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ON SETS CHARACTERIZING THE IDENTITY FUNCTION

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Dedicated to Professor Imre Kátai
on his 65th birthday

Abstract. We prove that if a function f with $f(4)f(9) \neq 0$ and a positive integer k satisfy the condition

$$f\left(n^2+m^2+k\right) = f(n^2) + f(m^2+k) \quad \text{for all} \quad n, m \in I\!N,$$

then $f(n) = n$ for all positive integers n, $(n, 2k) = 1$.

In this paper, we let $I\!N$, $I\!N_0$ and $\mathcal{P}$ stand for the set of positive integers, non-negative integers and prime numbers, respectively. We denote by $\mathcal{M}$ the set of all multiplicative functions f such that $f(1) = 1$. Furthermore, we deal with the set $\mathcal{B} \subset I\!N$ of non-negative integers which can be represented as a sum of two squares of integers and with $\mathcal{S}$ the set of all squares of positive integers.

Following C. Spiro [7], we say that subsets A and B of $I\!N$ are additive uniqueness sets (AU-sets) for $\mathcal{M}$ if there is exactly one element $f \in \mathcal{M}$ which satisfies

$$f(a+b) = f(a) + f(b) \quad \text{for all} \quad a \in A \quad \text{and} \quad b \in B.$$

In 1992 C. Spiro [7] showed that $A = B = \mathcal{P}$ are AU-sets for $\mathcal{M}$. In the paper [3] written jointly with J.-M. DeKoninck and I. Kátai we proved that $A = \mathcal{S}$ and $B = \mathcal{P}$ are also AU-sets for $\mathcal{M}$. For other results we refer to [1], [2] and

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[6]. For example, in [6] we proved that if a multiplicative function f satisfies the equation

$$f(n^2+m^2+3)=f(n^2+1)+f(m^2+2)$$

for all positive integers n and m, then either $f(n)=n$ or

$$f(n^2+1)=f(m^2+2)=f(n^2+m^2+3)=0 \quad \text{for all} \quad n,\ m\in I\!N.$$

Our purpose of this paper is to prove the following

Theorem. *Assume that $k\in I\!N$ and $f\in\mathcal{M}$ satisfy the condition*

$$f\left(n^2+m^2+k\right)=f(n^2)+f(m^2+k) \quad \textit{for all} \quad n,m\in I\!N.$$

If $f(4)f(9)\neq 0$, then

$$f(n)=n \quad \textit{for all} \quad n\in I\!N,\ (n,2k)=1.$$

If $f(9)=0$, then $k\equiv 2 \pmod 3$, $f(n^2)=\chi_3(n)$ for all $n\in I\!N$ and

$$(i) \qquad f(n^2+m^2+k)=\chi_3(n)+\chi_3(m)-1 \quad \textit{for all} \quad n\in I\!N,\quad m\in I\!N_0.$$

If $f(9)\neq 0$,$f(4)=0$ then either $k\equiv 3 \pmod 4$ and

$$(ii) \qquad f(n^2+m^2+k)=\chi_2(n)+\chi_2(m)-1 \quad \textit{for all} \quad n\in I\!N,\quad m\in I\!N_0,$$

or $k\equiv 0 \pmod 4$ and

$$(iii) \qquad f(n^2+m^2+k)=\chi_2(n)+\chi_2(m) \quad \textit{for all} \quad n\in I\!N,\quad m\in I\!N_0.$$

In the last two cases (ii) and (iii) we have $f(n^2)=\chi_2(n)$ for all $n\in I\!N$. Here χ_i denotes the principal character (mod i), that is

$$\chi_i(n)=\begin{cases}1, & \textit{if } (n,i)=1\\ 0 & \textit{if } i|n.\end{cases}$$

First we prove the following

Lemma 1. *Let a and b be non-negative integers and F be an arithmetical function, for which the condition*

$$(1) \qquad F(n^2+m^2+a+b)=F(n^2+a)+F(m^2+b)$$

is satisfied for all $n, m \in I\!N$. *For each* $j \in I\!N$ *let* $S_j := F(n^2 + a)$. *Then*

$$S_{n+12} = S_{n+9} + S_{n+8} + S_{n+7} - S_{n+5} - S_{n+4} - S_{n+3} + S_n \tag{2}$$

holds for all $n \in I\!N$ *and*

$$\begin{cases} S_7 &= 2S_5 - S_1, \\ S_8 &= 2S_5 + S_4 - 2S_1, \\ S_9 &= S_6 + 2S_5 - S_2 - S_1, \\ S_{10} &= S_6 + 3S_5 - S_3 - 2S_1, \\ S_{11} &= S_6 + 4S_5 - S_3 - S_2 - 2S_1, \\ S_{12} &= S_6 + 4S_5 + S_4 - S_2 - 4S_1. \end{cases} \tag{3}$$

Proof. Let F be an arithmetical function with the condition (1). From (1) we have

$$F(n^2 + a) + F(m^2 + b) = F(m^2 + a) + F(n^2 + b)$$

for all $n, m \in I\!N$, and so

$$F(n^2 + b) - F(n^2 + a) = F(1 + b) - F(1 + a) := D \quad \text{for all} \quad n \in I\!N.$$

Thus, we get from (1) that

$$F(n^2 + m^2 + a + b) = F(n^2 + a) + F(m^2 + a) + D \tag{4}$$

holds for all $n, m \in I\!N$. In the following for each $j \in I\!N$ let $S_j := F(j^2 + a)$. It follows from (4) that if the positive integers k, l, u and v satisfy the condition

$$k^2 + l^2 = u^2 + v^2,$$

then

$$F(k^2 + l^2 + a + b) = F(k^2 + a) + F(l^2 + a) + D =$$

$$= F(u^2 + v^2 + a + b) = F(u^2 + a) + F(v^2 + a) + D,$$

which shows that

$$k^2 + l^2 = u^2 + v^2 \quad \text{implies} \quad S_k + S_l = S_u + S_v. \tag{5}$$

Since

$$(2n + 1)^2 + (n - 2)^2 = (2n - 1)^2 + (n + 2)^2$$

and

$$(2n + 1)^2 + (n - 7)^2 = (2n - 5)^2 + (n + 5)^2$$

hold for all $n \in I\!N$, we get from (5) that

$$S_{2n+1} + S_{n-2} = S_{2n-1} + S_{n+2} \tag{6}$$

and

$$S_{2n+1} + S_{n-7} = S_{2n-5} + S_{n+5}.$$

These imply that

$$S_{n+5} - S_{n+2} + S_{n-2} - S_{n-7} = S_{2n-1} - S_{2n-5} =$$

$$= S_{n+1} - S_{n-3} + S_{2n-3} - S_{2n-5} = S_{n+1} - S_{n-3} + S_n - S_{n-4},$$

which proves (2).

Now we prove (3). Indeed, by using (6), we have

$$S_7 = S_{2.3+1} = 2S_5 - S_1,$$

$$S_9 = S_{2.4+1} = S_7 + S_6 - S_2 = S_6 + 2S_5 - S_2 - S_1$$

and

$$S_{11} = S_{2.5+1} = S_9 + S_7 - S_3 = S_6 + 4S_5 - S_3 - S_2 - 2S_1.$$

Finally, by using (5) and the facts

$$8^2 + 1^2 = 7^2 + 4^2, \quad 10^2 + 5^2 = 11^2 + 2^2 \quad \text{and} \quad 12^2 + 1^2 = 9^2 + 8^2,$$

we have

$$S_8 = S_7 + S_4 - S_1 = 2S_5 + S_4 - 2S_1,$$

$$S_{10} = S_{11} + S_2 - S_5 = S_6 + 3S_5 - S_3 - 2S_1$$

and

$$S_{12} = S_9 + S_8 - S_1 = S_6 + 4S_5 + S_4 - S_2 - 4S_1,$$

which completes the proof (3). Lemma 1 is proved.

Remark. It follows easily from (2) and (3) that

$$F(j^2+a) = j^2+a \quad \text{for all} \quad j \in I\!N \quad \text{if} \quad F(j^2+a) = j^2+a \quad \text{for} \quad j = 1, \cdots, 6.$$

Lemma 2. *Assume that $k \in I\!N$ and $f \in \mathcal{M}$ satisfy the condition*

$$f\left(n^2 + m^2 + k\right) = f(n^2) + f(m^2 + k) \quad \textit{for all} \quad n, m \in I\!N.$$

If $f(4)f(9) \neq 0$, *then*

$$f(\mu + k) = \mu + k \quad \textit{for all} \quad \mu \in \mathcal{B}$$

and

$$f(\nu) = \nu \quad \textit{for all} \quad \nu \in \mathcal{S},$$

where $\mathcal{B} \subset I\!N$ *is the set of non-negative integers which can be represented as a sum of two squares of integers and* $\mathcal{S}$ *is the set of all squares of positive integers.*

If $f(9)f(4) = 0$, *then statements* $(i) - (iii)$ *in the theorem are true.*

Proof. Assume that $k \in I\!N$ and $f \in \mathcal{M}$ satisfy the condition

$$f\left(n^2 + m^2 + k\right) = f(n^2) + f(m^2 + k) \quad \text{for all} \quad n, m \in I\!N. \tag{7}$$

We shall use the notations and the result of Lemma 1 with $a = 0$ and $b = k$. Let $S_j := f(j^2)$. It follows from (7) that

$$f(n^2 + m^2 + k) = f(n^2) + f(m^2) + D \quad \text{and} \quad f(n^2 + k) = f(n^2) + D \tag{8}$$

hold for all $n, m \in I\!N$, where $D = f(k+1) - f(1) = f(k+1) - 1$.

First we note from (8) that if $t^2 = u^2 + v^2$, then

$$f(t^2 + k) = f(t^2) + D \quad \text{and} \quad f(t^2 + k) = f(u^2) + f(v^2) + D.$$

Thus, we have

$$f(t^2) = f(u^2) + f(v^2) \quad \text{if} \quad t^2 = u^2 + v^2. \tag{9}$$

For each odd prime p let $t = (p^2 + 1)/2$ and $u = (p^2 - 1)/2$. Then

$$t^2 = u^2 + p^2 \quad \text{and} \quad (p, tu) = 1,$$

and so from (9) we have

$$f\left(t^2\right) = f\left(u^2\right) + f\left(p^2\right).$$

On the other hand, for each non-negative integer α

$$\left[p^\alpha t\right]^2 = \left[p^\alpha u\right]^2 + \left[p^{\alpha+1}\right]^2 \quad \text{and} \quad f\left[p^{2\alpha} t^2\right] = f\left[p^{2\alpha} u^2\right] + f\left[p^{2(\alpha+1)}\right],$$

which using the multiplicatity of f imply

$$f\left[p^{2(\alpha+1)}\right] = f(p^2) f\left(p^{2\alpha}\right) \quad \text{and} \quad f\left[p^{2\alpha}\right] = \left(f(p^2)\right)^\alpha,$$

consequently

$$S_{p^\alpha} = (S_p)^\alpha \quad \text{for all} \quad \alpha \in I\!N. \tag{10}$$

Similarly, by using $5^2 = 4^2 + 3^2$ and $17^2 = 15^2 + 8^2$, we also have

$$f\left(2^{2\alpha+4}\right) = f(2^4)f(2^{2\alpha}) \tag{11}$$

and

$$f\left(2^{2\alpha+6}\right) = f(2^6)f(2^{2\alpha}). \tag{12}$$

Since $5^2 = 4^2 + 3^2$ and $6 = 2 \cdot 3$, we have

$$\begin{cases} S_5 &= S_4 + S_3, \\ S_6 &= S_2 S_3. \end{cases} \tag{13}$$

Therefore, we infer from (3), (10)-(11) and (13) that

$$\begin{cases} S_8 &= S_2 S_4 = 3S_4 + 2S_3 - 2, \\ S_9 &= (S_3)^2 = S_2 S_3 + 2S_4 + 2S_3 - S_2 - 1, \\ S_{12} &= S_3 S_4 = S_2 S_3 + 5S_4 + 4S_3 - S_2 - 4. \end{cases} \tag{14}$$

On the other hand, we get from (8) and by using the multiplicatity of f that

$$\left[f\left(n^2\right) + D\right]\left[f\left(n^2\right) + 1 + D\right] = f\left(n^2 + k\right) f\left(n^2 + 1 + k\right) =$$

$$= f\left[\left(n^2 + k\right)^2 + n^2 + k\right] = f\left[\left(n^2 + k\right)^2\right] + f\left(n^2\right) + D,$$

which gives

$$f\left[\left(n^2 + k\right)^2\right] = \left[f\left(n^2\right) + D\right]^2$$

and so

$$S_{k+n^2} = (S_n + D)^2. \tag{15}$$

Case I: $S_3 = 0$.

We assume that $S_3 = 0$. In this case, from (14) we have

$$S_9 = 2S_4 - S_2 - 1 = 0 \quad \text{and} \quad S_{12} = 5S_4 - S_2 - 4 = 0,$$

consequently $S_2 = S_4 = 1$. Hence we infer from (14)-(15) that $S_3 = S_6 = S_{12} = 0$ and $S_1 = S_2 = S_4 = S_5 = S_7 = S_8 = S_{10} = S_{11} = 1$, which with (2) imply that the sequence $\{S_n\}_{n=1}^{\infty}$ is periodic, namely

$$S_n = \chi_3(n) = \begin{cases} 1, & \text{if } (n,3) = 1, \\ 0 & \text{if } 3|n. \end{cases} \tag{16}$$

Applying (15)-(16) with $n = 1$ and $n = 3$, we have

$$S_{k+3^2} = S_k = f(k^2) = D^2 \quad \text{and} \quad S_{k+1} = f\left[(k+1)^2\right] = (D+1)^2. \tag{17}$$

Thus, we infer from (16) and (17) that $k \not\equiv 1 \pmod 3$, consequently $(k+ +5, k+8) = (k-1, 3) = 1$. On the other hand, we have

$$f(k+5) = f(1)+f(2^2)+D = 2+D \quad \text{and} \quad f(k+8) = f(2^2)+f(2^2)+D = 2+D,$$

which imply that

$$(D+2)^2 = f(k+5)f(k+8) = f\left[(k+6)^2 + 2^2 + k\right] =$$

$$= f\left[(k+6)^2\right] + f(2^2) + D = S_k + D + 1 = D^2 + D + 1.$$

This implies $D = -1$. Therefore, we have $S_{k+1} = f\{(k+1)^2\} = (D+1)^2 = 0$, and (16) gives $k \equiv 2 \pmod 3$. So, the assertion (i) of Lemma 2 is proved.

Case II: $S_3 \neq 0$.

In the following we assume that $S_3 \neq 0$. Since $14^2 + 2^2 = 10^2 + 10^2$, $18^2 + 4^2 = 14^2 + 12^2$, we get from (2) and (13)-(14) that

$$S_{10} = S_2 S_3 + 3S_4 + 2S_3 - 2$$

and

$$S_{18} = S_{14} + S_{12} - S_4 = S_{12} + 2S_{10} - S_4 - S_2 =$$

$$= S_{12} + 2S_2S_3 + 5S_4 + 4S_3 - S_2 - 4 = 2S_{12} + S_2S_3.$$

Thus, by using the facts $S_{18} = S_2S_3^2$ and $S_3 \neq 0$, we infer

$$S_2S_3 = 2S_4 + S_2,$$

which with (14) implies that

$$S_4 = \left(\frac{S_3 - 1}{2}\right)^2 \quad \text{and} \quad S_3\left(\frac{S_3 - 1}{2}\right)^2 = 7\left(\frac{S_3 - 1}{2}\right)^2 + 8\left(\frac{S_3 - 1}{2}\right).$$

It is clear from the last relation that if $x = \left(\frac{S_3-1}{2}\right)$, then

$$(2x+1)x^2 - 7x^2 - 8x = 2x(x+1)(x-4) = 0.$$

Case II.1: $x = 0, S_3 = 1, S_4 = 0.$

In this case, it follows from (2) and (13)-(14) that the sequence $\{S_n\}_{n=1}^{\infty}$ is periodic, namely $S_n = S_{(n,4)}$. Therefore

$$S_{k+4} = S_k,\ S_{k+4^2} = S_k \quad \text{and} \quad S_{k+3^2} = S_{k+1}.$$

Thus, from (15) we have

$$\text{(18)} \qquad \begin{cases} S_k &= (S_2+D)^2 = D^2, \\ S_{k+1} &= (D+1)^2 \ . \end{cases}$$

If $k \equiv 1 \pmod 4$, then (18) gives $S_2 = 0$ and $D = -1$, in which case we have

$$\text{(19)} \qquad f\left[\left(\frac{k+3}{2}\right)^2 + 2^2 + k\right] = f\left[\left(\frac{k+5}{2}\right)^2\right],$$

that is $0 + 0 - 1 = 1$, a contradiction.

If $k \equiv 2 \pmod 4$, then (18) implies that

$$S_2 = D^2, \quad 1 = (D+1)^2 \quad \text{and} \quad S_2 = (S_2+D)^2.$$

The last relations imply $D = S_2 = 0$ or $D = -2, \quad S_2 = 4$. Since

$$\left(\frac{k}{2}\right)^2 + 1 + k = \left(\frac{k}{2}+1\right)^2,$$

which is impossible in the case $D = S_2 = 0$. Assume that $D = -2, \quad S_2 = 4$. Then

$$f(k+5) = f(1) + f(2^2) + D = 1 + 4 - 2 = 3$$

and

$$f(k+8) = f(2^2) + f(2^2) + D = 4 + 4 - 2 = 6,$$

which is a contradiction in the case $(k-1, 3) = 1$, because

$$18 = f(k+5)f(k+8) = f\left[(k+6)^2 + 2^2 + k\right] = f\left[(k+6)^2\right] + f(2^2) + D =$$

$$= S_{k+6} + S_2 + D = S_4 + S_2 + D = 0 + 4 - 2 = 2.$$

On the other hand, if $k \equiv 1 \pmod 3$, then

$$(k+37, k+40) = 1 \quad \text{and} \quad (k+37)(k+40) = (k+38)^2 + 6^2 + k,$$

which also is a contradiction, because

$$f(k{+}37) = f(6^2{+}1^2{+}k) = 4{+}1{-}2 = 3, \;\; f(k{+}40) = f(6^2{+}2^2{+}k) = 4{+}4{-}2 = 6$$

and

$$f\left[(k+38)^2 + 6^2 + k\right] = 0 + 4 - 2 = 2.$$

Finally, it remains to consider the cases $k \equiv 0 \pmod 4$ and $k \equiv 3 \pmod 4$. First, let $k \equiv 0 \pmod 4$. In this case (18) implies that $D = S_2 = 0$. Thus, we have proved that if $k \equiv 0 \pmod 4$, then $S_2 = 0$ and

$$f(n^2{+}m^2{+}k) = f(n^2){+}f(m^2) \;\text{ and }\; f(n^2) = \chi_2(n) \;\text{ for all }\; n \in I\!N, \;\; m \in I\!N_0.$$

Now let $k \equiv 3 \pmod 4$. In this case (18) implies $(D, S_2) = (-1, 0)$ or $(D, S_2) = (-1, 2)$. Assume that $(D, S_2) = (-1, 2)$, then it follows from (19) that

$$1 + 2 - 1 = f\left[\left(\frac{k+3}{2}\right)^2\right] + f(2^2) + D = f\left[\left(\frac{k+5}{2}\right)^2\right] = 0,$$

which is impossible. So

$$f(n^2 + m^2 + k) = f(n^2) + f(m^2) - 1 \;\text{ and }\; f(n^2) = \chi_2(n)$$

hold for all $n \in I\!N$, $m \in I\!N_0$. Thus, (ii) and (iii) are proved.

Case II.2: $x = -1, S_3 = -1, S_4 = 1$ and $S_5 = 0$.

In this case, it follows from $S_2 S_3 = 2S_4 + S_2$ that $S_2 = -1$. So, it follows from (2) that the sequence $\{S_n\}_{n=1}^{\infty}$ is also periodic, namely

$$S_n = S_m \;\text{ if }\; n \equiv m \pmod 5 \;\text{ and }\; S_j \in \{1, -1, -1, 1, 0\}$$

for all $j \in I\!N$. We infer from (15) that

$$S_k = D^2 \;\text{ since }\; S_{k+5^2} = S_k \;\text{ and }\; S_{k+5^2} = (S_5 + D)^2 = D^2,$$

consequently $S_k = D^2 \in \{0, 1\}$.

Assume first that $S_k = D = 0$. In this case, we have $k \equiv 0 \pmod 5$ and so $(k+8, k+13) = (k-2, 5) = 1$. Since

$$f(k+8) = f\left(2^2 + 2^2 + k\right) = S_2 + S_2 + D = -1 - 1 + 0 = -2$$

and

$$f(k+13) = f(2^2 + 3^2 + k) = S_2 + S_3 + D = -1 - 1 + 0 = -2,$$

we have

$$4 = f(k+8)f(k+13) = f\left((k+10)^2 + 2^2 + k\right) = S_{k+10} + S_2 + D = S_k - 1 = -1,$$

which is impossible.

Assume now that $S_k = D^2 = 1$. We get from (15) that

$$S_{k+1} = (S_1 + D)^2 = (D+1)^2 \in \{0, 1\},$$

and

$$S_{k+4} = (S_4 + D)^2 = (D-1)^2 \in \{0, 1\},$$

consequently $D \neq \pm 1$. This is a contradiction, because $D^2 = 1$.

Case II.3: $x = 4, S_3 = 9, S_4 = 16, S_5 = 25 \quad \text{and} \quad S_6 = 36.$

It is clear from (2)-(3) that in this case $S_j = f(j^2) = j^2$ for all $j \in I\!N$. It is remains to prove that $D = k$. Indeed, if k is odd, then

$$\left(\frac{k-1}{2}\right)^2 + k = \left(\frac{k+1}{2}\right)^2$$

with (8) implies

$$\left(\frac{k+1}{2}\right)^2 = f\left[\left(\frac{k+1}{2}\right)^2\right] = \left(\frac{k-1}{2}\right)^2 + D,$$

that is $D = k$. If k is even, then by

$$\left(\frac{k}{2}\right)^2 + 1 + k = \left(\frac{k}{2} + 1\right)^2,$$

we have

$$\left(\frac{k}{2} + 1\right)^2 = f\left[\left(\frac{k}{2}\right)^2 + 1 + k\right] = \left(\frac{k}{2}\right)^2 + 1 + D,$$

which implies that $D = k$. Finally, from $f(k+1) = f(1) + D = k + 1$ and

$$(k+1)f(k) = f(k+1)f(k) = f\left[k(k+1)\right] = f\left[k^2 + k\right] = f(k^2) + D = k^2 + k,$$

we have $f(k) = k$. This completes the proof of Lemma 2.

Now we prove our theorem. We will complete the proof of our theorem by showing the following

Lemma 3. *Let $\mathcal{B} \subset I\!N$ be the set of non-negative integers which can be represented as a sum of two squares of integers. If $k \in I\!N$ and $f \in \mathcal{M}$ satisfy the condition*

$$f\left(\mu + k\right) = \mu + k \quad \text{for all} \quad \mu \in \mathcal{B}, \tag{19}$$

then $f\left(n\right) = n$ for all $n \in I\!N, (n, 2k) = 1$.

Proof. Assume that $n \in I\!N$ with the condition $(n, 2k) = 1$. It follows from Theorem 1 of [5] that there are positive integers μ and ν such that

$$n(\mu + k) = \nu + k \quad \text{and} \quad (n, \mu + k) = 1.$$

Thus, from (19) we infer that

$$n(\mu + k) = \nu + k = f\left(\nu + k\right) = f\left[n(\mu + k)\right] =$$

$$= f(n)f\left(\mu + k\right) = f(n)\left(\mu + k\right),$$

which proves that

$$f(n) = n \quad \text{for all} \quad n \in I\!N,\ (n, 2k) = 1.$$

The proof of our Theorem is completed.

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