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A NEW BOUND FOR THE MINIMAL DIAMETER OF NINE COPLANAR CONGRUENT DISKS

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Abstract. Denote $d(n)$ the minimum diameter of a set of n points in the plane such that all the mutual distances between the points are at least 1; hence, $d(n)+1$ is the minimal diameter of the n circles of radius 0.5 drawn around the given points. The exact value of $d(n)$ is known up to 8. The best known estimation on $d(9)$ issued from a proposition of S. Vincze (1950): $d(9) \leq 2.58...$ In this paper we sharpen this value by using computer. We show a convex heptagon containing two further points inside such that all the mutual distances of the 9 points are at least 1 and $d(9) \leq 2.5693$.

1. Introduction

The *diameter* of a finite point set given in the plane is the maximal value of all the mutual distances between the points. In this paper we investigate point sets with the property of all the mutual distances between the points are at least 1; refer to this property of point sets in the plane as *property* $\mathcal{P}$. Hence, drawing a circle of radius 0.5 around all the points we get a set of congruent closed disks such that any two of them may touch each other at one common point of their boundaries or they must be disjoint. This view of the point sets with property $\mathcal{P}$ suggested the title of the paper and these two views are used equivalently in the following.

Considering n points of the kind $\mathcal{P}$ denote the minimal diameter of all these sets by $d(n)$. The exact value of $d(9)$ is not known up to date. Here we

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find a new upper bound for $d(9)$ by using a great amount of various kinds of computer support. This support is often surprising: in addition to conventional applications of computer it is also used as a 'practical experimenting device'. Another exciting and hard task is the preprocessing of mathematical problem so as its 'data space' could be the input of some computer applications or some algorithmical method. After this methodological preliminaries we continue with the short history of the problem.

We do not know exactly who was the first to ask the minimal diameter of a planar set of n points with property $\mathcal{P}$. However, thinking about the problem we often refer to a theorem of Bieberbach ([3], 1915) saying that among the convex planar regions of a given area the circle has the smallest diameter. We have also a nice theorem of Reinhardt from 1922 ([13]) that solves the minimal diameter problem for planar p-gons of all side lengths 1, where p is an odd prime. Vincze also investigated the problem of planar n-gons with side lengths 1 in the general case and constructed a non-regular convex octogon for which $d(8) \leq 2.58...$ applied ([15]) and it turned out later that an additional (9-th) point can be placed inside the octogon such that property $\mathcal{P}$ still remained and hence $d(9) \leq 2.58...$ was also provided. In 1951 Bateman and Erdős mentioned the problem explicitely ([1]) and showed that the exact value of $d(n)$ can be calculated easily up to $n = 6$ and also deduced in [1] that $d(7) = 2$. After a long period without new results A. Bezdek and F. Fodor showed in 1999 ([2]) that $d(8) = 1/(2\sin(\pi/14))$.

As mentioned above the exact value of $d(9)$ is not known. Moreover, the best known estimation proved up to date is the result of Vincze from 1950: $d(9) \leq 2.58...!$

Here we show that $d(9) \leq 2.5693$ is also valid and give a configuration that is an appropriate convex heptagon containing two further points inside, and for these 9 points the property $\mathcal{P}$ and the $d(9) \leq 2.5693$ inequality also applies. Hence, it also appears that one should think about configurations other than a convex octogon with an additional point inside when looking for a 9-point configuration with minimal diameter.

In Section 2 we specify the problem formally and summarize the known results in words and also in formulas.

In Section 3 we mention an experiment performed by using a computer CAD-system that leads us to an initial configuration of 9 points. Observing some geometric properties of this configuration the combinatorial cases can be reduced so as the diameter of the set still decreases and the restricted configuration can be the input of a simple discrete local minimum searching.

In Section 4 we present the discrete local minimum searching that leads us to the new bound. This last section is closed by a conjecture about the optimal configuration.

2. Problem, known results

Problem. Be given 9 points $P_1, P_2, \ldots, P_9$ in the plane such that all the mutual distances between them are at least 1: $|P_iP_j| \geq 1$, $i, j = 1, \ldots, 9$, $i \neq j$ (that is property $\mathcal{P}$ holds).

Find: Estimation on the smallest diameter of all the point sets of this kind that is smaller than 2.58... (the best known up to date).

More generally, denote $d(n)$ the smallest diameter of the n-th element point sets of property $\mathcal{P}$. Then the actually known results are the following:

- In case of $n = 2, 3, 4, 5$ the optimal configuration is the regular n-gon of side length 1.
- If $n = 6$ the optimal configuration is a regular pentagon constructed along a circle of radius 1 and centered at the fixed 6-th point.
- In case of $n = 7$, according to Bateman and Erdős ([1], 1951), the optimal result is the regular hexagon of diameter 2 with an additional 7-th point at the center of the 6-gon.
- For $n = 8$ A. Bezdek and F. Fodor showed in 1999 ([2]) that the optimal configuration is a regular heptagon of side length 1 with an additional 8-th point inside that is at least 1 unit distance from all the vertices of the 7-gon; hence, the solution is not unique.
- In case of $n = 9$ we have estimations for $d(9)$ only: the best known result up to date followed from the construction of Vincze that is a special non-regular octogon with an additional 9-th point inside, which is at least 1 unit distance from all the vertices of the 8-gon ([15], 1950).

Expressing the above by numerical values

$$d(2) = d(3) = 1;$$

$$d(4) = \sqrt{2}\ (= 1.4142...);$$

$$d(5) = (1 + \sqrt{5})/2\ (= 1.618...);$$

$$d(6) = 2\sin(72)\ (= 1.9...);$$

$$d(7) = 2;$$

$$d(8) = 1/(2\sin(\pi/14))\ (= 2.246...);$$

$$d(9) \leq 2.58...$$

In the following we show that

$$d(9) < 2.5693$$

ia also valid.

3. Computer experiments

In the course of history of science mathematicians also made trials every time. Gauss' experiments on investigating the properties of 'big triangles' are well known. He formed appropriate 'triangles' by directing sunrays by mirrors set in the top of hills. A less known - but closer to our topic - experiment is due to the English clergyman S. Hales about three hundred years ago. On investigating plant growth he also drew conclusions on space-filling by congruent bodies. He put dried peas into a pot and added water to it to fill the empty space. By putting the contents under pressure afterwards, the peas absorved water and expanded into the empty spaces. Observing samples of peas he found (thought) that they became regular dodecahedra. However, S. Hales did not know that regular dodecahedra do not fill space without gaps [5].

Here we can also start with an obvious trial: following intuitions one can configure nine coins of the same kind on the table so as to reach as narrow center point set as possible (denote its diameter $d'(9)$). Then the disk set (and the convex hull of the disk set) is also the narrowest possible: its diameter is $d'(9) + 1$.

One should not undervalue this non-mathematical trial: also forming (from the same kinds of coins) the Vincze configuration and continuously comparing to the current state of the experimental configurations, after a short trial period one can guess that a special kind of heptagons may be compared with the octogon of Vincze. However, one should also realize that the diameters of the two polygons are nearly the same when comparing them by compasses, for example; so this way of experimenting does not work further on. Hence, trials should be continued by using other methods that provide more (much more) accuracy, we should turn to a suitable computer program.

At this point of the 'experimental process' one can consider the following important moments.

- In our further investigations we should focus on a special kind of heptagons with two other points inside such that all nine point accomplish the property $\mathcal{P}$.
- Equivalently, nine non-intersecting disks of diameter 1 unit and centres at the previous nine points can be investigated.
- We should use an analytical computer program (for numerical accuracy) with an appropriate graphical interface (for visualisation).

In the remainder of this section first we specify the previous requirements on the computer system in details that lead us to computer CAD-systems as 'experimenting tools'; then model the nine-disk configuration suggested by the initial experimental steps above in such a CAD-system; and finally, deduce geometric properties of an initial disk configuration that we investigate analytically in the following section.

Below we list the most important characteristics according to a formal (informatical) terminology that a computer program should provide to be able to form valid nine-disk configurations in an efficient way.

- The system should be 'analytical' in the sense that it should provide arbitrary levels of numerical accuracy in representing defined and also calculated geometric entities.
- It should be the kind that can be manipulated by a multifunctional (computer) graphics interface that supports the definition of geometric objects, an arbitrary scaling of subviews and superviews of a view, regenerating views, ...
- It should also provide multifunctional constructing assistance like 'object snap' to defined points and also to points issued from certain relations of objects (such as intersections or touching points of circles,...), 'orthogonal moving',...
- Obviously, the system should provide the typical geometric transformations (moving, rotating, scaling, mirroring,...) of geometric objects - that can be performed also by graphical support.

The demanded above conditions are provided typically by general-purpose CAD-systems and hence writing in the following 'computer system', 'system', 'computer program',..., we mean a computer program of this kind.

In the following we define the promissing configuration of disks (found in the initial steps) in a computer CAD-system and observe some algorithmic properties of it to be able to define such configurations efficiently by the computer.

(1) We investigate only heptagons with two additional points inside such that for this set of points the property $\mathcal{P}$ holds.

(2) Suppose that the sides of the heptagon are all of 1 unit length except of one side (which has length more than 1 unit).
(3) It has an axis of symmetry s.
(4) Consequently, one vertex of the heptagon must be incident with this axis. Moreover, suppose that the points inside the heptagon are also incident with the axis of symmetry.
(5) We also suppose that the distance of these two inner points is also 1 unit length.
(6) Obviously, one should consider only the cases of such configurations when any vertex of the heptagon is in 2.58/2 unit distance to the axis of symmetry at most.

Properties (1)-(5) were suggested by the outputs of 'positioning coins'. Property (6) prevents us from investigating cases when the initial configuration was defined careless: there appears a known distance between the initial points equal or more than 2.58.

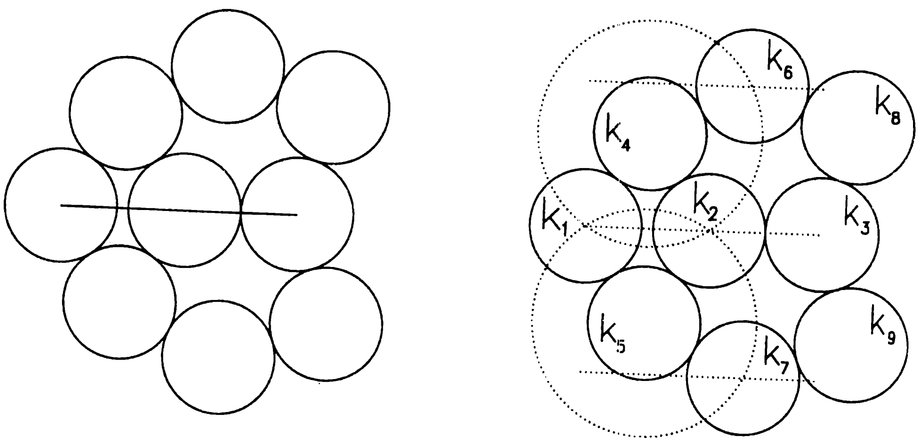

Figure 1. Close packing of nine disks in the plane

The left drawing of Figure 1 shows a general case of configurations of circles of diameter 1 of which the center points satisfy requirements (1)-(6). Number these circles according to the figure. Then the critical distances of the configuration are the ones appearing between the center points of circle-pairs 1-8, 5-8 and 6-7. Initializing and editing some adequate set of circles and measuring the mentioned distances shows that the distance of O_5O_8 dominates the diameter of the set and its value can be decreased by leaving circle k_4 to touch circles k_1 and k_2 (perhaps at the cost of increasing the less significant distance of O_1O_8).

This last observation may be true or false, one could prove or deny it geometrically. However, we do not worry about the truth-value of this assertion: we consider it initially and depending on the results of the calculations we can change or drop it afterwards.

(7) Let circle k_4 (and hence k_5, too) touch circles k_1 and k_2.

From now on any configuration satisfying properties (1)-(7) can be defined by two parameters.

- Denote $2t$ the distance of the two nearest quadrant points of circles k_1 and k_2.
- Denote h the distance of the center point of k_6 (and hence, the center point of k_7, too) from the axis of symmetry.

(The radius of any circle of the configuration is obviously 0.5.)

Then, after giving 'valid' values to t and h, a valid configuration can be constructed by the following 'algorithm' (construction steps).

(1) Draw the line s and choose a point O_1 on it as the center point of k_1.
(2) On the right side of k_1 set k_2 in $2t$ distance far from k_1 with center point O_2 on s.
(3) On the right side of k_2 set k_3 touching k_2 with center point O_3 on s.
(4) Construct k_4 with center point O_4 that touches k_1 and k_2.
(5) Construct a circle k_4' with center point O_4 and radius 1, and a line e in the halfplane bounded by s and containing O_4 that is parallel to s and runs h long far from s. Be O_6 = the right point of $k_4' \cap e$, the center point of k_6.
(6) Construct k_8 touching k_3 and k_6, with center point O_8.
(7) Construct similarly k_5, k_7 and k_9 on the other side of s.

4. Calculations

First of all perform the algorithm for some typical values of t and h!

For $t = 0.0$ and $h = 2.58/2.0$ (the 'natural' initial values of t and h) we get $d(9) = 2.58 \leq 2.58...$, hence the known upper bound for $d(9)$ is probably improved.

Be $t = 0.05$ (hence, the distance of k_1 and k_2 is 0.1) and $h = 2.57/2.0$. For these values we get $d(9) = 2.5729...$, which is already a definite improvement on the known results and hence it may be the subject of a new theorem.

However, we do not stop here. These results encourage us and the combinatorial complexity of the reduced problem enables us to perform a discrete local minimum searching on the diameters by using computer. For this purpose we should 'translate' the steps of the constructing algorithm to the 'language' of coordinate geometry.

First we define a Descartes coordinate system on the plane in which calculations can be performed conveniently. Be O_1 the origin and the half of s that contains O_2 the positive x-axis of this coordinate system. If $O_i(x_i, y_i)$ refers to point O_i one can get the following formulas by simple coordinate geometry calculations.

$$\begin{aligned}
&O_1(0,0),\\
&O_2(1+2t,0),\\
&O_3(2+2t,0),\\
&O_4(0.5+t,\sqrt{1-(0.5+t)^2}),\\
&O_5(x_4,-y_4),\\
&O_6(0.5+t+\sqrt{1-(h-y_4)^2},h),\\
&O_7(x_6,-y_6).
\end{aligned}$$

For the coordinates of O_8 and O_9 we use new variables. Be $u = (x_3 + x_6)/2.0$ and $v = (y_3 + y_6)/2.0$, hence $F(u, v)$ refers to the midpoint of the segment O_3O_6. Then the vector of direction of segment FO_8 is $\overline{r} = (v, 2 + 2t - u)$ and its length is $\sqrt{1-(2+2t-u)^2-v^2}$. Hence, the vectors of position of O_8 and O_9 are

$$\begin{aligned}
&O_8 : (u,v) + q\overline{r}/|\overline{r}|,\\
&O_9 : (x_8, -y_8).
\end{aligned}$$

Having these formulas one can organize a double-cycle on some discrete values of (t, h) using sufficiently small stepping distances (10^{-3}, 10^{-4} in unit length) to get a characteristic discrete local minimum value of function $d(.)$, which reduces to a two-variable function in this special kind of disk configurations on a reasonable domain (specified below). We can easily find a domain of ordered real pairs for the variables t and h:

- $0 \le t \le 0.58/2.0$, otherwise the distance of the circles k_1 and k_3 may exceed the known bound;

- $0.866 \le h \le 2.58/2.0$, because O_6 and O_7 cannot get nearer to the axis s than in the case when k_6 and k_7 touches k_2 and k_3, and again, we do not want to investigate cases when a known distance ($|O_6O_7|$) may exceed the 2.58... value.

Execute the algorithm when the variables t and h run across the specified intervals by the equidistant step 10^{-4}. In this procedure we get the following results.

- The discrete local minimum value that we were searching for is 2.5692553644.
- This minimum value appears at the value of variable-pair $(2t, 2h) = (0.0912,\ 2.5693)$.

Putting together the outputs of the experiments and the program execution we can form the following two theorems.

Theorem. *Between the configurations of nine coplanar unit-diameter disks corresponding to properties (1)-(7) there exist infinitely many ones for which the maximal value of all the mutual distances of the center points of the nine circles* < 2.58.

Proof. The coordinate functions connecting to the problem are all continuous uniformly on the domain of interest and hence the distance functions issued from them by additions, subtractions, multiplications, extractions of roots and maxima calculations are also of the same kind. Hence, the maximal value calculated for the investigated disk configurations can get all the real values between 2.5693 and 2.58.

Theorem. *For any configuration of nine coplanar points corresponding to property* $\mathcal{P}$, $d(9) < 2.5693$.

Proof. This is an obvious consequence of the output configuration of the discrete local minimum searched above.

One can easily reconstruct (and test) the described configuration in a CAD-system by using also the center points given below with coordinates rounded to four decimal digits.

$$O_1(0, 0),$$
$$O_2(1.0912, 0),$$
$$O_3(2.0912, 0),$$
$$O_4(0.5456, 0.8381),$$
$$O_5(0.5456, -0.8381),$$
$$O_6(1.4404, 1.2846),$$
$$O_7(1.4404, -1.2846),$$

$$O_8(2.3848, 0.9559),$$
$$O_9(2.3848, -0.9559).$$

After due investigation we guess that the following observation is also valid.

Conjecture. *The optimal positioning of nine disks in the domain of configurations defined in the first theorem is the one for which* $|O_1O_8| = |O_5O_8| = |O_6O_7| \approx 2.569255...$

One can find exciting topics on the disk-packings also in [4], [7] and [14]. There are useful results concerning the diameter of convex sets also in [6], [9] and [10]. One can efficiently combine computer programming and using a computer algebra system. In this case visualisation may also be supported by such a system (see [11], for example). In any case - and this is perhaps the main 'message' of this paper - a suitable CAD-system with a good computer graphics interface should be an aid (like a pocket-calculator used to be) or even an experimenting tool in many kinds of mathematical investigations. Making use of these kinds of computer support one can find useful ideas about similar topics in [8] and [12].

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