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DEGENERATE CENTER IN A PREDATOR–PREY SYSTEM WITH MEMORY

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Abstract. The purpose of this paper is to establish the occurrence of a denegerate center in a predator-prey system with memory due to Farkas et al [2], and described by a system of two differential equations with continuous delay. This study is done showing that the Liapunov coefficients of the system are null, by using a theorem due to Liapunov (see [1]). Finally, we construct a computer program for the calculation of these coefficients of similar problems.

1. Introduction

In this work we shall establish the occurrence of a degenerate center in a predator-prey system introduced in [2]. The model is described by a two-dimensional system

$$\begin{aligned} \dot{N}(t) &= \epsilon N(t)\left(1 - \frac{N(t)}{K} - \frac{P(t)\alpha}{\epsilon}\right), \\ \dot{P}(t) &= -\gamma P(t) + \beta P(t)\int_{-\infty}^{t} N(\tau)G(t-\tau)d\tau, \end{aligned} \tag{1}$$

where the parameters in (1) are all non-negative and represent

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- $N(t)$: *quantity of prey,*
- $P(t)$: *quantity of predator,*
- ϵ: *specific growth rate of prey,*
- α: *predation rate,*
- γ: *mortality of predator,*
- β: *conversion rate of the prey,*
- K: *carrying capacity of the environment,*
- $G(s) = ae^{-as}$: *density function.*

The introduction of the notation

$$Q(t) = a \int_{-\infty}^{t} N(\tau) e^{-a(t-\tau)} d\tau \tag{2}$$

transforms (1) into

$$\begin{aligned} \dot{N} &= \epsilon N \left(1 - \frac{N}{K} - \frac{P\alpha}{\epsilon}\right), \\ \dot{P} &= -\gamma P + \beta P Q, \\ \dot{Q} &= a(N - Q), \end{aligned} \tag{3}$$

where the last equation was obtained differentiating (2); we shall study (3) with $t \in [0, \infty)$ and $N, P, Q \geq 0$. The change of variables $N = Kn$, $P = Kp$, $Q = Kq$ and the introduction of the new time $t = \frac{s}{\epsilon}$, transforms (3) into

$$\begin{aligned} \frac{dn}{ds} &= n(1-n) - \frac{npK\alpha}{\epsilon}, \\ \frac{dp}{ds} &= -\frac{\gamma p}{\epsilon} + \frac{pqK\beta}{\epsilon}, \\ \frac{dq}{ds} &= \frac{a(n-q)}{\epsilon}. \end{aligned} \tag{4}$$

Farkas et al (see [2]) proved the occurrence of an Andronov-Hopf bifurcation in (3), restricting (4) to the two-dimensional center manifold to obtain

$$\dot{x} = \omega y + W \left\{ -\epsilon(1-\gamma b)x^2 - \epsilon(1-\gamma b) \left[\left(\frac{\gamma b}{\omega}\right)^2 - \frac{\gamma b(1-\gamma b)}{\omega^2 \epsilon b} \right] y^2 + \right.$$

$$+xy\left(1-2\gamma b-2\epsilon\gamma b^2\right)\frac{1-\gamma b}{\omega b}\Bigg\}+$$

$$+Wh(x,y)\left\{\left[-2\epsilon\left(\frac{\gamma b}{\omega}\right)^2-\epsilon+2\epsilon\gamma b\left(\frac{\gamma b}{\omega}\right)^2+2\epsilon\gamma b\right]x+\right.$$

$$\left.+y\left[1-2\epsilon b\left(\frac{\gamma b}{\omega}\right)^2-\epsilon b+\gamma b\frac{1-\gamma b}{\omega^2}\frac{(1-\gamma b)\gamma}{\omega}\right]\right\},$$

$$\dot{y}=-\omega x+W\left\{-\omega\epsilon^2 bx^2-\omega\epsilon^2 b\left[\left(\frac{\gamma b}{\omega}\right)^2-\gamma b\frac{1-\gamma b}{\epsilon b\omega^2}\right]y^2+\right.$$

(5)

$$\left.+\left[\frac{(1-\gamma b)\left(1+\left(\frac{\gamma b}{\omega}\right)^2+\epsilon b\right)}{b}-2\epsilon^2\gamma b^2\right]xy\right\}+$$

$$+Wh(x,y)\left\{-\left[2\left(\frac{\gamma b}{\omega}\right)^2\epsilon b+\epsilon b+1+\left(\frac{\gamma b}{\omega}\right)^2\right]\omega\epsilon x-\right.$$

$$\left.-\omega\epsilon^2 b\left[2\left(\frac{\gamma b}{\omega}\right)^3+\frac{\gamma b}{\omega}-\left(\frac{\gamma b}{\omega}\right)^2\frac{1-\gamma b}{b\omega\epsilon}+\frac{1-\gamma b}{b^2\omega\epsilon^2}+\left(\frac{\gamma b}{\omega}\right)^2\frac{1-\gamma b}{b^2\omega\epsilon^2}\right]y\right\},$$

where $b=\dfrac{1}{K\beta}$, $\omega=\left(\frac{(1-\gamma b-\gamma\epsilon b^2)}{\epsilon}\right)^{\frac{1}{2}}$. With the introduction of the new parameters

(6) $$u=\frac{\epsilon}{\gamma},\quad v=\frac{K\beta}{\gamma},\quad w=\frac{\gamma}{a}$$

the situation was considered in the three-dimensional parameter space u,v,w and the following surface F of bifurcation was obtained

(7) $$w(v^2-v-u)-v=0.$$

In [2] it was proved that when this surface is crossed, an Andronov-Hopf bifurcation occurs. It is supercritical (resp. subcritical) if the crossing is below or above the curve g, whose equation is

(8) $$2v-1-\left(\frac{8u^2+9u+2}{u+2}\right)^{\frac{1}{2}}=0.$$

In Section 2, we shall present some phase portrait executed with the software PHASER, whereby the occurrence of a degenerate center becomes apparent.

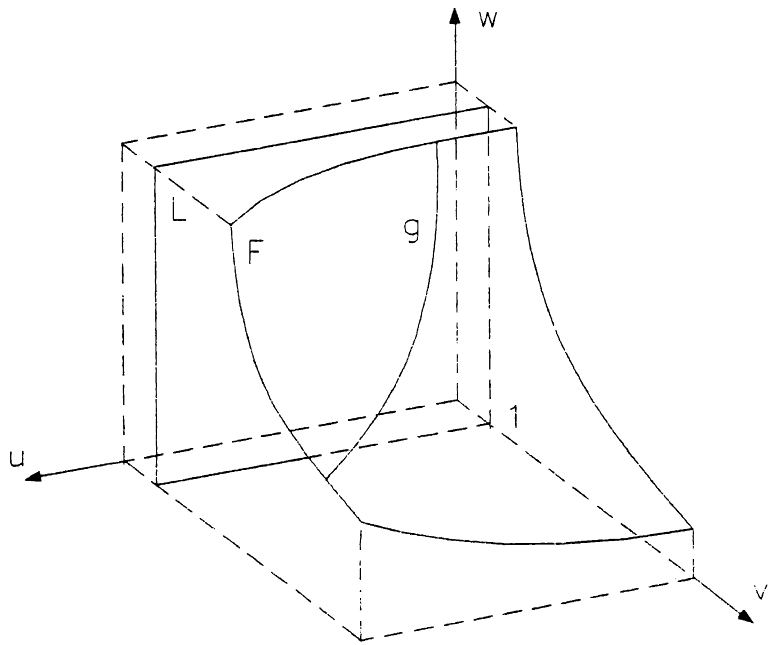

Figure 1. Bifurcation surface

In Section 3, we shall describe an algorithm establishing the Proposition. Finally, we shall give a program to calculate the Liapunov coefficients.

We are to choose now parameter values on the curve g that divides the bifurcation surface into two, a supercritical and a subcritical part. At these values the bifurcation shall be degenerate, neither supercritical, nor subcritical. The aim of the present paper is to show the character of this bifurcation.

Proposition 1.1. *System (5) with the parameters*

$$K = 0.7,\ \epsilon = 0.424264,\ \gamma = 0.3,\ \alpha = 0.5,\ \beta = 0.85714, \tag{9}$$
$$a = 0.087868,\ b = 1.666666389$$

admits a formal first integral of the form

$$F(x,y) = x^2 + y^2 + \sum_{k=3}^{\infty} F_k(x,y),$$

here $F_k(x,y) = \sum_{j=1}^{k+1} A_{kj} x^{k+1-j} y^{j-1}$ *is a homogeneous polynomial of degree* k, $k = 3,4,5,\dots,\ j = 1,2,\dots,\ k+1$.

As a consequence of Proposition 1.1, the system (3) admits a local center around the origin.

2. Simulations

From equation (8) $v = \frac{1}{2}\left(1 + \sqrt{\frac{8u^2+9u+2}{u+2}}\right)$; taking $u = 1$ we get $v = \frac{1}{2}\left(1 + \frac{\sqrt{57}}{3}\right)$. Replacing the values of u and v in (7) we obtained $w = = \frac{1}{2}\left(3+\sqrt{57}\right)$. Substituting u, v, w in (6) we get

$$\epsilon = \gamma,\ \beta = \frac{\gamma}{2K}\left(1 + \frac{\sqrt{57}}{3}\right),\ a = \frac{2\gamma}{3+\sqrt{57}}.$$

The parameters given in (9) determine a point on g. Considering these values in system (4) and the initial conditions

$$(n_1,p_1,q_1) = (0.7,0.7,0.7);\quad (n_2,p_2,q_2) = (0.6,0.6,0.6);$$
$$(n_3,p_3,q_3) = (0.5,0.5,0.5);\quad (n_4,p_4,q_4) = (0.4,0.4,0.4);$$
$$(n_5,p_5,q_5) = (0.2,0.2,0.2),$$

we present the following simulations, generated by the software PHASER. This suggests the existence of degenerate center.

In the simulation we consider $t = 0..100$ (Figure 1(a)) and $t = 0..500$ (Figure 1(b)). One can see that the center has a limit.

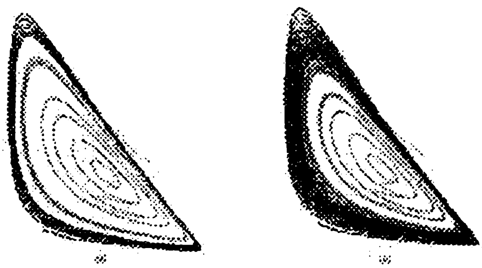

Figure 2. Local center

3. The algorithm

System (5) with the parameters (9) takes the form

$$
(10)\quad
\begin{aligned}
\dot{x} &= 0.321797436y - 0.2426410358x^2 + 0.2426410357y^2 - \\
&\quad - 0.7540176237xy + 0.5308179897x^3 + 0.4538306270x^2y - \\
&\quad - 0.4397447763xy^2 - 0.1708148734y^3, \\
\dot{y} &= -0.321797436x - 0.1104235269x^2 + 0.1104235267y^2 + \\
&\quad + 1.071068092xy + 0.53381710952x^3 + 1.002761197x^2y - \\
&\quad - 0.1500939754xy^2 - 0.7184908404y^3.
\end{aligned}
$$

Liapunov's theorem [1] implies that the formal series

$$
(11)\qquad F(x,y) = x^2 + y^2 + \sum_{k=3}^{\infty} F_k(x,y),
$$

where $F_k(x,y) = A_{k1}x^k + A_{k2}x^{k-1}y + \ldots + A_{kk}xy^{k-1} + A_{k,k+1}y^k$, can be determined such that $F'_{10}(x,y) = G_{2k}(x^2+y^2)^k + o((x^2+y^2)^k)$, with $k = 1,2,\ldots$ Here $F'_{10}(x,y)$ means the differentiation of F with respect to the system (10) and G_{2k} are the Liapunov coefficients for the equilibrium $(0,0)$ of (10). It can be proven that, if $G_{2k} = 0$, $k = 1,2,\ldots$ then $(0,0)$ is a local center and the system admits a first integral of the form (11). To achieve this objective, we shall construct an algorithm to determine these coefficients and $G_{2k} = 0$.

Adding the homogeneous polynomials $F_3(x,y)$ and $F_4(x,y)$ to $F_2(x,y) =$ $= x^2+y^2$ we obtain $F_{2,3,4}(x,y) = x^2+y^2+\sum_{j=1}^{4} A_{3j}x^{4-j}y^{j-1}+\sum_{j=1}^{5} A_{4j}x^{5-j}y^{j-1}$, with undetermined coefficients A_{3j}, $j = 1,\ldots,4$ and A_{4j}, $j = 1,\ldots,5$. To determine these coefficients, we calculate the derivative of $F_{2,3,4}(x,y)$ with respect to (10) and equal the coefficients from the terms of third and fourth order and the coefficients of the polynomial $G_4(x^2+y^2)^2 = G_4x^4+2G_4x^2y^2+$ $+G_4y^4$ to obtain the following system of linear equations:

$$
\begin{aligned}
0.3217974364A_{33}+0.2208470534 &= 0\\
0.4852820716-0.3217974364A_{32} &= 0\\
-1.7288823-0.6435948728A_{33}+0.9653923092A_{31} &= 0\\
-0.9653923092A_{34}+2.627418255+0.6435948728A_{32} &= 0
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
&0.3312705801A_{34}+0.3217974364A_{44}+0.2426410356A_{33}-\\
&\qquad -1.436981681 = G_4\\
&-0.3217974364A_{42}-0.1104235269A_{32}-0.7279231074A_{31}+\\
&\qquad +1.061635979 = G_4\\
&0.7279231068A_{31}+0.9653923092A_{42}+1.899495148A_{33}-\\
&-1.397611720A_{32}-0.9653923092A_{44}-0.3312705807A_{34}+\\
&\qquad +1.126032843 = 2G_4\\
&1.287189746A_{41}-2.262052870A_{31}+0.5857860204A_{32}-\\
&-0.6435948728A_{43}-0.2208470538A_{33}+1.974003446 = 0\\
&-1.287189746A_{45}-0.5331705699A_{33}+0.4852820712A_{32}+\\
&+3.213204276A_{34}+0.6435948728A_{43}-0.6418176996 = 0.
\end{aligned}
$$

Solving the first four equations we find

$$
\begin{aligned}
A_{31} &= 1.333331725, \quad A_{32} = -1.508035853,\\
A_{33} &= -0.6862921466, \quad A_{34} = 1.716249545.
\end{aligned}
$$

Substituting the values found for A_{3j}, $j = 1,\ldots,4$ into (12) the fifth, sixth and seventh equations can be solved independently of A_{41}, A_{43} and A_{45} and we get

$$A_{42} = 0.8004917008, \quad A_{44} = 3.216187222, \quad G_4 = -0.5330875000\times 10^{-6}.$$

We may consider $G_4 = 0$.

To find $G_6 = 0$, we need to determine the constants A_{41}, A_{43} and A_{45}. To achieve this, we add $F_5(x,y) = \sum_{j=1}^{6} A_{5j}x^{6-j}y^{j-1}$ to $F_{2,3,4}(x,y)$, we calculate the derivative with respect to (10) and we equal the coefficients from the terms of fifth order to zero to obtain a system of six equations with indeterminate coefficients A_{4j}, $j = 1,3,5$ and A_{5j} $j = 1,\ldots,6$. Grouping these equations with the last two in (12), we get

$$\begin{aligned}
1.287189746A_{41} - 1.773884131 - 0.6435948728A_{43} &= 0\\
-1.287189746A_{45} + 4.506930690 + 0.6435948728A_{43} &= 0\\
1.230835152 - 0.9705641432A_{41} - 0.3217974364A_{52} &= 0\\
-2.801720830 + 0.3217974365A_{55} + 0.4416941068A_{45} &= 0\\
-1.608987182A_{56} + 0.6435948730A_{54} + 0.4852820714A_{43} + &\\
+4.284272368A_{45} - 0.329257256 &= 0\\
1.608987183A_{51} - 3.016070495A_{43} - 0.6435948728A_{53} - &\\
-0.2208470538A_{43} - 1.755002194 &= 0\\
1.287189746A_{52} + 1.656854112A_{43} + 0.9705641428A_{41} - &\\
-0.9653923092A_{54} - 4.684711778 &= 0\\
0.9653923095A_{53} - 1.287188194A_{43} - 1.287189746A_{55} - &\\
-0.4416941076A_{43} + 16.92065858 &= 0.
\end{aligned}$$

Solving the previous system, in terms of A_{45}, we obtain

$$\begin{aligned}
A_{41} &= -2.123266258 + 0.9999999997A_{45};\\
A_{43} &= 2A_{45} - 7.002744862;\\
A_{51} &= -9.952764969 + 2.666663451A_{45};\\
A_{52} &= 10.22879885 - 3.016071706A_{45};\\
A_{53} &= 1.294079159A_{45} - 15.25558638;\\
A_{54} &= -5.367351123 + 0.4164273838A_{45};\\
A_{55} &= 8.706473428 - 1.372584293A_{45};\\
A_{56} &= -4.463654858 + 3.432499091A_{45}.
\end{aligned}$$

We must give A_{45} a value such that, $G_6 = 0$ and the terms of third order of $F'_{2,3,4}(x,y)_{(10)}$ be null. To achieve these results, we add $F_6(x,y) =$

$= \sum_{j=1}^{7} A_{6j}x^{7-j}y^{j-1}$ to $F_{2,3,4,5}(x,y)$, we substitute $A_{41}, A_{43}, A_{51}, A_{52}, A_{53}, A_{54}$, A_{55}, A_{56} depending of A_{45}, we calculate the derivative with respect to (10) and we equal the coefficients of the terms of sixth order in both sides of

$$G_6(x^2+y_2)^3 = G_6x^6 + 3G_6x^4y^2 + 3G_6x^2y^4 + G_6y^6$$

to obtain a system of seven equations with the constants indeterminate A_{45} and A_{6j}, $j = 1, \dots, 7$:

(13)

$$6.863773295 - 0.778892676A_{45} - 0.3217974364A_{62} = G_6$$
$$-0.901287442 - 1.311865361A_{45} + 0.3217974365A_{66} = G_6$$
$$19.82531980A_{45} - 76.95710798 - 0.9653923092A_{64} + 1.608987183A_{62} = 3G_6$$
$$-7.612567551A_{45} + 48.68689632 + 0.9653923095A_{64} - 1.608987182A_{66} = 3G_6$$
$$-1.930784618A_{67} + 17.72934095A_{45} - 35.18266588 + 0.6435948730A_{65} = 0$$
$$32.67569982 - 6.694469985A_{45} - 0.6435948728A_{63} + 1.930784619A_{61} = 0$$
$$-1.162157565A_{45} + 29.52817523 + 1.287189746A_{63} - 1.287189746A_{65} = 0.$$

The first four equations in (13) can be solved in terms of A_{45} independently of the constants A_{61}, A_{63}, A_{65} and A_{67}, and $G_6 = 0$ if

$$A_{45} = -0.8767770729;$$

consequently:

$$A_{41} = -3.000043331; \quad A_{43} = -8.756299008; \quad A_{45} = -0.8767770729;$$
$$A_{51} = -12.29083434; \quad A_{52} = 12.87322137; \quad A_{53} = -16.39020532;$$
$$A_{54} = -5.732465106; \quad A_{55} = 9.909923867; \quad A_{56} = -7.473191364;$$
$$A_{62} = 23.45167388; \quad A_{64} = 58.63528289; \quad A_{66} = -0.773548826.$$

There are some constants to determine: A_{61}, A_{63}, A_{65} and A_{67}. Observe that, to obtain $G_6 = 0$ we solved a system of seven equations in terms of A_{45}, where the first four equations could be solved independently of the constants in the last three equations, and after we give an adequate value to A_{45}.

We may generalize the idea, i.e. for $k = 1, 2, \dots$ we can obtain $G_{2k} = 0$ solving a system with $2k+1$ equations attributing an adequate value to $A_{2k-2,2k-1}$. This follows.

Once $G_{2k-2} = 0$ has been determined by equating the terms of order $2k-2$ in both sides of $F'_{2,\ldots,2k-2}(x,y) = G_{2k-2}(x^2+y^2)^{\frac{2k-2}{2}}$ and giving a value to $A_{2k-4,2k-3}$, we also find the values of the constants $A_{2k-4,j}$, $j = 1,3,\ldots,2k-5$, $A_{2k-3,j}$, $j = 1,2,\ldots,2k-2$ and $A_{2k-2,j}$, $j = 2,4,\ldots,2k-2$, we have $A_{2k-2,j}$, $j = 1,3,\ldots,2k-1$ to determine. To do this, we add $F_{2k}(x,y) = \sum_{j=1}^{2k+1} A_{2k,j}x^{2k+1-j}y^{j-1}$ to $F_{2,\ldots,2k-1}(x,y)$ and find

$$F_{2,\ldots,2k}(x,y) = x^2 + y^2 + \sum_{j=1}^{4} A_{3j}x^{4-j}y^{j-1} + \ldots + \sum_{j=1}^{2k+1} A_{2k,j}x^{2k+1-j}y^{j-1}.$$

We substitute the values of $A_{2k-4,j}$, $j = 1,2,\ldots,2k-3$, $A_{2k-3,j}$, $j = 1,2,\ldots,2k-2$ and $A_{2k-2,j}$, $j = 2,4,\ldots,2k-2$; then we calculate $F'_{2,\ldots,2k-1}(x,y)$ and equate the coefficients of the **term of third order** of $F'_{2,\ldots,2k-1}(x,y)$ to zero to obtain a system of $2k$ equations with undetermined coefficients **given previously**. In this system we solve $A_{2k-2,j}$, $j = 1,3,\ldots,2k-3$ and $A_{2k-1,j}$, $j = 1,2,\ldots,2k$ in terms of $A_{2k-2,2k-1}$. To conclude add

$$F_{2k}(x,y) = \sum_{j=1}^{2k+1} A_{2k,j}x^{2k+1-j}y^{j-1}$$

to $F_{2,\ldots,2k-1}(x,y)$ and find

$$F_{2,\ldots,2k}(x,y) = x^2 + y^2 + \sum_{j=1}^{4} A_{3j}x^{4-j}y^{j-1} + \ldots + \sum_{j=1}^{2k+1} A_{2k,j}x^{2k+1-j}y^{j-1}.$$

We substitute the constants depending of $A_{2k-2,2k-1}$ into $F_{2,\ldots,2k}(x,y)$ and calculate $F'_{2,\ldots,2k}(x,y)$; equating the coefficients of the terms of order $2k$ in both sides of the equation $F'_{2,\ldots,2k}(x,y) = G_{2k}(x^2+y^2)^{\frac{2k}{2}}$ we determine a system of $2k+1$ equations depending on the constants $A_{2k-2,2k-1}$ and $A_{2k,j}$, $j = 1,\ldots,2k+1$. The first $k+1$ equations depend on $A_{2k-2,2k-1}$, G_{2k} and $A_{2k,j}$, $j = 2,4,\ldots,2k$ and can be solved independently of $A_{2k,j}$, $j = 1,3,\ldots,2k+1$. Solving these equations in terms of $A_{2k-2,2k-1}$ we obtain G_{2k} depending on $A_{2k-2,2k-1}$. Attributing a value to $A_{2k-2,2k-1}$ such that $G_{2k} = 0$, we find also the constants that we were to determine in the two previous steps, $A_{2k-2,j}$, $j = 1,3,\ldots,2k-3$, $A_{2k-1,j}$, $j = 1,2,\ldots,2k$ and $A_{2k,j}$, $j = 2,4,\ldots,2k$. We are to determine $A_{2k,j}$, $j = 1,3,\ldots 2k+1$, so that $G_{2k+2} = 0$.

In this way we determine $G_{2k} = 0,\ k = 1, 2, \ldots$. Then system (10) admits a formal first integral and, consequently, a local center around origin.

4. The program

In this section, we give a program realized with MAPLE-VIII which computes the Liapunov coefficients for the system (10) and for similar systems.

restart; $n = 12$:

$w := 0.3217974364 : a_1 := -0.2426410358 : b_1 := -0.7540176233 :$

$c_1 := 0.2426410356 : d_1 := 0.5308179897 : e_1 := 0.4538306273 :$

$f_1 := -0.4397447761 : g_1 := -0.1708148739 : a_2 := -0.1104235269 :$

$b_2 := 1.071068092 : c_2 := 0.1104235267 : d_2 := 0.5331710957 :$

$e_2 := 1.002761197 : f_2 := -0.1500939759 : g_2 := -0.7184908406 :$

$x_1 := w * y(t) + a_1 * x(t)^2 + b_1 * x(t) * y(t) + c_1 * y(t)^2 + d_1 * x(t)^3 + e_i * x(t) \hat{} 2 * y(t) + f_1 * x(t) * y(t) \hat{} 2 + g_2 * y(t) \hat{} 3 :$

$y_1 := -w*x(t)+a_2*y(t)+a_2*x(t) \hat{} 2+b_2*x(t)*y(t)+c_2*y(t) \hat{} 2+d_2*x(t) \hat{} 3+ e_2 * x(t) \hat{} 2 * y(t) + f_2 * x(t) * y(t) \hat{} 2 + g_2 * y(t) \hat{} 3 :$

```
for i from 3 by 1 to n do
  F[i](t):=(sum(A[i,j]*x(t)^(i+1-j)*y(t)^(j-1), j=1..i+1));
  if irem(i,2)=0 then
      GK[i](t):=G[i]*(x(t)^2+y(t)^2)^(i/2);
    else
  end if:
end do:
for i from 3 by 1 to n do
  Eqs[i]:= :
  if irem(i,2)=0 then
    a[i]:=seq(A[i,j], j=0..i):A[i,0]:=G[i]:
    else a[i]:=seq(A[i,j], j=1..i+1):
  end if:
end do:
F(t):=x(t)^2+y(t)^2:
for 1 from 3 by 1 to n do
  F(t):=F(t)+F[l](t):
```

```
    FP(t):=(subs(diff(x(t),t)=x1,diff(y(t),t)=y1,diff(F(t),t))):
    for i from 0 by 1 to 1 do
      for j from 0 by 1 to l do
        if i+1>=3 then
          EqFP[i,j]:=coeff(coeff(FP(t),y(t),j),x(t),i)=
            coeff(coeff(GK[i+j](t),y(t),j),x(t),i):
          else
        end if:
      end do:
    end do:
    for k from 3 by 1 to l do
      for i from 0 by 1 to l do
        for j from 0 by 1 to l do
          if k=i+j then
            Eqs[k]:=Eqs[k] union EqFP[i,j]:
            else
          end if:
        end do:
      end do:
    end do:
    for i from 3 by 1 to 1 l do
      if irem(i,2)=0 and i<>4 then
          assign(solvefor[a[i]](Eqs[i])):
          assign(solvefor[A[i-2,i-1]](G[i]=0)):
        else
          assign(solvefor[a[i]](Eqs[i])):
      end if:
    end do:
  end do:
  unassign('i,'j,'k,'l'):
  GG:=seq(G[2*i],i=2..n/2):
  for i from 3 by 1 to n-1 do
    FL[i]:=(sum(A[i,j]*x^(i+1-j)*y^(j-1), j=1..i+1)):
  end do:
  FLT:=x^2+y^+(sum(FL[k],k=3..n-1)):
  print("The Liapunov coefficients", 'G[4]','G[6]',"...",'G[n]'"are:" ,):
```

```
print(GG):
# print("The first integral is:"):
# print(FLT):
```

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