SOME REMARKS ON THE φ AND ON THE σ FUNCTIONS

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Abstract. Some theorems are proved for the functions $n-\varphi(n)$, $\sigma(n)-$ $-n$, $\varphi_k(n)$, $\sigma_k(n)$, where $\varphi(n)$ is Euler's totient function, $\sigma(n)$ is the sum of divisors function, φ_k and σ_k are the k'th iterate of φ and σ.

1. Let $\varphi(n)$ be Euler's totient function, and $\sigma(n)$ be the sum of divisors of n.

We shall use the following notation: $\omega(n)$ = number of distinct prime factors of n; $\varphi_k(n) = k$-fold iterate of $\varphi(n)$, $\sigma_k(n) = k$-fold iterate of $\sigma(n)$;

$$\psi(n) := n - \varphi(n); \quad \rho(n) := \sigma(n) - n;$$

$\mathcal{P}$ = set of primes. p, q with or without suffixes always denote prime numbers.

Furthermore we shall write $x_1 := \log x$, $x_2 = \log x_1, \ldots$.

W. Sierpinski asked in 1959 ([12], pp. 200-201) whether there exist infinitely many positive integers not of the form $\psi(n)$. J. Browkin and A. Schinzel [15] proved that none of the numbers $2^k \cdot 509203$ $(k = 1, 2, \ldots)$ belong to $\psi(\mathbb{N})$. Erdős proved earlier in [14] that there are infinitely many integers not of the form $\rho(n)$.

The second named author asked whether are there infinitely many n for which $\omega(n) = k$ = fixed and $\psi(n)$ = prime, or not. If $n = p$, then $\psi(p) = 1$, if

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$n = p^2$, then $\psi(p^2) = p$. If $n = pq,\ p \neq q$, then $\psi(n) = p + q - 1$ which can be prime for appropriate choices of p, q.

By using Vinogradov's method for the odd Goldbach problem one can get the following

Lemma 1. *Let* $\varepsilon_x \to 0$, *slowly,* $E_x := e^{x_1^{\varepsilon_x}}$, $E_x \le U < V < x$, $\Delta U = \frac{U}{(\log U)^\kappa}$, $\Delta V = \frac{V}{(\log V)^\kappa}$, *where* κ *is a large constant, and let*

(1.1)
$$M[U, \Delta U;\ V, \Delta V] := \#\{p + q - 1 \in \mathcal{P},\ p \in [U, U + \Delta U],\ q \in [V, V + \Delta V]\}.$$

Then

(1.2)
$$M(U, \Delta U;\ V, \Delta V) = \frac{c\pi([U, U + \Delta U])\pi([V, V + \Delta V])}{\log(U + V)} \left(1 + O\left(\frac{1}{\log 4}\right)\right),$$

where c *is an absolute positive constant.*

We omit the proof.

Hence one can deduce the following

Theorem 1. *Let* $M(x)$ *be the number of those* $n \le x$, *for which* $\omega(n) = 2$, $\psi(n) \in \mathcal{P}$ *holds. Then*

$$M(x) = c\frac{x}{x_1^2} x_2 (1 + o_x(1)). \tag{1.3}$$

Proof. Let $U_0 = E_x,\ U_{j+1} = U_j + \Delta U_j,\ V_j = U_j$. Let us estimate those prime tuples p, q for which

$$pq < x,\ p < E_x,\ p + q - 1 \in \mathcal{P}. \tag{1.4}$$

By [1], Corollary 2.4.1, we obtain that for every fixed p, the number of those $q < \frac{x}{p}$ for which $q \in \mathcal{P},\ (p-1) + q \in \mathcal{P}$, is less than

$$c_i \frac{(p-1)}{\varphi(p-1)} \frac{x/p}{\log^2 x/p},$$

which by summing up to $p < x^{\varepsilon_x}$, is less than

$$c_2 \frac{x}{x_1^2} \log\log E_x = O\left(\frac{x}{x_1^2} \varepsilon_x \cdot x_2\right). \tag{1.5}$$

Let

$$A_1 = \sum_{\substack{i \le j \\ U_i V_j < x}} M(U_i, U_i + \Delta U_i;\ V_j, V_j + \Delta V_j), \tag{1.6}$$

$$A_2 = \sum_{\substack{i \le j \\ (U_i+\Delta U_i)(V_i+\Delta V_i) < x}} M(U_i, U_i + \Delta U_i;\ V_j + \Delta V_j). \tag{1.7}$$

The difference $A_1 - A_2$ is clearly less than the number of those $p, q \in \mathcal{P}$ for which $p + q - 1 \in \mathcal{P}$, and

$$p \in [U_i, U_i + \Delta U_i], \quad q \in [V_j, V_j + \Delta V_j], \tag{1.8}$$

for such choices of i, j for which

$$U_i V_j < x,\ (U_i + \Delta U_i)(V_j + \Delta V_j) > x. \tag{1.9}$$

Let i, j be fixed, so that (1.9) is satisfied. If p, q is such a couple for which (1.8) holds, then

$$\frac{x - c_2 x}{(\log U_i)^\kappa} < pq < x + \frac{c_2 x}{(\log U_i)^\kappa}, \tag{1.10}$$

and we have to estimate those $q \in \mathcal{P}$, for which

$$\frac{x}{p} - \frac{c_2 x}{p(\log U_i)^\kappa} < q < \frac{x}{p} + c_2 \frac{x}{p(\log U_i)^\kappa},$$

$q + p - 1 \in \mathcal{P}$ holds, then sum over $p \in (U_i, U_i + \Delta U_i)$.

Then, by sieve ([1], Corollary 2.4.1) this is less than

$$\sum_{p \in [U_i, U_i + \Delta U_i]} \frac{x}{\varphi(p)(\log U_i)^\kappa (\log x)^2} \ll \frac{x}{x_1^2} \cdot \frac{1}{(\log U_i)^{2\kappa}}. \tag{1.11}$$

Furthermore

$$\sum_{x > n > E_x} \frac{1}{n(\log n)^\kappa} \gg \sum_{n \in [U_i, U_i + \Delta U_i]} \frac{1}{U_i(\log U_i)^\kappa}(\Delta U_i) = \sum \frac{1}{(\log U_i)^{2\kappa}}, \tag{1.12}$$

and the right hand side is bounded in x, therefore

$$A_1 - A_2 = O\left(\frac{x}{x_1^2}\right).$$

To complete the proof of Theorem 1, it remains to apply Lemma 1 to (1.6). The proof is completed.

Similar theorem can be proved for $\rho(n)$.

2. Let m be such an integer for which $(m, \varphi(m)) = 1$. Let $p \in \mathcal{P},\ (p, m) = 1$. Then

$$\psi(mp) = mp - (p-1)\varphi(m) = p(m - \varphi(m)) + \varphi(m). \tag{2.1}$$

Remark. According to the Hardy-Littlewood conjecture, if $(A, B) = 1,\ A, B > 0$, then in the set

$$\{Ap + B \mid p \in \mathcal{P}\},$$

there exist infinitely many primes.

Erdős [2] investigated the set of those m for which $(m, \varphi(m)) = 1$. He proved that the set $\{m \mid (m, \varphi(m)) = 1,\ m \le x\}$ is almost the same as $\{m \mid m \le x,\ p(m) > x_2\}$ and proved that

$$\#\{m \le x \mid (m, \varphi(m)) = 1\} = (1 + o_x(1))x \prod_{p < x_2}\left(1 - \frac{1}{p}\right).$$

One can prove easily that for every fixed $k \ge 2$ there exist infinitely many m with $\omega(m) = k$ for which $(m, \varphi(m)) = 1$. Consequently, if the Hardy-Littlewood conjecture holds, then for every fixed integer $k \ge 2$ there exist infinitely many n for which $\psi(n) = \text{prime}$, $\omega(n) = k + 1$.

Similar assertion can be proved for $\sigma(n)$.

3. As we mentioned earlier, Erdős [2] proved that

$$x^{-1}\#\{n \le x,\ (n, \varphi(n)) = 1\} = (1 + o_x(1)) \prod_{p < x_2} (1 - 1/p).$$

We shall investigate the set

$$\mathcal{B}_{k+1} = \{n \le x,\ (n, \varphi_{k+1}(n)) = 1\}. \tag{3.1}$$

For $k \ge 1$ let $w_k(x) = \prod_{p<x_2^k} \left(1 - \frac{1}{p}\right)$.

For a fixed prime Q let $\kappa_0, \kappa_1, \dots$ be a sequence of completely additive functions defined for primes p as follows:

$$\kappa_0(p) = \begin{cases} 1 & \text{if } p = Q \\ 0 & \text{if } p \ne Q \end{cases}, \quad \kappa_{j+1}(p) = \sum_{\substack{q \in \mathcal{P} \\ q \mid p-1}} \kappa_j(q).$$

Let

$$\rho_k(Q)(= \rho_k(Q|x)) = \prod_{\substack{p<x \\ \kappa_{k+1}(p) \ne 0 \\ p \in \mathcal{P}}} (1 - 1/p)\,. \tag{3.2}$$

To emphasize the value Q on which κ_j depend, we write $\kappa_j(p|Q)$ instead of $\kappa_j(p)$.

Let

$$N_k(Q|x) = \#\{n \le x \mid Q \nmid \varphi_{k+1}(n)\}. \tag{3.3}$$

In a paper of Indlekofer and Kátai [3] the following two theorems are proved, which will be quoted now as Lemma 2 and Lemma 3.

Lemma 2. *Let $x_3 x_2 \le Q \le x_2^2$. Then*

$$N_1(Q|x) = x\rho_1(Q)\left(1 + O\left(\frac{x_2 x_3}{Q}\right)\right),$$
$$\log \frac{1}{\rho_1(Q)} = \frac{x_2^2}{2Q} + O\left(\frac{x_2^3}{Q^2} + \frac{x_2 \log Q}{Q}\right).$$

Lemma 3. *Let $\varepsilon > 0$, $k \ge 2$ be fixed, $x_2^{k+\varepsilon} \le Q \le x_2^{k+1-\varepsilon}$. Then*

$$N_k(Q|x) = \rho_k(Q)x\left(1 + O\left(\frac{1}{x_2}\right)\right),$$

and, moreover

$$\log \frac{1}{\rho_k(Q)} = A_{k+1}(x) + O\left(\frac{1}{Q}\right) + O\left(\frac{x_2^{2k+1}}{Q^2}\right),$$
$$A_{k+1}(x) = \frac{x_2^{k+1}}{(k+1)!(Q-1)} + O\left(\frac{x_2^{k+\varepsilon/2}}{Q}\right).$$

We shall say that $p_0, p_1, \ldots, p_h$ is a chain of primes if $p_{j+1} - 1 \equiv 0 \pmod{p_j}$ $(j = 0, \ldots, h-1)$ holds.

We shall give an upper estimate for $N_k(Q|x)$ for those Q which satisfy the conditions given in the Lemmas 2 and 3.

Let $M_k(Q, x)$ be the number of those $n \le x$, for which $p(n) = Q$, and $Q \not| \varphi_{k+1}(n)$. Let $Q, q_0, \ldots, q_k$ be an arbitrary chain of primes, i.e. $q_{k-1} | q_k - -1, \ldots, q_0 \mid q_1 - 1,\ Q | q_0 - 1$. If $Q \not| \varphi_{k+1}(n)$, then clearly $(q_k, n) = 1$. Hence we obtain that

$$M_k(Q, x) \le \frac{cx}{Q} \prod_{p<Q} \left(1 - \frac{1}{p}\right) \cdot \rho_k(Q), \tag{3.4}$$

where $\rho_k(Q)$ is defined in (3.2).

Let us estimate $\rho_k(Q)$. Let $\tilde{\kappa}_j$ be a truncation of κ_j, more exactly let $\kappa_0 = \tilde{\kappa}_0$, and

$$\tilde{\kappa}_{j+1}(p) = \sum_{\substack{q|p-1 \\ q<p^{1/6}}} \tilde{\kappa}_j(q). \tag{3.5}$$

Let

$$A_V = \sum_{q<V} \frac{\tilde{\kappa}_k(q)}{q-1}, \quad B_V^2 = \sum_{q<V} \frac{\tilde{\kappa}_k^2(q)}{q-1}.$$

From the Bombieri-Vinogradov inequality (see e.g. in [10]) one can deduce the following Turán-Kubilius type inequality:

$$\sum_{p\in[V,2V]} (\kappa_{k+1}(p) - A_V)^2 \ll \frac{V}{\log V} \cdot B_V^2 + O\left(\frac{V}{(\log V)^D}\right), \tag{3.6}$$

where D is an arbitrary large constant.

Assume that $k = 1$. Then $A_V = B_V^2$, and so

$$\#\{p \in [V, 2V],\ \kappa_2(p) = 0\} \le c\frac{V}{(\log V)} \cdot \frac{1}{A_V} + O\left(\frac{V}{(\log V)^D A_V}\right). \tag{3.7}$$

Furthermore

$$A_V = \sum_{q<(2V)^{1/6}} \frac{\tilde{\kappa}_1(q)}{q-1} = \sum_{\substack{Q<q<(2V)^{1/6} \\ q-1\equiv 0 (\mathrm{mod}\ Q)}} \frac{1}{q-1}.$$

Assume that $V^{1/6} \geq \exp(Q^{1/T})$, where T is an arbitrary fixed number. Then, from the Siegel-Walfisz theorem we obtain that

$$A_V \geq \frac{1}{(Q-1)} \int\limits_{\exp(Q^{1/T})}^{(2V)^{1/6}} \frac{1}{u \log u} du - c_1,$$

with an absolute constant c_1. Then

$$A_V \geq \frac{1}{(Q-1)} \left\{ \log\log(2V)^{1/6} - \frac{1}{T} \log Q \right\} - c_1. \tag{3.8}$$

Let $Q \leq x_2/x_3$, V_0 be defined so that

$$\frac{\log\log(2V_0)^{1/6}}{Q-1} = \frac{1}{\varepsilon_1},$$

where ε_1 is an arbitrary (small) positive constant. Then

$$A_V \geq \frac{1}{2\varepsilon_1}, \tag{3.9}$$

if ε_1 is small enough.

Thus

$$\sum_{\substack{V_0 < p < x \\ \tilde{\kappa}_2(p) \neq 0}} 1/p = x_2 - \log\log V_0 - \sum_{\tilde{\kappa}_2(p)=0} 1/p + O(1),$$

and

$$\sum_{\substack{\tilde{\kappa}_2(p)=0 \\ V_0 < p < x}} 1/p \leq 2c\,\varepsilon_1 (x_2 - \log\log V_0),$$

consequently

$$w_2(x) \leq \exp(-(1-\varepsilon_2)x_2) \tag{3.10}$$

with an arbitrary $\varepsilon_2 > 0$.

We proved the following

Lemma 4. *Let $Q \leq x_2/x_3$, ε_2 be an arbitrary positive number. Then*

$$M_1(Q,x) \leq \frac{cx}{Q \log Q} \exp(-(1-\varepsilon_2)x_2). \tag{3.11}$$

Assume now that $k \geq 2$, and that $Q \leq x_2^{k-\varepsilon}$, where ε is an arbitrary small positive constant.

For integers l and m let

$$\delta(x, m, l) = \sum_{\substack{p \leq x \\ p \equiv l \pmod{m}}} 1/p.$$

Lemma 5. *For $l = 1$ or -1 and $m \leq x$, $x \geq 3$ we have*

$$\delta(x, m, l) \leq \frac{c_1 x_2}{\varphi(m)}, \tag{3.12}$$

where c_1 is an absolute constant.

For $l = 1$, this is Eq. (3.1) of [4]. The proof of (3.1) in $l = -1$ is the same, so we omit.

Let

$$A_j(y) = \sum_{p \leq y} \frac{\kappa_j(p)}{p}, \quad D_j^2(y) = \sum_{p \leq y} \frac{\kappa_j^2(p)}{p}. \tag{3.13}$$

In [4] the following assertion has been proved.

Lemma 6. *With some constant c_j, for $z > e^2$, we have*

$$A_j(z) < c_j \frac{(\log\log z)^j}{Q}. \tag{3.14}$$

Let m be an arbitrary positive integer, and let $y > e^{2^{j-2}Q^2}$. Then

$$A_{j+1}(y) = \frac{(\log\log y)^{j+1}}{(j+1)!(Q-1)} + O\left(\frac{(\log\log y)^j}{Q^{(m-1)/m}}\right). \tag{3.15}$$

The constants implied by the error terms may depend on j and m.

Since

$$D_{k+1}^2(y) = \sum_{p \leq y} \left(\sum_{q | p-1} \kappa_k(q)\right)^2 =$$
$$= \sum_q \kappa_p^2(q)\delta(y, q, 1) + \sum_{q_1 \neq q_2} \kappa_k(q_1)\kappa_k(q_2)\delta(y, q_1 q_2, 1),$$

by Lemma 5 and (3.13) we obtain that

$$\begin{aligned} D_{k+1}^2(y) &\le cD_k^2(y)\log\log y + c(\log\log y)A_k^2(y) \le \\ &\le cD_k^2(y)\log\log y + c\frac{(\log\log y)^{2k+1}}{Q^2}. \end{aligned} \tag{3.16}$$

We have

$$D_1^2(y) = A_1(y) \le c\frac{\log\log y}{Q},$$

whence

$$D_2^2(y) \ll \frac{(\log\log y)^2}{Q} + \frac{(\log\log y)^3}{Q^2},$$
$$D_3^2(y) \ll \frac{(\log\log y)^3}{Q} + \frac{(\log\log y)^5}{Q^2},$$

and in general

$$D_{k+1}^2(y) \le c\frac{(\log\log y)^{k+1}}{Q} + \frac{(\log\log y)^{2k+1}}{Q^2}. \tag{3.17}$$

From (3.6) we obtain that

$$\begin{aligned} &\#\{p \in [V, 2V],\ \kappa_{k+1}(p) = 0\} \ll \\ &\ll \frac{V}{\log V} \cdot \frac{B_V^2}{A_V^2} + O\left(\frac{V}{(\log V)^D A_V^2}\right) \end{aligned} \tag{3.18}$$

we can substitute $A_V = A_{k+1}(V)$, $B_V^2 = D_{k+1}(V)$.

From (3.18), (3.17), (3.15) we can deduce that

$$\begin{aligned} &\frac{1}{\frac{V}{\log V}}\#\{p \in [V, 2V],\ \kappa_{k+1}(p) = 0\} \ll \\ &\ll \frac{Q}{(\log\log V)^{k+1}} + \frac{1}{\log\log V}, \end{aligned}$$

whenever $V \ge V_0$ and $V_0 > e^{2^{j-2}Q^2}$ (see Lemma 6).

Let V_1 be so defined that

$$\frac{Q}{(\log\log V_1)^{k+1}} = \varepsilon_1,$$

where ε_1 is a small constant, i.e. $V_1 = \exp\left(\exp\left(\left(\frac{Q}{\varepsilon_1}\right)\right)^{1/k+1}\right)$.

Then, for $V > V_1$ we have

$$\#\{p \in [V, 2V],\ \kappa_{k+1}(p) \neq 0\} \geq (1 - 2\varepsilon_1)\frac{V}{\log V},$$

whence we can deduce that

$$\log w_k(x) = -\sum_{\kappa_{k+1}(p)\neq 0} \frac{1}{p} \leq -(1-2\varepsilon_1)\int_{V_1}^{x} \frac{du}{u\log u} + O(1) =$$
$$= -(1-2\varepsilon_1)(\log\log x - \log\log V_1) + O(1) \leq (1-3\varepsilon_1)x_2.$$

Thus the following assertion is true.

Lemma 7. *Let* $k \geq 2$, $Q \leq x_2^{k-\varepsilon}$, $\varepsilon > 0$ *be fixed. Let* $\varepsilon_1 > 0$ *be another arbitrary small constant. Then*

$$M_k(Q, x) \leq \frac{cx}{Q\log Q}\exp(-(1-\varepsilon_1)x_2).$$

Lemma 8. *Let*

$$S(x|Q) := \sum_{\substack{n\leq x\\ p(n)=Q}} 1.$$

Then, uniformly in $1 \leq Q \leq x_1$ *(say), we have*

$$S(x|Q) = \frac{x}{Q}\prod_{p<Q}\left(1 - \frac{1}{p}\right)(1 + o_x(1)) =$$
$$= \frac{x}{Q}\frac{e^{-\gamma}}{(\log Q)}(1 + o_x(1)),$$

see [1].

Now we shall prove the following

Theorem 2. *Let* $k \geq 1$. *Then*

$$\frac{1}{x}\#\{n \leq x \mid (n, \varphi_{k+1}(n)) = 1\} = \frac{(1 + o_x(1))e^{-\gamma}}{(k+1)x_3}.$$

Proof. By using the notation (3.1), we have

$$\#\mathcal{B}_{k+1} = \sum_{Q<x_2^{k+1}} \#\left(\mathcal{B}_{k+1}^{(Q)}\right) + \#\left(\mathcal{B}_{k+1}^{*}\right) = \sum\nolimits_1 + \#\left(\mathcal{B}_{k+1}^{*}\right).$$

where

$$\mathcal{B}_{k+1}^{(Q)} = \{n \in \mathcal{B}_{k+1},\ p(n) = Q\}, \quad \mathcal{B}_{k+1}^{*} = \{n \in \mathcal{B}_{k+1},\ p(n) > x_2^{k+1}\}.$$

Assume that $k \geq 2$. We split $\sum_1 = \sum^{(1)} + \sum^{(2)} + \sum^{(3)} + \sum^{(4)}$, where in $\sum^{(1)}$, $Q \leq x_2^{k-\varepsilon}$, in $\sum^{(2)}$ $x_2^{k-\varepsilon} \leq Q \leq x_2^{k+\varepsilon}$, in $\sum^{(3)}$ $x_2^{k+\varepsilon} \leq Q \leq x_2^{k+1-\varepsilon}$, and in $\sum^{(4)}$, $x_2^{k+1-\varepsilon} \leq Q \leq x_2^{k+1+\varepsilon}$.

From Lemma 8 we obtain that

$$\sum\nolimits^{(2)} \ll x \sum_{x_2^{k-\varepsilon} \leq Q \leq x_2^{k+\varepsilon}} \frac{1}{Q \log Q} \ll \varepsilon \frac{x}{x_3},$$

and similarly that

$$\sum\nolimits^{(4)} \ll \varepsilon \frac{x}{x_3}.$$

From Lemma 7 we have $\sum^{(1)} \ll x \exp\left(-\frac{x_2}{2}\right)$, say, and by Lemma 3, that

$$\sum\nolimits^{(3)} \ll x/x_2^2.$$

Thus

$$\#(\mathcal{B}_{k+1}) \leq \#(\mathcal{B}_{k+1}^{*}) + O\left(\frac{\varepsilon x}{x_3}\right).$$

Finally we shall prove that

$$p(n) > x_2^{k+1+\varepsilon}, \quad n \leq x$$

implies that $n \in \mathcal{B}_{k+1}$, for all but a small percentage of the integers.

If $n \in \mathcal{B}_{k+1},\ p(n) > x_2^{k+1}$ and $(n, \varphi_{k+1}(n)) \neq 1$, then there is a prime number Q for which $Q|n$, and $Q|\varphi_{k+1}(n)$.

Thus either $Q^2|\varphi_k(n)$ or there is some $q_0 \equiv 1 (\text{mod } Q)$ for which $q_0|\varphi_k(n)$. In the first case $Q|\varphi_k(n)$ obviously holds. Thus always exists a chain of primes $Q \to q_0 \to \ldots \to q_j$, such that $n = Qm,\ q_j|m$.

Thus

$$E := \#\left\{n \le x,\ p(n) > x_2^{k+1+\varepsilon},\ (n, \varphi_{k+1}(n)) \ne 1\right\} \le$$
$$\le c \sum_{Q > x_2^{k+1+\varepsilon}} \sum_{j=0}^{k} \#\left\{n = Qm \le x,\ p(m) > x_2^{k+1+\varepsilon},\ q_j | m\right\},$$

where q_j is the final term in the chain $Q \to q_0 \to \ldots \to q_j$.

Let

$$E_Q^{(j)} := \{n = Qm \le x,\ p(m) > x_2^{k+1+\varepsilon},\ q_j \mid m\}.$$

Then

$$E_Q^{(j)} \ll \frac{x}{Q} \prod_{p<Q} \left(1 - \frac{1}{p}\right) \cdot \sum \frac{1}{q_j}.$$

Since

$$\sum \frac{1}{q_j} \le cx_2 \sum \frac{1}{q_{j-1}} \le \ldots \le \frac{cx_2^{j+1}}{Q},$$

therefore

$$E_Q^{(j)} \ll \frac{x}{Q^2} \frac{x_2^{j+1}}{\log Q},$$

and so

$$\sum_{Q > x_2^{k+1}} \sum_{j=0}^{k} E_Q^{(j)} \ll \frac{x}{x_2^\varepsilon} \sum \frac{1}{Q \log Q} \ll \frac{x}{x_2^\varepsilon}.$$

Thus

$$E = O\left(\frac{x}{x_2^\varepsilon}\right).$$

We proved that

$$\#(\mathcal{B}_{k+1}) = \#\left\{n \le x \mid p(n) > x_2^{k+1}\right\} + o(1)\frac{x}{x_3}.$$

Hence the theorem readily follows.

The case $k = 1$ is similar, somewhat easier.

4. By using the above method, one can prove the assertions formulated in the following

Theorem 3. *Let $k \geq 1,\ l, h \neq 0$ be fixed integers,*

$$S_k^{(l)}(x) = \#\{n \leq x,\ (n, \varphi_k(n+l)) = 1\},$$
$$T_k^{(l)}(x) = \#\{n \leq x,\ (n, \sigma_k(n+l)) = 1\},$$
$$R_k^{(l,h)}(x) = \#\{p \leq x,\ (p+h,\ \sigma_k(p+l)) = 1\},$$
$$Q_k^{(l+h)}(x) = \#\{p \leq x,\ (p+h,\ \varphi_k(p+l)) = 1\}.$$

Then

$$\frac{S_k^{(l)}(x)}{x} = (1+o_x(1))\frac{e^{-\gamma}}{kx_3}, \quad \frac{T_k^{(l)}(x)}{x} = (1+o_x(1))\frac{e^{-\gamma}}{kx_3},$$
$$\frac{R_k^{(l,h)}(x)}{\operatorname{li} x} = (1+o_x(1))\frac{e^{-\gamma}}{kx_3}, \quad \frac{Q_k^{(l,h)}(x)}{\operatorname{li} x} = (1+o_x(1))\frac{e^{-\gamma}}{kx_3},$$

$\gamma =$ *Euler's constant.*

5. Let
$$N(x) := \#\{n \,|\, \varphi(n) \leq x\}.$$

Erdős proved in [5] that $\dfrac{N(x)}{x} \to A\ (\neq 0)$, and in [6] he noted:

Let $\alpha(n)$ be a nonnegative multiplicative function, and assume that there exists its density function $f(x)$. Let

$$M(x) = \#\{n \mid n\alpha(n) \leq x\}.$$

Then
$$\frac{M(x)}{x} \to \int_0^\infty f(u)du.$$

Bateman [7] proved that

$$N(x) = Ax + O\left(x \cdot \exp\left(-c\sqrt{x_1 \cdot x_2}\right)\right),$$
$$A = \frac{\zeta(2)\zeta(3)}{\zeta(6)},$$

with some $c > 0$, by using analytic method. Later Balazard and Smati [8] deduced the same result with elementary method.

We can see that, for $k \geq 1$,

$$\frac{\varphi_{k+1}(n)}{\varphi_k(n)} \sim w_k(x) = \prod_{p < x_2^k} (1 - 1/p),$$

for almost all $n \leq x$. This was observed first by Erdős [11].

Let us write

$$\frac{\varphi_{k+1}(n)}{\varphi_k(n)} = w_k(x)\Gamma_k(n), \tag{5.1}$$

$$\Gamma_k(n) = \prod_{\substack{p \mid \varphi_k(n) \\ p > x_2^k}} (1 - 1/p) \cdot \prod_{\substack{p \nmid \varphi_k(n) \\ p \leq x_2^k}} \frac{1}{1 - 1/p}. \tag{5.2}$$

Then

$$\varphi_{k+1}(n) = w_k(x) \dots w_1(x)\Gamma_k(n) \dots \Gamma_1(n)\varphi(n), \tag{5.3}$$

and so $\varphi_{k+1}(n) \leq x$ holds if and only if

$$\Gamma_k(n) \dots \Gamma_1(n)\varphi(n) \leq \frac{x}{w_k(x) \dots w_1(x)} = Y, \tag{5.4}$$

where

$$Y = (1 + o_x(1))k! x_3^k x. \tag{5.5}$$

If $\varphi_{k+1}(n) \leq x$, then $n \leq c\,x\,x_2^{k+1}$, which directly follows from the inequality

$$\varphi(n) \geq \frac{cn}{\log\log n}.$$

Thus $\varphi_{k+1}(n) \leq x$ implies that $n \leq Z = c\,x\,x_2^{k+1}$ for a suitable c.

Let $\varepsilon > 0$ be an arbitrary constant. We shall prove that

$$\frac{1}{Y} \#\{n < Z \mid |\Gamma_k(n) \dots \Gamma_1(n) - 1| > \varepsilon\} \to 0, \tag{5.6}$$

as $x \to \infty$.

Hence, by the analogon of Erdős's theorem directly follows

Theorem 4. *Let $k \geq 1$ be an arbitrary fixed integer. Then*

$$\#\{n \mid \varphi_{k+1}(n) \leq x\} = (1 + o_x(1))\#\left\{\varphi(n) \leq k! x_3^k x\right\}.$$

Proof. It remains to prove (5.6). If

$$|\Gamma_k(n) \dots \Gamma_1(n) - 1| > \varepsilon,$$

then, for some $j \in \{1, \dots, k\}$.

$$|\Gamma_j(n) - 1| > \delta, \quad \text{where } 1 + \delta = (1 + \varepsilon)^{1/k}. \tag{5.7}$$

Let $\sqrt{x} \leq U \leq x^2$, and count those integers $n \in [U, 2U]$ for which (5.7) holds. Let $\varepsilon_1 > 0$ be a small constant. We have

$$\sum_{x_2^{j-\varepsilon_1} < p < x_2^{j+\varepsilon_1}} \frac{1}{p} \leq \log \frac{j + \varepsilon_1}{j - \varepsilon_1} + O\left(\frac{1}{x_3}\right) \leq \frac{3}{j}\varepsilon_1,$$

whenever x is large enough. Thus

$$\Gamma_j(n) = e^{-\beta_j(n)} \cdot e^{\gamma_j(n)} \cdot e^{0(\varepsilon_1)}, \tag{5.8}$$

where

$$\beta_j(n) = -\sum_{\substack{p > x_2^{j+\varepsilon_1} \\ p \mid \varphi_j(n)}} \log(1 - 1/p), \tag{5.9}$$

$$\gamma_j(n) = \sum_{\substack{p \nmid \varphi_j(n) \\ p \leq x_2^{j+\varepsilon_1}}} \log \frac{1}{1 - 1/p}. \tag{5.10}$$

Let

$$B_j(U) = \sum_{U \leq n \leq 2U} \beta_j(n), \tag{5.11}$$

$$C_{j,r}(U) = \sum_{U \leq n \leq 2U} \gamma_j^r(n). \tag{5.12}$$

By sieve method one can prove that

$$\sum_{\substack{p\nmid \varphi_j(n)\\ n\in[U,2U]}} 1 \ll \frac{U}{(\log U)^c} \quad \text{if} \quad p \le (\log\log u)^{j-\varepsilon_1},$$

whith a suitable constant $c>0$ (see (4.2) and Theorem 3.4 in [9]), whence

$$B_j(U) \ll \frac{Ux_3}{x_1^c} \tag{5.13}$$

follows.

Let us estimate (5.12). We have

$$\begin{aligned} C_{j,r}(U) \ll & \sum_{x_2^{j+1+\varepsilon_1}<q_1,\dots,q_r} \frac{1}{q_1\dots q_r} \sum_{\substack{q_1\dots q_r\mid\varphi_j(n)\\ n\in[U,2U]}} 1+ \\ & + \sum_{s=1}^{r-1} \sum_{x_2^{j+1+\varepsilon_1}<q_1<\dots<q_s} \frac{1}{q_1^{a_1}\dots q_s^{a_s}} \sum_{\substack{q_1\dots q_s\mid\varphi_j(n)\\ n\in[U,2U]}} 1 = \\ = & \sum\nolimits_1 + \sum\nolimits_2 . \end{aligned}$$

The main contribution of $\sum_{\substack{q_1\dots q_r\mid\varphi_j(n)\\ n\in[U,2U]}} 1$ is smaller than

$$\sum \frac{U}{\pi_1\dots\pi_r},$$

where π_ν is the tail of the chain of primes

$$q_\nu \to p_\nu^{(1)} \to p_\nu^{(2)} \to \dots \to p_\nu^{(j-1)} \to \pi_\nu$$

for every ν, thus

$$\sum \frac{1}{\pi_\nu} \le \frac{(\log\log U)^j}{q_\nu} \ll \frac{x_2^j}{q_j},$$

consequently

$$C_{j,r}(U) \ll Ux_2^{jr}\left(\sum \frac{1}{q^2}\right)^r .$$

Since

$$\sum_{q>x_2^{j+\varepsilon_1}} \frac{1}{q^2} \ll \frac{1}{x_2^{j+\varepsilon_1}\cdot x_3},$$

we obtain that

$$C_{j,r}(U) \ll U \cdot x_2^{-r\varepsilon_1},$$

and so

$$\sum_{n\le Z} \gamma_j^r(n) \ll \frac{Z}{x_2^{r\varepsilon_1}}. \tag{5.14}$$

Since r can be arbitrary large, therefore

$$\#\{n \le Z \mid \gamma_j(n) > \delta\} \ll \frac{Z}{x_2^{2k+1}}.$$

Hence the theorem is straightforward.

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