_{n < N} |f(n)|^\alpha \right)^{\frac{1}{\alpha}}, \tag{1}$$

and let

$$\mathcal{L}^\alpha := \{ f \mid f : \mathbb{N}_0 \to \mathbb{C},\ \|f\|_\alpha < \infty \}.$$

An arithmetical function[1] $f : \mathbb{N}_0 \to \mathbb{C}$ is called *uniformly summable* in case

$$\lim_{K \to \infty} \sup_{N \ge 1} \frac{1}{N} \sum_{\substack{n < N \\ |f(n)| \ge K}} |f(n)| = 0.$$

The set of all uniformly summable functions, denoted by $\mathcal{L}^*$, is a proper subset of $\mathcal{L}^1$. Obviously $(\alpha > 1)$

$$\mathcal{L}^\alpha \underset{\neq}{\subset} \mathcal{L}^* \underset{\neq}{\subset} \mathcal{L}^1.$$

In [10] K.-H. Indlekofer has given a complete characterization of the asymptotic behaviour of the sums $\sum_{n \le x} f(n)$ $(x \to \infty)$ for uniformly summable multiplicative functions. Putting

$$\rho(n) = \begin{cases} \frac{f(p)}{|f(p)|} & \text{if } f(p) \ne 0, \\ 1 & \text{otherwise,} \end{cases}$$

he proves the following

Theorem B. (Indlekofer [10]). *Let $f \in \mathcal{L}^*$ be multiplicative and $\|f\|_1 > 0$. Then the following two assertions hold.*

[1] If f is defined on $\mathbb{N}$ we extend f to $\mathbb{N}_0$ by putting $f(0) = 0$.

(i) If there exists a constant $a_0 \in \mathbb{R}$ such that the series

$$\sum_p \frac{(1 - \mathrm{Re}\varrho(p)p^{-ia})}{p} \tag{2}$$

converges for $a = a_0$ then there exists a constant $c_0 \in \mathbb{C}$ such that, as $x \to \infty$,

$$\frac{1}{x}\sum_{n\le x} f(n) = x^{ia_0} \exp\left(\sum_{p\le x} \frac{f(p)p^{-ia_0} - 1}{p}\right)(c_0 + o(1)),$$

where

$$c_0 = \frac{1}{1+ia_0}\prod_p \left(1 - \frac{1}{p}\right)\left(1 + \sum_{k=1}^{\infty} \frac{f(p^k)}{p^{k(1+ia)}}\right)\exp\left\{\frac{1 - f(p)p^{-ia_0}}{p}\right\}.$$

If

$$A^*(x) := \sum_{p\le x} \frac{\mathrm{Im} f(p)p^{-ia_0}}{p},$$

then

$$\lim_{x\to\infty} \sup_{x\le y\le x^2} |A^*(y) - A^*(x)| = 0.$$

(ii) If the series (2) diverges for all $a \in \mathbb{R}$ then the mean-value $M(f)$ of f exists and equals zero.

We will extend results of this kind to q-multiplicative functions. For this let $q \ge 2$ be an integer and $\mathbb{A} = \{0, 1, \ldots, q-1\}$. The *$q$-ary expansion* of some $n \in \mathbb{N}_0$ is defined as the unique sequence $\varepsilon_0(n), \varepsilon_1(n), \ldots$ for which

$$n = \sum_{r=0}^{\infty} \varepsilon_r(n) q^r, \quad \varepsilon_r(n) \in \mathbb{A} \tag{3}$$

holds. $\varepsilon_0(n), \varepsilon_1(n), \ldots$ are called the *digits* in the q-ary expansion of n. In fact, $\varepsilon_r(n) = 0$ if $r > \dfrac{\log n}{\log q}$.

A function $f : \mathbb{N}_0 \to \mathbb{C}$ is called *q-multiplicative* if $f(0) = 1$, and for every $n \in \mathbb{N}_0$,

$$f(n) = \prod_{r=0}^{\infty} f(\varepsilon_r(n) q^r). \tag{4}$$

A classical theorem of H. Delange [3] asserts that for q-multiplicative function f with $|f(n)| \le 1$, where $N_x = \left\lfloor \dfrac{\log x}{\log q} \right\rfloor$,

$$m(x) := \frac{1}{x} \sum_{n<x} f(n) = \prod_{r=0}^{N_x-1} \frac{1}{q} \left(\sum_{a\in\mathbb{A}} f(aq^r) \right) + o(1)$$

as $x \to \infty$.

From this he deduced that $\lim_{x\to\infty} |m(x)|$ always exists and equals

$$\prod_{r=0}^{\infty} \left| \frac{1}{q} \sum_{a\in\mathbb{A}} f(aq^r) \right|,$$

which is nonzero if and only if

$$\sum_{a\in\mathbb{A}} f(aq^r) \ne 0 \qquad (\text{for all } \ r \in \mathbb{N}_0) \tag{5}$$

and

$$\sum_{r=0}^{\infty} \sum_{a\in\mathbb{A}} \mathrm{Re}\,(1 - f(aq^r)) < \infty. \tag{6}$$

Furthermore, he proved that $\lim_{x\to\infty} m(x)$ exists and is nonzero if and only if (5) holds and the series

$$\sum_{r=0}^{\infty} \sum_{a\in\mathbb{A}} (1 - f(aq^r)) \tag{7}$$

is convergent.

The aim of this paper is to study the behaviour of the sums

$$\frac{1}{N} \sum_{n<N} f(n) \qquad \text{and} \qquad \frac{1}{N} \sum_{n<N} |f(n)|^\alpha$$

as $N \to \infty$, $\alpha > 0$, where $f \in \mathcal{L}^*$ is q-multiplicative.

2. Main results

We use the following notations.

Let $\widetilde{\Pi_{R,\alpha}} := \prod_{r<R}(1+\widetilde{u_{r,\alpha}})$ and $\Pi_R := \prod_{r<R}(1+u_r)$ with $\widetilde{u_{r,\alpha}} :=$ $:= \frac{1}{q}\sum_{a=1}^{q-1}(|f(aq^r)|^\alpha - 1)$ and $u_r := \frac{1}{q}\sum_{a=1}^{q-1}(f(aq^r)-1)$, respectively.

Definition 1. A function g is said to be *finitely distributed* if there are positive constants c_1 and c_2, and an unbounded sequence of real numbers $x_1 < < x_2 < \ldots$, so that for each x_j at least k positive integers $a_1 < a_2 < \ldots < < a_k \le x_j$ may be found, with $k \ge c_1 x_j$, so that

$$|g(a_m) - g(a_n)| \le c_2 \qquad 1 \le m \le n \le k.$$

The following theorem describes a complete characterization of q-multiplicative uniformly summable functions.

Theorem 1. *Let f be a q-multiplicative function. Then the following assertions are equivalent.*

(i) $f \in \mathcal{L}^*$ and $\|f\|_1 > 0$.

(ii) Let $\alpha > 0$. The series

$$\sum_{r=0}^{\infty}\frac{1}{q}\sum_{a=0}^{q-1}(|f(aq^r)|^\alpha - 1)^2 \tag{8}$$

is convergent, and for some constants $c_1(\alpha), c_2(\alpha) \in \mathbb{R}$, for all R and for some sequence $\{R_i\}$, $R_i \to \infty$, the inequalities

$$\sum_{r<R}\frac{1}{q}\sum_{a=0}^{q-1}(|f(aq^r)|^\alpha - 1) \le c_1(\alpha) < \infty \tag{9}$$

and

$$\sum_{r<R_i}\frac{1}{q}\sum_{a=0}^{q-1}(|f(aq^r)|^\alpha - 1) \ge c_2(\alpha) > -\infty \tag{10}$$

hold.

(iii) $f \in \mathcal{L}^\alpha$ and $\|f\|_\alpha > 0$ for all $\alpha > 0$.

The mean behaviour of such functions is given in

Theorem 2. *Let $f \in \mathcal{L}^*$ be a q-multiplicative function and $\|f\|_1 > 0$. Further, let $q^{R-1} \le N < q^R$, $R \in \mathbb{N}$. Then, as $N \to \infty$,*

$$\frac{1}{N}\sum_{n<N} f(n) = \Pi_R + o(1)$$

and, for every $\alpha > 0$,

$$\frac{1}{N}\sum_{n<N} |f(n)|^\alpha = \widetilde{\Pi_{R,\alpha}} + o(1).$$

An immediate consequence is the following

Corollary 1. *Let f be q-multiplicative. Then the following assertions hold.*

(i) Let $f \in \mathcal{L}^*$. If the mean-value $M(f)$ of f exists and is different from zero then the series

$$\sum_{r=0}^{\infty}\sum_{a=0}^{q-1}(f(aq^r) - 1) \tag{11}$$

and

$$\sum_{r=0}^{\infty}\sum_{a=0}^{q-1}|f(aq^r) - 1|^2 \tag{12}$$

converge and

$$\sum_{a=0}^{q-1} f(aq^r) \ne 0 \quad \text{for each } r \in \mathbb{N}_0.$$

(ii) If the series (11) and (12) converge then $f \in \mathcal{L}^*$, the mean-value $M(f)$ of f exists,

$$M(f) = \prod_{r=0}^{\infty}\left(\frac{1}{q}\sum_{a=0}^{q-1} f(aq^r)\right)$$

and $\|f - f_R\|_1 \to 0$ as $R \to \infty$, where

$$f_R(n) = \prod_{r\le R} f(\varepsilon_r(n)q^r) \qquad 0 \le \varepsilon_r(n) < q.$$

(iii) Let $f \in \mathcal{L}^*$. If the mean-value $M(f)$ of f exists and is different from zero then the mean-value $M(|f|^\alpha)$ of $|f|^\alpha$ exists for each $\alpha > 0$ (and is different from zero).

The case of mean value zero is contained in

Corollary 2. *Let $f \in \mathcal{L}^*$ be q-multiplicative. Then the mean-value $M(f)$ of f is zero if and only if* $\Pi_R = o(1)$ as $R \to \infty$.

Let us now turn to q-additive functions. Here the main results are as follows.

Theorem 3. *Let g be q-additive. Then the following assertions hold.*

(i) If g is finitely distributed, then the series $\sum_{r=0}^{\infty}\sum_{a=0}^{q-1}(g(aq^r))^2$ converges.

(ii) If, for some $\alpha(x)$,

$$\frac{1}{x}\sharp\{n \le x : g(n) - \alpha(x) \le y\} \Rightarrow G(y),$$

where G is a distribution function, then g is finitely distributed.

(iii) Let $\sum_{r=0}^{\infty}\sum_{a=0}^{q-1}(g(aq^r))^2$ converge and put $\alpha(x) = \sum_{r<N_x}\frac{1}{q}\sum_{a=0}^{q-1}g(aq^r)$, $N_x :=$ $:= \left\lfloor \frac{\log x}{\log q} \right\rfloor$. Then

$$\frac{1}{x}\sharp\{n \le x : g(n) - \alpha(x) \le y\} \Rightarrow G(y),$$

where G is some distribution function.

Assertion (iii) of Theorem 3 has already been proved by J. Coquet (see [1], Theorem II. 4).

3. Preliminary results

To prove our main theorem, we need to show the following lemmata.

Lemma 1. *Let $f \in \mathcal{L}^*$ be q-multiplicative and $\|f\|_1 > 0$. Then*

$$\sum_{r=0}^{\infty}\frac{1}{q}\sum_{a=0}^{q-1}(|f(aq^r)|^\alpha - 1)^2 < \infty$$

for all $\alpha > 0$.

Proof. Because of $\|f\|_1 > 0$ we can find a sequence $\{x_i\}$ such that $\sum_{\substack{n<x_i \\ \varepsilon<|f(n)|^\alpha<K}} 1 \gg x_i$, as $i \to \infty$ for some suitable $\varepsilon, K > 0$. We define an q-additive function g by

$$g(aq^r) = \begin{cases} \log(|f(aq^r)|^\alpha) & \text{if } f(aq^r) \neq 0, \\ 1 & \text{if } f(aq^r) = 0. \end{cases}$$

Then $\sum_{\substack{n<x_i \\ -c_1<g(n)<c_2}} 1 \asymp x_i$ with $c_1 = \log 1/\varepsilon$ and $c_2 = \log K$.

For real numbers t, define the functions

$$H(x,t) = \sum_{n<x} \exp(itg(n)),$$

for any $x > 0$.

Delange proved that the limit $l(t) = \lim_{x\to\infty} \frac{1}{x}|H(x,t)|$ always exists and $l(t) \neq 0$ holds if and only if

$$\sum_{r=0}^{\infty} \frac{1}{q} \sum_{a=1}^{q-1} (1 - \cos(tg(aq^r)))$$

converges. Further, define the function D by

$$D(\nu) = \begin{cases} \left(\dfrac{\sin \pi\nu}{\pi\nu}\right)^2 & \text{if } \nu \neq 0, \\ 1 & \text{if } \nu = 0. \end{cases}$$

Then, for each real number y,

$$\int_{-\infty}^{\infty} e^{2\pi i\nu y} D(\nu) d\nu = \begin{cases} 1 - |y| & \text{if } |y| \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Interchanging summation and integration shows that for positive λ

$$\int_{-\infty}^{\infty} \lambda |H(x,t)|^2 D(\lambda t) dt = \sum_{\substack{n_1,n_2\leq x \\ |g(n_1)-g(n_2)|\leq\lambda}} (1 - \lambda^{-1}|g(n_1) - g(n_2)|).$$

We divide by x_i, let $x_i \to \infty$, and apply Lebesgue's theorem for dominated convergence. If λ is sufficiently large then

$$\int_{-\infty}^{\infty} \lambda l(t)^2 D(\lambda t)dt > 0.$$

More exactly, if $g(n)$ satisfies the condition given in the definition of finitely distributed functions, and if $\lambda \geq 2c_2$, then the value of this integral is at least as large as $c_1^2/2$.

It follows that there is a set E, of positive Lebesgue measure, on which $l(t) > 0$.

Now $\sum_{r=0}^{\infty}(1-\cos(tg(aq^r))) < \infty$ for every $1 \leq a \leq q-1$ and for all $t \in E$. It means $\sum_{r=0}^{\infty}(1-\cos(tg(aq^r))) \leq c$ for all $t \in E^*$ where E^* is some subset of E and $m(E^*) > 0$. This is equivalent to $\sum_{r=0}^{\infty} \sin^2\left(\frac{t}{2}g(aq^r)\right) \leq c < \infty$ for all $t \in E^*$.

In view of the inequality $\sin^2(x \pm y) \leq 2\sin^2 x + 2\sin^2 y$ and applying Steinhaus's lemma[2] we can find a $T > 0$, such that for all $1 \leq a \leq q-1$ and for $|t| \leq T$

$$\sum_{r=0}^{\infty}(1-\cos(tg(aq^r))) \leq 4c < \infty. \tag{13}$$

Integrating (13) from 0 to T and multiplying with $1/T$, we have

$$\sum_{r=0}^{\infty} h(Tg(aq^r)) \leq 4c < \infty, \tag{14}$$

where $h(u) = 1 - \dfrac{\sin u}{u}$ for $u \neq 0$ and $h(0) = 0$.

Since $h(u) \geq 0$ for all real number u and $h(u) \geq 1/2$ for $u \geq 2$, we conclude that $|g(aq^r)| \geq 2/T$ for only finitely many r. Thus, there exists $M_a > 0$ such

[2] (See [4] Lemma (1.1). The differences generated by a set of real numbers of positive measure, cover an open interval about the origin.)

that $|g(aq^r)| \le M_a$ for all $r \ge 0$, and there exists $m_a > 0$ so that $h(u) \ge m_a u^2$ for $|u| \le TM_a$.

Hence

$$\sum_{r=0}^{\infty}(g(aq^r))^2 \le \frac{2q\log 2}{m_a T^2},$$

and the series $\sum_{r=0}^{\infty}\frac{1}{q}\sum_{a=1}^{q-1}(g(aq^r))^2$ converges. Since $(\log|x|)^2 \asymp (|x|-1)^2$ if $||x|-1| \le 1/2$, the proof of Lemma 1 is finished.

Lemma 2. *Let f be q-multiplicative and $R \in \mathbb{N}$. Then*

$$\sum_{n=0}^{q^R-1}|f(n)|^\alpha = q^R\widetilde{\Pi_{R,\alpha}}$$

for every $\alpha > 0$, and

$$\sum_{n=0}^{q^R-1} f(n) = q^R\Pi_R.$$

Proof. Induction over R yields the following formulas

$$\sum_{n=0}^{q^{R+1}-1}|f(n)|^\alpha = \sum_{a=0}^{q-1}\left(\sum_{l=0}^{q^R-1}|f(aq^R+l)|^\alpha\right)$$

and

$$\sum_{n=0}^{q^{R+1}-1} f(n) = \sum_{a=0}^{q-1}\left(\sum_{l=0}^{q^R-1} f(aq^R+l)\right),$$

which prove Lemma 2.

Lemma 3. *Let $f \in \mathcal{L}^*$ be q-multiplicative and $\|f\|_1 > 0$. Then*

$$\widetilde{\Pi_{R,\alpha}} = (c(\alpha,|f|)+o(1))\exp\left(\sum_{r<R}\widetilde{u_{r,\alpha}}\right)$$

for all $\alpha > 0$ with some constant $c(\alpha,|f|) \in \mathbb{R}$.

Proof. It is easy to see that, because of the convergence of the series in Lemma 1

$$\widetilde{\Pi_{R,\alpha}} = \prod_{r<R}(1+\widetilde{u_{r,\alpha}}) =$$

$$= \exp\left(\sum_{r<R} \log(1+\widetilde{u_{r,\alpha}})\right) =$$
$$= \exp\left(\sum_{r<R} \widetilde{u_{r,\alpha}} + O\left(\sum_{r<R} (\widetilde{u_{r,\alpha}})^2\right)\right) =$$
$$= (c(\alpha,|f|)+o(1)) \exp\left(\sum_{r<R} \widetilde{u_{r,\alpha}}\right)$$

for all $\alpha > 0$ and some constant $c(\alpha,|f|) \in \mathbb{R}$.

Lemma 4. *Let $f \in \mathcal{L}^*$ be q-multiplicative with $\|f\|_1 > 0$ and $\alpha > 0$. Then there exist some constants $c_1(\alpha)$, $c_2(\alpha) \in \mathbb{R}$ such that*

$$\sum_{r<R} \frac{1}{q} \sum_{a=0}^{q-1} (|f(aq^r)|^\alpha - 1) \le c_1(\alpha) < \infty \tag{15}$$

for all R and

$$\sum_{r<R_i} \frac{1}{q} \sum_{a=0}^{q-1} (|f(aq^r)|^\alpha - 1) \ge c_2(\alpha) > -\infty \tag{16}$$

for some sequence $\{R_i\}$, $R_i \to \infty$.

Proof. By Lemma 3, we get the inequalities (15) and (16) for $\alpha = 1$, since $f \in \mathcal{L}^1$ and $\|f\|_1 > 0$. Now, let $\alpha > 0$, and let $||f(aq^r)| - 1| \le \frac{1}{2}$. Then

$$|f(aq^r)|^\alpha - 1 = (|f(aq^r)| - 1 + 1)^\alpha - 1 =$$
$$= \alpha(|f(aq^r)| - 1) + O((|f(aq^r)| - 1)^2),$$

which implies the inequalities (15) and (16) for all $\alpha > 0$.

Remark 1. Let $f \in \mathcal{L}^*$ be q-multiplicative with $\|f\|_1 > 0$ and $\alpha > 0$. If $\sum_{n<x} |f(n)|^\alpha \asymp x$ then

$$\sum_{r<R} \frac{1}{q} \sum_{a=0}^{q-1} (|f(aq^r)|^\alpha - 1) = O(1)$$

as $R \to \infty$.

The next lemma shows a general method for getting upper estimates.

Lemma 5. *Let f be q-multiplicative and $q^{R-1} \le N < q^R$ with $R \in \mathbb{N}$. Then, for every $h \in \mathbb{N}$,*

$$\left|\sum_{n<N} f(n)\right| \le \sum_{r=1}^{h} \left| q^{R-r}\Pi_{R-r} \prod_{t=1}^{r-1} f(\varepsilon_{R-t}(N)q^{R-t}) \sum_{a=0}^{\varepsilon_{R-r}(N)-1} f(aq^{R-r})\right| +$$

$$+\left(\prod_{r=R-h}^{R-1} |f(\varepsilon_r(N)q^r)|\right)\cdot O(q^{R-h}),$$

where the O-constant depends only on f.

Proof. Let $N = cq^{R-1}+b$ where $1 \le c < q$ and $b = \sum_{r<R-1} \varepsilon_r(N)q^r < q^{R-1}$ where $0 \le \varepsilon_r(N) \le q-1$. Then

$$\sum_{n<N} f(n) = \sum_{a=0}^{c-1}\left(\sum_{l=0}^{q^{R-1}-1} f(aq^{R-1}+l)\right) + \sum_{l=0}^{b-1} f(cq^{R-1}+l) =$$

$$= \sum_{a=0}^{c-1} f(aq^{R-1}) \sum_{l=0}^{q^{R-1}-1} f(l) + f(cq^{R-1}) \sum_{l=0}^{b-1} f(l) =$$

$$= q^{R-1}\Pi_{R-1} \sum_{a=0}^{c-1} f(aq^{R-1}) +$$

$$+ q^{R-2}\Pi_{R-2} f(cq^{R-1}) \sum_{a=0}^{\varepsilon_{R-2}(N)-1} f(aq^{R-2}) +$$

$$+ q^{R-3}\Pi_{R-3} f(cq^{R-1}) f(\varepsilon_{R-2}(N)q^{R-2}) \sum_{a=0}^{\varepsilon_{R-3}(N)-1} f(aq^{R-3}) +$$

$$\vdots$$

$$+ q^{R-h}\Pi_{R-h} f(cq^{R-1}) f(\varepsilon_{R-2}(N)q^{R-2}) \cdots$$

$$\cdots f(\varepsilon_{R-h+1}(N)q^{R-h+1}) \sum_{a=0}^{\varepsilon_{R-h}(N)-1} f(aq^{R-h}) +$$

$$+ f(cq^{R-1}) f(\varepsilon_{R-2}(N)q^{R-2}) \cdots$$

$$\cdots f(\varepsilon_{R-h+1}(N)q^{R-h+1}) f(\varepsilon_{R-h}(N)q^{R-h}) \sum_{l=0}^{b_h-1} f(l),$$

where $b_h < q^{R-h}$ and $\left|\sum_{l=0}^{b_h-1} f(l)\right| \le \sum_{l=0}^{q^{R-h}-1} |f(l)| = O(q^{R-h})$.

In the following lemmata 6, 7 and 8 we collect some more properties of q-multiplicative functions $f \in \mathcal{L}^*$ with $\|f\|_1 > 0$.

Lemma 6. *Let $f \in \mathcal{L}^*$ be q-multiplicative and $\|f\|_1 > 0$. Then the series*

$$\sum_{r=0}^{\infty} \frac{1}{q} \sum_{a=0}^{q-1} |f(aq^r) - 1|^2$$

is convergent if and only if

$$\sum_{r<R_i} \frac{1}{q} \sum_{a=0}^{q-1} (\mathrm{Re} f(aq^r) - 1) \ge c_3 > -\infty$$

for some constant $c_3 \in \mathbb{R}$ and some sequence $\{R_i\}$, $R_i \uparrow \infty$.

Proof. We have

$$\begin{aligned}\sum_{r<R} \frac{1}{q} \sum_{a=0}^{q-1} |f(aq^r) - 1|^2 =& \sum_{r<R} \frac{1}{q} \sum_{a=0}^{q-1} (|f(aq^r)| - 1)^2 + \\ &+ 2 \sum_{r<R} \frac{1}{q} \sum_{a=0}^{q-1} (|f(aq^r)| - 1) - \\ &- 2 \sum_{r<R} \frac{1}{q} \sum_{a=0}^{q-1} (\mathrm{Re} f(aq^r) - 1) = \\ =& \sum_1 + 2 \sum_2 - 2 \sum_3 .\end{aligned}$$

By Lemma 1, $\sum_1$ is convergent and, by Lemma 4, $\sum_2$ is bounded from above for some sequence $\{R_i\}$, $R_i \to \infty$. Thus Lemma 6 holds true.

Lemma 7. *Let $f \in \mathcal{L}^*$ be q-multiplicative and $\|f\|_1 > 0$. If*

$$\sum_{r<R_i} \frac{1}{q} \sum_{a=0}^{q-1} (\mathrm{Re} f(aq^r) - 1) \ge c_3 > -\infty$$

for some constant c_3 and for some sequence $\{R_i\}$, $R_i \uparrow \infty$, then

$$\Pi_R := \prod_{r<R} (1 + u_r) = (c(f) + o(1)) \exp\left(\sum_{r<R} u_r\right)$$

with some constant $c(f) \neq 0$.

Proof. If $\sum_{r<R_i} \frac{1}{q} \sum_{a=0}^{q-1} (\mathrm{Re} f(aq^r) - 1) \geq c_3 > -\infty$ for some constant c_3 and for some sequence $\{R_i\}$, $R_i \uparrow \infty$, then by Lemma 6

$$\sum_{r=0}^{\infty} |u_r|^2 \leq \sum_{r=0}^{\infty} \frac{1}{q} \sum_{a=0}^{q-1} |f(aq^r) - 1|^2 < \infty,$$

and we obtain

$$\begin{aligned}\Pi_R := \prod_{r<R} (1 + u_r) = \\ &= \exp\left(\sum_{r<R} u_r + O\left(\sum_{r<R} |u_r|^2\right)\right) = \\ &= (c(f) + o(1)) \exp\left(\sum_{r<R} u_r\right)\end{aligned}$$

with some constant $c(f) \neq 0$.

Lemma 8. *Let* $f \in \mathcal{L}^*$ *be* q*-multiplicative and* $\|f\|_1 > 0$. *If*

$$\lim_{R\to\infty} \sum_{r<R} \frac{1}{q} \sum_{a=0}^{q-1} (\mathrm{Re} f(aq^r) - 1) = -\infty,$$

then $\Pi_R \to 0$ *as* $R \to \infty$.

Proof. Obviously

$$|\Pi_R| = \exp\left(\sum_{r<R} \log|1 + u_r|\right)$$

and

$$\begin{aligned}\log|1 + u_r| &= \frac{1}{2} \log((1 + \mathrm{Re}\ u_r)^2 + (\mathrm{Im}\ u_r)^2) = \\ &= \frac{1}{2} \log(1 + 2\mathrm{Re}\ u_r + |u_r|^2) \leq \\ &\leq \mathrm{Re}\ u_r + \frac{1}{2}|u_r|^2 .\end{aligned}$$

Since

$$|u_r|^2 \le \frac{q-1}{q} \cdot \frac{1}{q} \sum_{a=0}^{q-1} |f(aq^r) - 1|^2 =$$
$$= \frac{q-1}{q} \left\{ \frac{1}{q} \sum_{a=0}^{q-1} (|f(aq^r)| - 1)^2 + \frac{2}{q} \sum_{a=0}^{q-1} (|f(aq^r)| - 1) - 2\mathrm{Re}\ u_r \right\},$$

we observe

$$\mathrm{Re}\ u_r + \frac{1}{2} \left(\frac{q-1}{q} \cdot (-2\mathrm{Re}\ u_r) \right) = \frac{1}{q} \mathrm{Re}\ u_r,$$

which implies

$$|\Pi_R| \ll \exp\left(\sum_{r<R} \frac{1}{q} \cdot \frac{1}{q} \sum_{a=0}^{q-1} (\mathrm{Re} f(aq^r) - 1) \right),$$

and the assertion of Lemma 8 follows.

Remark 2. Let $f \in \mathcal{L}^*$ be q-multiplicative and $\|f\|_1 > 0$. Then by Lemma 7 and Lemma 8 $\Pi_R = o(1)$ if and only if

$$\sum_{r<R} \frac{1}{q} \sum_{a=0}^{q-1} (\mathrm{Re} f(aq^r) - 1) \to -\infty$$

as $R \to \infty$.

Observing that q-additive functions are sums of "almost independent random variables", we prove the following inequality which is interesting in itself.

Turán-Kubilius inequality for q-additive functions

Let $g : \mathbb{N}_0 \to \mathbb{C}$ be q-additive, $cq^{R-1} \le N < (c+1)q^{R-1}$ with $R \in \mathbb{N}$ and some $c \in \mathbb{N}$ with $0 < c < q$. Put

$$E_R(g) = \sum_{r=0}^{R-2} \frac{1}{q} \sum_{a=0}^{q-1} g(aq^r)$$

and

$$E_{R,c}(g) = E_R(g) + \frac{1}{c} \sum_{a=1}^{c} g(aq^{R-1}).$$

Then

$$\text{(17)}\quad \frac{1}{N}\sum_{n<N}|g(n)-E_{R,c}(g)|^2 \le 2\left(\sum_{r=0}^{R-2}\frac{1}{q}\sum_{a=0}^{q-1}|g(aq^r)|^2+\frac{1}{c}\sum_{a=1}^{c}|g(aq^{R-1})|^2\right).$$

The result is well-known (see for example M. Peter and J. Spilker [11]). We give here a new proof of (17) which is much shorter than the proof present in [11].

Proof.

$$\frac{1}{N}\sum_{n<N}(g(n)-E_{R,c}(g))^2 \le$$
$$\le\frac{1}{N}\sum_{n<(c+1)q^{R-1}}(g^*(n)-E_{R,c}(g))^2 \le$$
$$\le\frac{c+1}{c}\cdot\frac{1}{(c+1)q^{R-1}}\sum_{n<(c+1)q^{R-1}}|g^*(n)-E_{R,c}(g)|^2,$$

where $g^*(aq^r)=g(aq^r)$ for $r<R-1,\ 0\le a\le q-1$ or $r=R-1,\ 0\le a\le c$ and $g^*(aq^r)=0$ for $r>R-1$, $0\le a\le q-1$ or $r=R-1$, $c<a\le q-1$.

Since in the Laplace space $\{0,1,\dots,(c+1)q^{R-1}\}$ a q-additive function is a sum of independent random variables, we obtain

$$\frac{1}{N}\sum_{n<N}|g(n)-E_{R,c}(g)|^2 \le 2\left(\sum_{r=0}^{R-2}\frac{1}{q}\sum_{a=0}^{q-1}|g(aq^r)|^2+\frac{1}{c}\sum_{a=1}^{c}|g(aq^{R-1})|^2\right).$$

Using the Turán-Kubilius inequality we prove

Lemma 9. *Let $f\in\mathcal{L}^*$ be q-multiplicative, $\|f\|_1>0$ and $q^{R-1}\le N<q^R$ where $R\in\mathbb{N}$. Further, let*

$$\sum_{r=0}^{\infty}\frac{1}{q}\sum_{a=0}^{q-1}|f(aq^r)-1|^2<\infty.$$

Then, for any $h\in\mathbb{N}$,

$$\left|\frac{1}{N}\sum_{n<N}f(n)-\Pi_R\right|\le\tilde{c}q^{-h}+o(1)$$

as $N\to\infty$, with some constant $\tilde{c}\in\mathbb{R}$ depending only on f.

Proof. Put

$$f_R(n) = \prod_{r=0}^{R} f(e_r(n)q^r).$$

Then, for any $h \in \mathbb{N}$,

$$\left|\frac{1}{N}\sum_{n<N} f(n) - \Pi_R\right| \leq \frac{1}{N}\sum_{n<N} |f(n) - f_{R-h}(n)| +$$

$$+ \frac{1}{N}\left|\sum_{n<N} f_{R-h}(n) - N\Pi_{R-h+1}\right| + |\Pi_{R-h+1} - \Pi_R| =:$$

$$=: \sum\nolimits_1 + \sum\nolimits_2 + \triangle.$$

Ad $\sum_1$:

We choose $r_0 \in \mathbb{N}$ so that $|f(aq^r) - 1| \leq \frac{1}{10}$, for all $r > r_0$, $0 \leq a < q$ r, $a \in \mathbb{N}$ and define the function g_R

$$g_R(n) := \begin{cases} \sum\limits_{r>R} \log f(e_r(n)q^r) & \text{for } R \geq r_0, \\ 0 & \text{for } R < r_0. \end{cases}$$

Then the functions g_R are q-additive. Now,

$$\frac{1}{N}\sum_{n<N} |f(n) - f_{R-h}(n)| =$$

$$= \frac{1}{N}\sum_{n<N} |f_{R-h}(n)||\exp(g_{R-h}(n)) - 1| \leq$$

$$\leq \frac{1}{N}\sum_{n<N} |g_{R-h}(n)|(|f(n)| + |f_{R-h}(n)|) \leq \left(\frac{1}{N}\sum_{n<N} |g_{R-h}(n)|^2\right)^{1/2} \times$$

$$\times \left(\left(\frac{2}{N}\sum_{n<N} |f(n)|^2\right)^{1/2} + \left(\frac{2}{N}\sum_{n<N} |f_{R-h}(n)|^2\right)^{1/2}\right).$$

Applying the Turán-Kubilius inequality for q-additive functions, we obtain

$$\frac{1}{N}\sum_{n<N} |g_{R-h}(n)|^2 \leq$$

$$\leq \frac{2}{N} \sum_{n<N} \left| g_{R-h}(n) - \sum_{R-h<r<R} \frac{1}{q} \sum_{a=0}^{q-1} g_{R-h}(aq^r) \right|^2 +$$

$$+ \frac{2}{N} \sum_{n<N} \left| \sum_{R-h<r<R} \frac{1}{q} \sum_{a=0}^{q-1} g_{R-h}(aq^r) \right|^2 \leq$$

$$\leq 4 \left(\sum_{R-h<r<R-1} \frac{1}{q} \sum_{a=0}^{q-1} |\log f(aq^r)|^2 + \frac{1}{c} \sum_{a=1}^{c} |\log f(aq^{R-1})|^2 \right) +$$

$$+ 2 \left| \sum_{R-h<r<R} \frac{1}{q} \sum_{a=0}^{q-1} \log f(aq^r) \right|^2,$$

where $cq^{R-1} \leq N < (c+1)q^{R-1}$, with some integer c, $0 < c < q$.

Now, h is fixed and $\log f(aq^r) \to 0$ for $r \to \infty$, so that

$$\lim_{R\to\infty} \overline{\lim_{N\to\infty}} \frac{1}{N} \sum_{n<N} |g_{R-h}(n)|^2 = 0.$$

Using Lemmata 1, 3 and 4 for $\alpha = 2$ shows f, $f_{R-h} \in \mathcal{L}^2$, and thus

$$\frac{1}{N} \sum_{n<N} |f(n) - f_{R-h}(n)| = o(1).$$

Ad $\sum_2$:

For all $0 \leq a < q$, $0 \leq n < q^{R-h+1}$

$$f_{R-h}(aq^{R-h+1} + n) = f(n)$$

and for all $l \in \mathbb{N}$

$$\sum_{n=0}^{lq^{R-h+1}-1} f_{R-h}(n) = l \sum_{n=0}^{q^{R-h+1}-1} f(n) = lq^{R-h+1}\Pi_{R-h+1}.$$

Further, for $N = lq^{R-h+1}$ we obtain

$$\frac{1}{N} \sum_{n<N} f_{R-h}(n) - \Pi_{R-h+1} = 0$$

and for $lq^{R-h+1} < N < (l+1)q^{R-h+1}$, $l \geq 1$ we conclude

$$\begin{aligned}
&\left|\sum_{n<N} f_{R-h}(n) - N\Pi_{R-h+1}\right| = \\
=&\left|-(N-lq^{R-h+1})\Pi_{R-h+1} + \sum_{n=lq^{R-h+1}}^{N-1} f_{R-h}(n)\right| = \\
=&\left|-(N-lq^{R-h+1})\Pi_{R-h+1} + f_{R-h}(lq^{R-h+1}) \sum_{n=0}^{N-lq^{R-h+1}-1} f(n)\right| \leq \\
\leq & c(N-lq^{R-h+1}) < \\
< & cq^{R-h+1}
\end{aligned}$$

with some constant c depending only on f.

Ad $\triangle$: Obviously (cf. proof of Lemma 8)

$$\begin{aligned}
|\Pi_R - \Pi_{R-h+1}| = |\Pi_{R-h+1}| &\left|\left(\prod_{r=R-h+1}^{R-1} \frac{1}{q}\sum_{a=0}^{q-1} f(aq^r)\right) - 1\right| \leq \\
&\leq c \sum_{r=R-h+1}^{R-1} \left|\frac{1}{q}\sum_{a=0}^{q-1}(f(aq^r)-1)\right|.
\end{aligned}$$

Since h is fixed and $f(aq^r)$ tends to 1 as r runs to infinity, we have $|\Pi_R - -\Pi_{R-h}| = o(1)$ as $R \to \infty$.

4. Proof of the main results

Proof of Theorem 1. The implication (i) $\Rightarrow$ (ii) is proved as follows.

If $f \in \mathcal{L}^*$ and $\|f\|_1 > 0$ we conclude, by Lemma 1, that the series (8) is convergent. Lemma 4 shows the inequalities (9) and (10) for all $\alpha > 0$.

Proof of (ii) $\Rightarrow$ (iii).

By Lemma 2 and the convergence of (8) we show as in the proof of Lemma 3

$$\frac{1}{q^R}\sum_{n=0}^{q^R-1} |f(n)|^\alpha = \widetilde{\Pi_{R,\alpha}} = (c(\alpha,|f|)+o(1))\exp\left(\sum_{r<R} \widetilde{u_{r,\alpha}}\right)$$

for all $\alpha > 0$ and some constant $c(\alpha, |f|) \in \mathbb{R}$. Observing, if $q^{R-1} \le N < q^R$

$$\frac{1}{N} \sum_{n<N} |f(n)|^\alpha \ll \frac{1}{q^R} \sum_{n<q^R} |f(n)|^\alpha = \widetilde{\Pi_{R,\alpha}}$$

and the inequality (9) gives $f \in \mathcal{L}^\alpha$ and (10) implies $\|f\|_\alpha > 0$.

The implication (iii) $\Rightarrow$ (i) is obvious.

Proof of Corollary 1. (i) Let $f \in \mathcal{L}^*$ be q-multiplicative. If the mean-value $M(f)$ of f exists and is nonzero then obviously $\|f\|_1 > 0$. We know that (see the proof of Lemma 8)

$$|\Pi_R| \ll \exp\left(\sum_{r<R} \frac{1}{q^2} \sum_{a=0}^{q-1} (\mathrm{Re} f(aq^r) - 1)\right).$$

Further, $\sum_{r=0}^{\infty} \frac{1}{q} \sum_{a=0}^{q-1} (\mathrm{Re} f(aq^r) - 1) > c_3 > -\infty$ for some constant $c_3 \in \mathbb{R}$, since the mean-value $M(f)$ of f exists and is different from zero.

By Lemma 6 the series (12) converges, and Lemma 7 yields

$$\begin{aligned}\Pi_R := \prod_{r<R} \frac{1}{q} \sum_{a=0}^{q-1} f(aq^r) = \\ =(c(f) + o(1)) \exp\left(\sum_{r<R} \frac{1}{q} \sum_{a=0}^{q-1} (f(aq^r) - 1)\right),\end{aligned}$$

with some constant $c(f) \ne 0$.

Since the mean-value $M(f)$ of f exists and is nonzero, the series (11) converge and $\sum_{a=0}^{q-1} f(aq^r) \ne 0$ for each $r \in \mathbb{N}_0$.

(ii) If the series (11) and (12) converge then the infinite product $\lim_{R\to\infty} \Pi_R$ exists and is zero if and only if a factor equals zero. Thus $0 < \widetilde{\Pi_{R,1}}$ for all R and

$$\sum_{r<R} \frac{1}{q} \sum_{a=0}^{q-1} (|f(aq^r)| - 1) > c_4 > -\infty$$

for some constant $c_4 \in \mathbb{R}$. Now

$$\begin{aligned}\sum_{r<R} \frac{1}{q} \sum_{a=0}^{q-1} |f(aq^r) - 1|^2 = &\sum_{r<R} \frac{1}{q} \sum_{a=0}^{q-1} (|f(aq^r)| - 1)^2 + \\ &+ 2\sum_{r<R} \frac{1}{q} \sum_{a=0}^{q-1} (|f(aq^r)| - 1) - \\ &- 2\sum_{r<R} \frac{1}{q} \sum_{a=0}^{q-1} (\mathrm{Re} f(aq^r) - 1)\end{aligned}$$

holds, and the convergence of the series (11) and (12) shows that the series

$$\sum_{r<R} \frac{1}{q} \sum_{a=0}^{q-1} (|f(aq^r)| - 1)^2$$

and

$$\sum_{r<R} \frac{1}{q} \sum_{a=0}^{q-1} (|f(aq^r)| - 1)$$

converge. Then, by Theorem 1 we have $f \in \mathcal{L}^\alpha$ and $\|f\|_\alpha > 0$.

Furthermore by Lemma 6 and Lemma 9 we know that the mean-value $M(f)$ of f exists and $M(f) = \prod_{r=0}^{\infty} \left(\frac{1}{q} \sum_{a=0}^{q-1} f(aq^r) \right)$.

A small modification of the proof for the estimate of $\sum_1$ in Lemma 9 yields, because of the convergence of the series (11) and (12), that $\|f - f_R\|_1 \to 0$ as $R \to \infty$.

(iii) Using Theorem 1 and the same arguments as above we conclude that the series

$$\sum_{r=0}^{\infty} \sum_{a=0}^{q-1} (|f(aq^r)|^\alpha - 1)$$

and

$$\sum_{r=0}^{\infty} \sum_{a=0}^{q-1} (|f(aq^r)|^\alpha - 1)^2$$

converge, and thus the mean-value $M(|f|^\alpha)$ of $|f|^\alpha$ exists for each $\alpha > 0$ (and is different from zero).

Proof of Theorem 2.

First, we assume that $\Pi_R = o(1)$. Then, by Lemma 5 $\frac{1}{N}\sum_{n<N} f(n) = o(1)$.

Now, let $\Pi_R \neq o(1)$. Then by Lemma 6 and Lemma 9 we have $\frac{1}{N}\sum_{n<N} f(n) = \Pi_R + o(1)$.

Furthermore $\widetilde{\Pi_{R,\alpha}} \neq o(1)$, because of $0 < \|f\|_1 \leq \|f\|_\alpha$ for all $\alpha > 0$. Then, by Lemma 1 and Lemma 9 the second assertion of Theorem 2 follows.

The proof of Corollary 2 is obvious.

Proof of Theorem 3.

Ad (i) The assertion is an immediate consequence of the proof of Lemma 1.

Ad (ii) We choose the number γ sufficiently large, and such that $\pm\gamma$ are continuity points of the limiting distribution of $g(n) - \alpha(x)$. Then

$$S := \frac{1}{x}\sharp\{n \leq x : g(n) - \alpha(x) \leq \gamma\} > \frac{1}{2}.$$

Moreover, let m and n be any two elements of S, then

$$|g(m) - g(n)| \leq |g(m) - \alpha(x)| + |\alpha(x) - g(n)| \leq 2\gamma,$$

from which it is clear that $g(n)$ is finitely distributed.

Ad (iii) Let

$$\varphi_x(t) := \frac{1}{x}\sum_{n<x} e^{itg(n)}.$$

Then we shall prove that, for all $t \in \mathbb{R}$,

$$\varphi_x(t)e^{-it\alpha(x)} \to \varphi(t) \qquad (x \to \infty),$$

where $\varphi(t)$ is continuous at $t = 0$.

By Theorem 2 we have

$$\frac{1}{x}\sum_{n<x} e^{itg(n)} = \prod_{r<N_x}\left(1 + \frac{1}{q}\sum_{a=1}^{q-1}\left(e^{itg(aq^r)} - 1\right)\right) + o(1).$$

Let $u_r(t) = \frac{1}{q}\sum_{a=1}^{q-1}\left(e^{itg(aq^r)} - 1\right)$ and $v_r(t) = \frac{it}{q}\sum_{a=1}^{q-1} g(aq^r)$. For $|t| \leq T$ we obtain

$$|u_r(t)| \leq \frac{T}{q}\sum_{a=1}^{q-1}|g(aq^r)|,$$

$$|u_r(t)|^2 \le \frac{T^2(q-1)}{q^2}\sum_{a=1}^{q-1}(g(aq^r))^2$$

and

$$|u_r(t)-v_r(t)| \le \frac{T^2}{2q}\sum_{a=1}^{q-1}(g(aq^r))^2.$$

Hence the infinite product $\prod_{r=0}^{\infty}(1+u_r(t))e^{-v_r(t)}$ is uniformly convergent for $t\in[-T,T]$ and defines the characteristic function of a distribution function G.

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