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GENERALIZED CONVOLUTIONS OF THE INTEGRAL KONTOROVICH–LEBEDEV, FOURIER SINE AND COSINE TRANSFORMS

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Abstract. Generalized convolutions of the integral Kontorovich-Lebedev, Fourier sine and cosine transforms are introduced and studied. Applications of these new convolutions to solving systems of integral equations are suggested.

I. Introduction

In 1941 Churchill R.V. introduced the convolution of the Fourier cosine transforms ([2])

$$(1)\qquad \left(f \underset{F_c}{*} g\right)(x) := \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} f(u)[g(x+u) + g(|x-u|)]du,\ x > 0,$$

which satisfies the factorization equality

$$F_c\left(f \underset{F_c}{*} g\right)(y) = (F_c f)(y)(F_c g)(y), \quad \forall\, y > 0.$$

At the same moment, for the first time, Churchill introduced the generalized convolution of the Fourier sine transform and the cosine Fourier transform ([2])

$$(2)\qquad \left(f \underset{1}{*} g\right)(x) := \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} f(u)[g(|x-u|) - g(x+u)]du,\ x > 0,$$

which satisfies the factorization equality

$$F_s\left(f \underset{1}{*} g\right)(y) = (F_s f)(y)(F_c g)(y), \quad \forall\, y > 0.$$

Subsequently, in 1990 Yakubovich S.B. published a series of papers devoted to the generalized convolutions of several index integral transforms, such as integral transforms of Mellin type [15], integral transforms of Kontorovich-Lebedev type [17], the G transform [10] and the H transforms [18].

In 1998 Kakichev V.A. and Nguyen Xuan Thao proposed a construction method for defining the generalized convolutions of the tree arbitrary integral transforms ([6]). In recent years, several generalized convolutions of integral transforms were published [7-9]. We mention here the generalized convolution of the Fourier sine and cosine transforms ([8])

$$(3) \quad \left(f \underset{2}{*} g\right)(x) := \frac{1}{\sqrt{2\pi}} \int\limits_0^{+\infty} f(u)[\mathrm{sign}(u-x)g(|u-x|) + g(u+x)]du, \ x > 0,$$

which satisfies the factorization identity

$$F_c\left(f \underset{2}{*} g\right)(y) = (F_s f)(y)(F_s g)(y), \quad \forall\, y > 0.$$

Convolutions of integral transforms and their applications attract attention of many mathematicians and have been announced, e.g. in [3, 4, 11, 12, 13, 15, 16]. In this paper we define two generalized convolutions of the integral Kontorovich-Lebedev, Fourier sine and cosine transforms and apply them to solving two systems of integral equations.

II. Generalized convolutions of the integral Kontorovich-Lebedev, sine and cosine Fourier transforms

Definition. The generalized convolutions with the weight $\gamma(y) = \mathrm{sh}^{-1}(\pi y)$ of the functions f and g with respect to the integral Kontorovich-Lebedev, Fourier sine and cosine transforms are respectively defined as follows

$$(4) \quad \begin{aligned}\left(f \overset{\gamma}{\underset{1}{*}} g\right)(x) = \frac{1}{\pi^2} \int\limits_0^{+\infty}\int\limits_0^{+\infty} \Big[&\mathrm{sh}(x+v)e^{-u.\mathrm{ch}(x+v)} + \\ &+\mathrm{sh}(x-v)e^{-u.\mathrm{ch}(x-v)}\Big] f(u)g(v)dudv, \quad x > 0,\end{aligned}$$

$$\left(f \overset{\gamma}{\underset{2}{*}} g\right)(x) = \frac{1}{\pi^2} \int_0^{+\infty} \int_0^{+\infty} \Big[\mathrm{sh}(x+v)e^{-u.\mathrm{ch}(x+v)} - \tag{5}$$
$$-\mathrm{sh}(x-v)e^{-u.\mathrm{ch}(x-v)}\Big] f(u)g(v)dudv.$$

Theorem 1. *Suppose that* $f \in L\left(\frac{1}{x}, \mathbb{R}_+\right)$, $g \in L(\mathbb{R}_+)$. *Then, the convolutions* $\left(f \overset{\gamma}{\underset{1}{*}} g\right)(x)$, $\left(f \overset{\gamma}{\underset{2}{*}} g\right)(x)$ *belong to* $L(\mathbb{R}_+)$ *and respectively satisfy the factorization equalities*

$$F_s\left(f \overset{\gamma}{\underset{1}{*}} g\right)(y) = \gamma(y)\left(K^{-1}f\right)(y)(F_c g)(y), \quad \forall\, y > 0, \tag{6}$$

$$F_c\left(f \overset{\gamma}{\underset{2}{*}} g\right)(y) = \gamma(y)\left(K^{-1}f\right)(y)(F_s g)(y), \quad \forall\, y > 0. \tag{7}$$

Here K^{-1} *is the inverse Kontorovich-Lebedev transform ([1]).*

Proof. Since $u.\mathrm{sh}(x+v)e^{-u.\mathrm{ch}(x+v)} \to 0$ when $u, v \to +\infty$, we have

$$\int_0^{+\infty} \int_0^{+\infty} \left|\mathrm{sh}(x+v)e^{-u.\mathrm{ch}(x+v)} + \mathrm{sh}(x-u)e^{-u.\mathrm{ch}(x-v)}\right| .|f(v)||g(v)dudv \leq$$
$$\leq C \int_0^{+\infty} \frac{1}{u}|f(u)|du \int_0^{+\infty} |g(v)|dv < +\infty.$$

Because

$$\int_0^{+\infty} \mathrm{sh}(x+v)e^{-u.\mathrm{ch}(x+v)}dx = \frac{1}{u}e^{-u.\mathrm{ch}v} \tag{8}$$

and

$$\int_0^{+\infty} \left|\mathrm{sh}(x-v)e^{-u.\mathrm{ch}(x-v)}\right| dx = \int_v^{+\infty} \mathrm{sh}(x-v)e^{-u.\mathrm{ch}(x-v)}dx -$$
$$- \int_0^{v} \mathrm{sh}(x-v)e^{-u.\mathrm{ch}(x-v)}dx = \tag{9}$$
$$= -\frac{1}{u}e^{-u.\mathrm{ch}(x-v)}\Big|_v^{+\infty} + \frac{1}{u}.e^{-u.\mathrm{ch}(x-v)}\Big|_0^{v} =$$
$$= 2\frac{e^{-u}}{u} - \frac{e^{-u.\mathrm{ch}v}}{u}$$

it follows from (8), (9) that

$$\int_0^{+\infty} \left|\left(f \overset{\gamma}{*} g\right)\right|(x)dx \leq \frac{1}{\pi^2} \int_0^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \Big[\mathrm{sh}(x+v).e^{-u.\mathrm{ch}(x+v)} +$$

$$+|\mathrm{sh}(x-v)|e^{-u.\mathrm{ch}(x-v)}\Big]\,|f(u)|g(v)dudvdx =$$

$$= \frac{1}{\pi^2}\int_0^{+\infty}\int_0^{+\infty} |f(u)||g(v)|\Bigg\{\int_0^{+\infty}\Big[\mathrm{sh}(x+v)e^{-u.\mathrm{ch}(x+v)} +$$

$$+|\mathrm{sh}(x-v)|e^{-u.\mathrm{ch}(x-v)}\Big]\,dx\Bigg\}dudv \leq$$

$$\leq \frac{2}{\pi^2}\int_0^{+\infty}\int_0^{+\infty} |f(u)||g(v)|\frac{e^{-u}}{u}dudv =$$

$$= \frac{2}{\pi^2}\int_0^{+\infty} e^{-u}\frac{1}{u}|f(u)|du \int_0^{+\infty} |g(v)|dv \leq$$

$$\leq \int_0^{+\infty}\frac{1}{u}|f(u)|du\int_0^{+\infty}|g(v)|dv < +\infty.$$

Thus, $\left(f \overset{\gamma}{*} g\right)(x) \in L(\mathbb{R}_+)$.

Now we prove that the generalized convolution (4) satisfies the factorization equality (6). Since

$$\gamma(y)\left(K^{-1}f\right)(y)(F_c g)(y) =$$

$$= \mathrm{sh}^{-1}(\pi y)\sqrt{\frac{2}{\pi}}\int_0^{+\infty}\int_0^{+\infty}\cos(yv)\frac{2}{\pi^2}y\,\mathrm{sh}(\pi y)K_{iy}(u)\frac{1}{u}f(u)g(v)dudv =$$

$$= \frac{2}{\pi^2}\sqrt{\frac{2}{\pi}}\int_0^{+\infty}\int_0^{+\infty}\cos(yv)f(u)g(v)\frac{1}{u}yK_{iy}(u)dudv,$$

from the formula 1 ([1] p.130) we get

(10)
$$\begin{aligned}
&\gamma(y)\left(K^{-1}f\right)(y)(F_cg)(y)=\\
=&\frac{2}{\pi^2}\sqrt{\frac{2}{\pi}}\int_0^{+\infty}\int_0^{+\infty}\cos(yv)f(u)g(v)\left\{\int_0^{+\infty}\sin(y\alpha)\mathrm{sh}\alpha.e^{-u\mathrm{ch}\alpha}d\alpha\right\}dudv=\\
=&\frac{\sqrt{2}}{\pi^2\sqrt{\pi}}\int_0^{+\infty}\int_0^{+\infty}\left\{\int_0^{+\infty}\left[\sin y(\alpha+v)+\sin y(\alpha-v)\right]\mathrm{sh}\alpha.e^{-u\mathrm{ch}\alpha}d\alpha\right\}=\\
=&f(u)g(v)dudv.
\end{aligned}$$

Further,

(11)
$$\begin{aligned}
&\int_0^{+\infty}[\sin y(\alpha+v)+\sin y(\alpha-v)]\mathrm{sh}\alpha.e^{-u\mathrm{ch}\alpha}d\alpha=\\
&=\int_v^{+\infty}\sin(yt)\mathrm{sh}(t-v)e^{-u.\mathrm{ch}(t-v)}dt+\int_{-v}^{+\infty}\sin(yt)\mathrm{sh}(t+v)e^{-u.\mathrm{ch}(t+v)}dt=\\
&=\int_0^{+\infty}\sin(yt)[\mathrm{sh}(t+v)e^{-u.\mathrm{ch}(t+v)}+\mathrm{sh}(t-v)e^{-u\mathrm{ch}(t-v)}]dt-\\
&\quad-\int_0^{v}\sin(yt)\mathrm{sh}(t-v)e^{-u.\mathrm{ch}(t-v)}dt+\int_{-v}^{0}\sin(yt)\mathrm{sh}(t+v)e^{-u.\mathrm{ch}(t+v)}dt.
\end{aligned}$$

As

(12)
$$\begin{aligned}
&\int_{-v}^{0}\sin(yt)\mathrm{sh}(t+v)e^{-u.\mathrm{ch}(t+v)}dt=\int_{v}^{0}\sin(ys)\mathrm{sh}(v-s)e^{-u.\mathrm{ch}(v-s)}ds=\\
&\quad=\int_0^{v}\sin(ys)\mathrm{sh}(s-v)e^{-u.\mathrm{ch}(s-v)}ds,
\end{aligned}$$

it follows from (10), (11), (12) that

$$\gamma(y)\left(K^{-1}f\right)(y)(F_c g)(y) = \sqrt{\frac{2}{\pi}}\int\limits_0^{+\infty}\sin(yt)\Bigg\{\frac{1}{\pi^2}\int\limits_0^{+\infty}\int\limits_0^{+\infty}\Big[\mathrm{sh}(t+v)e^{-u.\mathrm{ch}(t+v)}+$$

$$+\mathrm{sh}(t-v)e^{-u.\mathrm{ch}(t-v)}\Big]f(u)g(v)dudv\Bigg\}dt =$$

$$= F_s\left(f\overset{\gamma}{*}g\right)(y).$$

Thus, the factorization equality (6) is proved.

Similarly,

$$\gamma(y)\left(K^{-1}f\right)(y)(F_s g)(y) = \frac{2}{\pi^2}\sqrt{\frac{2}{\pi}}\int\limits_0^{+\infty}\int\limits_0^{+\infty}\sin(yv)f(u)g(v)\frac{1}{u}yK_{iy}(u)dudv =$$

$$= \frac{2}{\pi^2}\sqrt{\frac{2}{\pi}}\int\limits_0^{+\infty}\int\limits_0^{+\infty}\sin(yv)f(u)g(v)\left\{\int\limits_0^{+\infty}\sin(y\alpha)\mathrm{sh}\alpha.e^{-u.\mathrm{ch}\alpha}d\alpha\right\}dudv =$$

$$= \frac{\sqrt{2}}{\pi^2.\sqrt{\pi}}\int\limits_0^{+\infty}\int\limits_0^{+\infty}\Bigg\{\int\limits_0^{+\infty}[\cos y(\alpha-v)-$$

$$-\cos y(v+\alpha)]\mathrm{sh}\alpha.e^{-u.\mathrm{ch}\alpha}d\alpha\Bigg\}f(u)g(v)dudv =$$

$$= \frac{\sqrt{2}}{\pi^2.\sqrt{\pi}}\int\limits_0^{+\infty}\int\limits_0^{+\infty}f(u)g(v)\Bigg\{\int\limits_{-v}^{+\infty}\cos(yt)\mathrm{sh}(t+v)e^{-u\mathrm{ch}(t+v)}dt-$$

$$-\int\limits_v^{+\infty}\cos(yt)\mathrm{sh}(t-v)e^{-u.\mathrm{ch}(t-v)}dt\Bigg\}dudv =$$

$$= \frac{\sqrt{2}}{\pi^2\sqrt{\pi}}\int\limits_0^{+\infty}\int\limits_0^{+\infty}f(u)g(v)\Bigg\{\int\limits_0^{+\infty}\cos(yt)\Big[\mathrm{sh}(t+v)e^{-u.\mathrm{ch}(t+v)}-$$

$$-\mathrm{sh}(t-v)e^{-u.\mathrm{ch}(t-v)}\Big]dt+$$

$$+\int_{-v}^{0}\cos(yt)\mathrm{sh}(t+v)e^{-u.\mathrm{ch}(t+v)}dt+$$

$$\left.+\int_{0}^{v}\cos(yt)\mathrm{sh}(t-v)e^{-u.\mathrm{ch}(t-v)}dt\right\}dudv.$$

Taking the equality

$$\int_{-v}^{0}\cos(yt)\mathrm{sh}(t+v)e^{-u\mathrm{ch}(t+v)}dt+\int_{0}^{v}\cos(yt)\mathrm{sh}(t-v)e^{-u.\mathrm{ch}(t-v)}dt=0$$

into account, we obtain that

$$\gamma(y)\left(K^{-1}f\right)(y)(F_s g)(y)=$$

$$=\sqrt{\frac{2}{\pi}}\int_{0}^{+\infty}\cos(yt)\left\{\frac{1}{\pi^2}\int_{0}^{+\infty}\int_{0}^{+\infty}\left[\mathrm{sh}(t+v)e^{-u.\mathrm{ch}(t+v)}-\right.\right.$$

$$\left.\left.-\mathrm{sh}(t-v)e^{-u.\mathrm{ch}(t-v)}\right]f(u)g(v)dudv\right\}dt=$$

$$=F_c\left(f\overset{\gamma}{\underset{2}{*}}g\right)(y).$$

The equality (7) is proved.

With the same reasonings we conclude that the convolution (5) exists and belongs to $L(\mathbb{R}_0)$.

Theorem 2. *The convolutions (4), (5) are non-commutative, not associative and the following identities hold*

$$1)\quad f\overset{\gamma}{\underset{1}{*}}\left(g\overset{\gamma}{\underset{2}{*}}h\right)=g\overset{\gamma}{\underset{1}{*}}\left(f\overset{\gamma}{\underset{2}{*}}h\right),$$

$$2)\quad f\overset{\gamma}{\underset{2}{*}}\left(g\overset{\gamma}{\underset{1}{*}}h\right)=g\overset{\gamma}{\underset{2}{*}}\left(f\overset{\gamma}{\underset{1}{*}}h\right),$$

$$3)\quad f\overset{\gamma}{\underset{1}{*}}\left(g\underset{F_c}{*}h\right)=\left(f\overset{\gamma}{\underset{1}{*}}g\right)\underset{1}{*}h,$$

$$4)\quad f\overset{\gamma}{\underset{2}{*}}\left(g\underset{1}{*}h\right)=\left(f\overset{\gamma}{\underset{2}{*}}g\right)\underset{F_c}{*}h,$$

$$5)\quad f\overset{\gamma}{\underset{1}{*}}\left(g\underset{2}{*}h\right)=h\underset{2}{*}\left(f\overset{\gamma}{\underset{1}{*}}g\right).$$

Proof. Theorem 1 implies that the convolutions (4), (5) are non-commutative and not associative. It remains to prove 1) - 5). We have

$$\begin{aligned} F_s\left(f \overset{\gamma}{\underset{1}{*}} \left(g \overset{\gamma}{\underset{2}{*}} h\right)\right)(y) &= \mathrm{sh}^{-1}(\pi y)\left(K^{-1}f\right)(y).F_c\left(g \overset{\gamma}{\underset{2}{*}} h\right)(y) = \\ &= \mathrm{sh}^{-1}(\pi y)\left(K^{-1}f\right)(y).\mathrm{sh}^{-1}(\pi y)\left(K^{-1}g\right)(y)(F_s h)(y) = \\ &= \mathrm{sh}^{-1}(\pi y)\left(K^{-1}g\right)(y).F_c\left(f \overset{\gamma}{\underset{2}{*}} h\right)(y) = F_s\left(g \overset{\gamma}{\underset{1}{*}} \left(f \overset{\gamma}{\underset{2}{*}} h\right)\right)(y). \end{aligned}$$

It follows

$$f \overset{\gamma}{\underset{1}{*}} \left(g \overset{\gamma}{\underset{2}{*}} h\right) = g \overset{\gamma}{\underset{1}{*}} \left(f \overset{\gamma}{\underset{2}{*}} h\right).$$

Similarly, we can prove the remaining identities. As an example, we prove 3). We have

$$\begin{aligned} F_s\left(f \overset{\gamma}{\underset{1}{*}} \left(g \underset{F_c}{*} h\right)\right)(y) &= \mathrm{sh}^{-1}(\pi y)\left(K^{-1}f\right)(y).F_c\left(g \underset{F_c}{*} h\right)(y) = \\ &= \mathrm{sh}^{-1}(\pi y)\left(K^{-1}f\right)(y).(F_c g)(y)(F_c h)(y) = \\ &= F_s\left(f \overset{\gamma}{\underset{1}{*}} g\right)(y)(F_c h)(y) = F_s\left(\left(f \overset{\gamma}{\underset{1}{*}} g\right) \underset{1}{*} h\right)(y). \end{aligned}$$

Hence

$$f \overset{\gamma}{\underset{1}{*}} \left(g \underset{F_c}{*} h\right) = \left(f \overset{\gamma}{\underset{1}{*}} g\right) \underset{1}{*} h.$$

Remark. *The following identities hold*

$$1) \quad \left(f \overset{\gamma}{\underset{1}{*}} \left(g \overset{\gamma}{\underset{2}{*}} h\right)\right) \underset{1}{*} k = f \overset{\gamma}{\underset{1}{*}} \left(g \overset{\gamma}{\underset{2}{*}} \left(h \underset{1}{*} k\right)\right),$$

$$2) \quad \left(f \overset{\gamma}{\underset{2}{*}} \left(g \overset{\gamma}{\underset{1}{*}} h\right)\right) \underset{F_c}{*} k = f \overset{\gamma}{\underset{2}{*}} \left(g \overset{\gamma}{\underset{1}{*}} \left(h \underset{F_c}{*} k\right)\right).$$

Proof. It follows from Theorem 1 that

$$\begin{aligned} F_s\left(\left(f \overset{\gamma}{\underset{1}{*}} (g \overset{\gamma}{\underset{2}{*}} h)\right) \underset{1}{*} k)(y\right) &= F_s\left(f \overset{\gamma}{\underset{1}{*}} \left(g \overset{\gamma}{\underset{2}{*}} h\right)\right)(y)(F_c k)(y) = \\ &= \mathrm{sh}^{-1}(\pi y)\left(K^{-1}f\right)(y).(F_c)\left(g \overset{\gamma}{\underset{2}{*}} h\right)(y)(F_c k)(y) = \\ &= \mathrm{sh}^{-1}(\pi y)\left(K^{-1}f\right)(y).\mathrm{sh}^{-1}(\pi y)\left(K^{-1}g\right)(y)(F_s h)(y)(F_c k)(y) = \\ &= \mathrm{sh}^{-2}(\pi y)\left(K^{-1}f\right)(y).\left(K^{-1}g\right)(y)F_s\left(h \underset{1}{*} k\right) = \\ &= \mathrm{sh}^{-1}(\pi y)\left(K^{-1}f\right)(y).F_c\left(g \overset{\gamma}{\underset{2}{*}} \left(g \underset{1}{*} k\right)\right) = \\ &= F_s\left(f \overset{\gamma}{\underset{1}{*}} \left(g \overset{\gamma}{\underset{2}{*}} \left(h \underset{1}{*} k\right)\right)\right). \end{aligned}$$

Hence

$$\left(f \overset{\gamma}{\underset{1}{*}} \left(g \overset{\gamma}{\underset{2}{*}} h\right)\right) \underset{1}{*} k = f \overset{\gamma}{\underset{1}{*}} \left(g \overset{\gamma}{\underset{2}{*}} \left(h \underset{1}{*} k\right)\right).$$

Similarly we can prove 2).

Theorem 3. *For the generalized convolutions (4), (5), we have the identity*

(13)

$$\left(f \overset{\gamma}{\underset{1}{*}} g\right)(x) + \left(f \overset{\gamma}{\underset{2}{*}} g\right)(x) =$$
$$= \frac{\sqrt{2}}{\pi\sqrt{\pi}} \int_0^{+\infty} f(u) \left[\left(\mathrm{sh}t.e^{-u.\mathrm{ch}t} \underset{F_c}{*} g(t)\right)(x) - \left(\mathrm{sh}t.e^{-u\mathrm{ch}t} \underset{1}{*} g(t)\right)(x)\right] du.$$

Proof. We have

(14)

$$\left(f \overset{\gamma}{\underset{1}{*}} g\right)(x) =$$
$$= \frac{1}{\pi^2} \int_0^{+\infty}\int_0^{+\infty} \left[\mathrm{sh}(x+v)e^{-u\mathrm{ch}(x+v)} + \mathrm{sh}(|x-v|)e^{-u.\mathrm{ch}(x-v)}\right] f(u)g(v)dudv+$$
$$+ \frac{2}{\pi^2} \int_0^{+\infty}\int_x^{+\infty} \mathrm{sh}(x-v)e^{u\mathrm{ch}(x-v)} f(u)g(v)dudv =$$
$$= \frac{\sqrt{2}}{\pi\sqrt{\pi}} \int_0^{+\infty} f(u) \left(\mathrm{sh}t.e^{-u.\mathrm{ch}t} \underset{F_c}{*} g(t)\right)(x)du+$$
$$+ \frac{2}{\pi^2} \int_0^{+\infty}\int_x^{+\infty} \mathrm{sh}(x-v)e^{-u.\mathrm{ch}(x-v)} f(u)g(v)dudv.$$

On the other hand,

$$\left(f \overset{\gamma}{\underset{2}{*}} g\right)(x) =$$
$$= \frac{1}{\pi^2} \int_0^{+\infty}\int_0^{+\infty} \left[\mathrm{sh}(x+v)e^{-u.\mathrm{ch}(x+v)} - \mathrm{sh}(|x-v|)e^{-u.\mathrm{ch}(x-v)}\right] f(u)g(v)dudv-$$
$$- \frac{2}{\pi^2} \int_0^{+\infty}\int_x^{+\infty} \mathrm{sh}(x-v)e^{-u.\mathrm{ch}(x-v)} f(u)g(v)dudv =$$

$$(15)\quad \begin{aligned} &= -\frac{1}{\pi^2}\int\limits_0^{+\infty} f(u)\left(\text{sh}t.e^{-u.\text{ch}t} \underset{1}{*} g(t)\right)(x)du - \\ &- \frac{2}{\pi^2}\int\limits_0^{+\infty}\int\limits_x^{+\infty} \text{sh}(x-v)e^{-u.\text{ch}(x-v)} f(u)g(v)dudv. \end{aligned}$$

From (14), (15) we obtain (13).

III. Application

Theorem 4. *Consider the system of integral equations*

$$(16)\quad \begin{aligned} f(x)+\lambda_1\int\limits_0^{+\infty}\int\limits_0^{+\infty}\theta_1(x,u,v)\varphi(u)g(v)dudv = h(x), \\ \lambda_2\int\limits_0^{+\infty}\theta_2(x,v)f(u)du + g(x) = k(x),\ x>0. \end{aligned}$$

Here,

$$\begin{aligned} \theta_1(x,u,v) &= \frac{1}{\pi^2}\left[\text{sh}(x+v)e^{-u\text{ch}(x+v)} + \text{sh}(x-v)e^{-u\text{ch}(x+v)}\right], \\ \theta_2(x,u) &= \frac{1}{\sqrt{2\pi}}[\text{sign}(u-x)\psi(|u-x|) - \psi(u+x)], \end{aligned}$$

$\varphi \in L\left(\frac{1}{x},\mathbb{R}_+\right)$, $\psi, h, k \in L(\mathbb{R}_+)$ *are given functions,* λ_1,λ_2 *are given constants and* f, g *are the unknowns. Then, with the condition*

$$1-\lambda_1\lambda_2 F_c\left(\varphi \overset{\gamma}{*} \psi\right)(y) \neq 0,$$

the system (16) has a solution

$$f(x) = h(x) + \left(h \underset{1}{*} q\right)(x) - \lambda_1\left(\varphi \overset{\gamma}{\underset{1}{*}} k\right)(x) - \lambda_1\left(\varphi \overset{\gamma}{\underset{1}{*}}\left(q \underset{F_c}{*} k\right)\right)(x) \in L(\mathbb{R}_+),$$

$$g(x) = k(x) + \left(q \underset{F_c}{*} k\right)(x) - \lambda_2\left(\psi \underset{2}{*} h\right)(x) - \lambda_2\left(q \underset{F_c}{*}\left(\psi \underset{2}{*} h\right)\right)(x) \in L(\mathbb{R}_+)$$

with $q(x) \in L(\mathbb{R}_+)$ *satisfying*

$$(F_c q)(y) = \frac{\lambda_1 \lambda_2 F_c \left(\varphi \overset{\gamma}{\underset{2}{*}} \psi\right)(y)}{1 - \lambda_1 \lambda_2 F_c (\varphi \overset{\gamma}{\underset{2}{*}} \psi)(y)}.$$

Proof. Using (10) and the factorization equalities, we obtain the linear system

$$\begin{aligned} (F_s f)(y) + \lambda_1 \mathrm{sh}^{-1}(\pi y)(K^{-1}\varphi)(y).(F_c g)(y) &= (F_s h)(y), \\ \lambda_2.(F_s \psi)(y).(F_s f)(y) + (F_c g)(y) &= (F_c k)(y). \end{aligned}$$

We have

$$\begin{aligned} \Delta &= \begin{vmatrix} 1 & \lambda_1 \mathrm{sh}^{-1}(\pi y).(K^{-1}\varphi)(y) \\ \lambda_2 (F_s \psi)(y) & 1 \end{vmatrix} = \\ &= 1 - \lambda_1 \lambda_2 F_c \left(\varphi \overset{\gamma}{*} \psi\right)(y) \neq 0, \end{aligned}$$

$$\begin{aligned} \Delta_1 &= \begin{vmatrix} (F_s h)(y) & \lambda_1 \mathrm{sh}^{-1}(\pi y).\left(K^{-1}\varphi\right)(y) \\ (F_c k)(y) & 1 \end{vmatrix} = \\ &= (F_s h)(y) - \lambda_1 \mathrm{sh}^{-1}(\pi y).\left(K^{-1}\varphi\right)(y).(F_c k)(y), \end{aligned}$$

$$\begin{aligned} \Delta_2 &= \begin{vmatrix} 1 & (F_s h)(y) \\ \lambda_2 (F_s \psi)(y) & (F_s h)(y) \end{vmatrix} = \\ &= (F_c k)(y) - \lambda_2 (F_s \psi)(y)(F_s k)(y). \end{aligned}$$

Hence

$$(F_s f)(y) = \frac{1}{\Delta}\left[(F_s h)(y) - \lambda_1 \mathrm{sh}^{-1}(\pi y).\left(K^{-1}\varphi\right)(y).(F_c k)(y)\right] =$$

$$= \left[1 + \frac{\lambda_1 \lambda_2 F_c \left(\varphi \overset{\gamma}{\underset{2}{*}} \psi\right)(y)}{1 - \lambda_1 \lambda_2 F_c (\varphi \overset{\gamma}{\underset{2}{*}} \psi)(y)}\right]\left[(F_s h)(y) - \lambda_1 \mathrm{sh}^{-1}(\pi y)\left(K^{-1}\varphi\right)(y)(F_c k)(y)\right].$$

In virtue of Wiener-Levi's theorem there is a function $q(x) \in L(\mathbb{R}_+)$ such that

$$\frac{\lambda_1 \lambda_2 F_c \left(\varphi \overset{\gamma}{\underset{2}{*}} \psi\right)(y)}{1 - \lambda_1 \lambda_2 F_c \left(\varphi \overset{\gamma}{\underset{2}{*}} \psi\right)(y)} = (F_c q)(y).$$

From this and (1), (2), (4) we get

$$(F_sf)(y) = \Big[\ 1 + (F_cq)(y)\ \Big]\Big[\ (F_sh)(y) - \lambda_1 \mathrm{sh}^{-1}(\pi y)\left(K^{-1}\varphi\right)(y)(F_ck)(y)\ \Big] =$$
$$= (F_sh)(y) + (F_cq)(y)(F_sh)(y) - \lambda_1 F_s\left(\varphi \overset{\gamma}{\underset{1}{*}} k\right)(y) -$$
$$- \lambda_1 \mathrm{sh}^{-1}(\pi y)\left(K^{-1}\varphi\right)(y).F_c\left(q \underset{F_c}{*} k\right)(y) =$$
$$= (F_sh)(y) + F_s\left(h \underset{1}{*} q\right)(y) - \lambda_1 F_s\left(\varphi \overset{\gamma}{\underset{1}{*}} k\right)(y) -$$
$$- \lambda_1 F_s\left(\varphi \overset{\gamma}{\underset{1}{*}}\left(q \underset{F_c}{*} k\right)\right)(y).$$

From the last equation and Theorem 1 we obtain

$$f(x) = h(x) + \left(h \underset{1}{*} q\right)(x) - \lambda_1\left(\varphi \overset{\gamma}{\underset{1}{*}} k\right)(x) - \lambda_1\left(\varphi \overset{\gamma}{\underset{1}{*}}\left(q \underset{F_c}{*} k\right)\right)(x) \in L(\mathbb{R}_+).$$

Similarly,

$$(F_cg)(y) = \frac{1}{\Delta}\left[(F_ck)(y) - \lambda_2(F_s\psi)(y)(F_sh)(y)\right] =$$
$$= \Big[\ 1 + (F_cq)(y)\ \Big]\Big[\ (F_ck)(y) - \lambda_2(F_s\psi)(y)(F_sh)(y)\ \Big] =$$
$$= (F_ck)(y) + F_c\left(q \underset{F_c}{*} k\right)(y) - \lambda_2 F_c\left(\psi \underset{2}{*} h\right)(y) -$$
$$- \lambda_2(F_cq)(y).F_c\left(\psi \underset{2}{*} h\right)(y) =$$
$$= (F_ck)(y) + F_c\left(q \underset{F_c}{*} k\right)(y) - \lambda_2 F_c\left(\psi \underset{2}{*} h\right)(y) - \lambda_2 F_c\left(q \underset{F_c}{*}\left(\psi \underset{2}{*} h\right)\right)(y).$$

Hence

$$g(x) = k(x) + \left(q \underset{F_c}{*} k\right)(x) - \lambda_2\left(\psi \underset{2}{*} h\right)(x) - \lambda_2\left(q \underset{2}{*} h\right)(x) \in L(\mathbb{R}_+).$$

By the same way we can prove the following result.

Theorem 5. *Consider the system of integral equations*

$$f(x) + \lambda_1 \int_0^{+\infty} \theta_3(x,u)g(u)du = h(x),$$

(17)

$$\lambda_2 \int_0^{+\infty}\int_0^{+\infty} \theta_4(x,u,v)\psi(u)f(v)dudv + g(x) = k(x).$$

Here,

$$\theta_3(x,u) = \frac{1}{\sqrt{2\pi}}\left[\varphi(|x-u|) - \varphi(x+u)\right],$$
$$\theta_4(x,u,v) = \frac{1}{\pi^2}\left[\mathrm{sh}(x+v)e^{-\mathrm{ch}(x+v)} - \mathrm{sh}(x-v)e^{-u\mathrm{ch}(x-v)}\right].$$

The functions $\psi \in L\left(\frac{1}{x}, \mathbb{R}_+\right)$; $\varphi, h, k \in L(\mathbb{R}_+)$ *are given,* λ_1, λ_2 *are given constants, and* f, g *are unknown functions. Then, with the condition*

$$1 - \lambda_1\lambda_2.F_c\left(\psi \underset{F_c}{*} \varphi\right)(y) \neq 0$$

the system (17) has a solution

$$f(x) = h(x) + \left(h \underset{1}{*} q\right)(x) - \lambda_1\left(\varphi \underset{1}{*} k\right)(x) - \lambda_1\left(\varphi \underset{1}{*}\left(q \underset{F_c}{*} k\right)\right)(x) \in L(\mathbb{R}_+),$$
$$g(x) = k(x) + \left(q \underset{F_c}{*} k\right)(x) - \lambda_2\left(\psi \overset{\gamma}{\underset{2}{*}} h\right)(x) - \lambda_2\left(\psi \overset{\gamma}{\underset{2}{*}}\left(h \underset{1}{*} q\right)\right)(x) \in (\mathbb{R}_+)$$

with $q(x) \in L(\mathbb{R}_+)$ *satisfying the condition*

$$(F_c q)(y) = \frac{\lambda_1\lambda_2 F_c\left(\psi \overset{\gamma}{\underset{2}{*}} \varphi\right)(y)}{1 - \lambda_1\lambda_2 F_c\left(\psi \overset{\gamma}{\underset{2}{*}} \varphi\right)(y)}.$$

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