ON CERTAIN ARITHMETIC FUNCTIONS INVOLVING EXPONENTIAL DIVISORS

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Dedicated to Professor Imre Kátai on the ocassion of his 65th birthday

1. Introduction

Let $n > 1$ be an integer of canonical form $n = \prod_{i=1}^{r} p_i^{a_i}$. The integer d is called an *exponential divisor* of n if $d = \prod_{i=1}^{r} p_i^{c_i}$, where $c_i | a_i$ for every $1 \le i \le r$, notation: $d|_e n$. By convention $1|_c 1$. This notion was introduced by M.V. Subbarao [9]. Note that 1 is not an exponential divisor of $n > 1$, the smallest exponential divisor of $n > 1$ is its squarefree kernel $\kappa(n) = \prod_{i=1}^{r} p_i$.

Let $\tau^{(e)}(n) = \sum_{d|_e n} 1$ and $\sigma^{(e)}(n) = \sum_{d|_e n} d$ denote the number and the sum of exponential divisors of n, respectively. The integer $n = \prod_{i=1}^{r} p_i^{a_i}$ is called *exponentially squarefree* if all the exponents a_i $(1 \le i \le r)$ are squarefree. Let $q^{(e)}$ denote the characteristic function of exponentially squarefree integers. Properties of these functions were investigated by several authors, see [1], [2], [3], [5], [8], [9], [12].

Two integers $n, m > 1$ have common exponential divisors iff they have the same prime factors and in this case, i.e. for $n = \prod_{i=1}^{r} p_i^{a_i}$, $m = \prod_{i=1}^{r} p_i^{b_i}$, $a_i,\ b_i \ge$

The research was supported by the fund of Applied Number Theory Research Group of the Hungarian Academy of Sciences.

≥ 1 $(1 \leq i \leq r)$, the *greatest common exponential divisor* of n and m is

$$(n,m)_{(e)} = \prod_{i=1}^{r} p_i^{(a_i,b_i)}.$$

Here $(1,1)_{(e)} = 1$ by convention and $(1,m)_{(e)}$ does not exist for $m > 1$.

The integers $n, m > 1$ are called *exponentially coprime*, if they have the same prime factors and $(a_i, b_i) = 1$ for every $1 \leq i \leq r$, with the notation of above. In this case $(n,m)_{(e)} = \kappa(n) = \kappa(m)$. 1 and 1 are considered to be exponentially coprime. 1 and $m > 1$ are not exponentially coprime.

For $n = \prod_{i=1}^{r} p_i^{a_i} > 1$, $a_i \geq 1$ $(1 \leq i \leq r)$, denote by $\phi^{(e)}(n)$ the number of integers $\prod_{i=1}^{r} p_i^{c_i}$ such that $1 \leq c_i \leq a_i$ and $(c_i, a_i) = 1$ for $1 \leq i \leq r$, and let $\phi^{(e)}(1) = 1$. Thus $\phi^{(e)}(n)$ counts the number of divisors d of n such that d and n are exponentially coprime.

It is immediately, that $\phi^{(e)}$ is a prime independent multiplicative function and for $n > 1$

$$\phi^{(e)}(n) = \prod_{i=1}^{r} \phi(a_i),$$

where ϕ is the Euler-function. Exponentially coprime integers and function $\phi^{(e)}$ were introduced by J. Sándor [6]. He showed that

$$\limsup_{n\to\infty} \frac{\log \phi^{(e)}(n) \log\log n}{\log n} = \frac{\log 4}{5}.$$

We consider the functions $\tilde{\sigma}$ and $\tilde{P}$ defined as follows. Let $\tilde{\sigma}(n)$ be the sum of those divisors d of n such that d and n are exponentially coprime. Function $\tilde{\sigma}$ is multiplicative and for every prime power p^a,

$$\tilde{\sigma}(p^a) = \sum_{\substack{1 \leq c \leq a \\ (c,a)=1}} p^c.$$

Here $\tilde{\sigma}(p) = \tilde{\sigma}(p^2) = p$, $\tilde{\sigma}(p^3) = p + p^2$, $\tilde{\sigma}(p^4) = p + p^3$, etc.

Furthermore let $\tilde{P}(n)$ be given by

$$\tilde{P}(n) = \sum_{\substack{1 \leq j \leq n \\ \kappa(j)=\kappa(n)}} (j,n)_{(c)},$$

representing an analogue of Pillai's function $P(n) = \sum_{j=1}^{n} (j, n)$.

Function $\tilde{P}$ is also multiplicative and for every prime power p^a,

$$\tilde{P}(p^a) = \sum_{1 \le c \le a} p^{(c,a)} = \sum_{d|a} p^d \phi(a/d),$$

here $\tilde{P}(p) = p,\ \tilde{P}(p^2) = p + p^2,\ \tilde{P}(p^3) = 2p + p^3,\ \tilde{P}(p^4) = 2p + p^2 + p^4$, etc.

We call an integer $n = \prod_{i=1}^{r} p_i^{a_i}$ *exponentially k-free* if all the exponents a_i $(1 \le i \le r)$ are k-free, i.e. are not divisible by the k-th power of any prime $(k \ge 2)$. Let $q_k^{(e)}$ denote the characteristic function of exponentially k-free integers.

The aim of this paper is to investigate the functions $\phi^{(e)}(n),\ \tilde{\sigma}(n),\ \tilde{P}(n)$ and $q_k^{(e)}(n)$. The estimate given for the sum $\sum_{n \le x} q_k^{(e)}(n)$ generalizes the result of J. Wu [12] concerning exponentially squarefree integers. Our main results are formulated in Section 2, their proofs are given in Section 3.

Our estimates for $\sum_{n \le x} (\tilde{\sigma}(n))^u$ and $\sum_{n \le x} q_k^{(e)}(n)$ are consequences of a general result due to V. Sita Ramaiah and D. Suryanarayana [7], the proof of which uses the estimate of A. Walfisz [11] concerning k-free integers and is simpler than the proof given by J. Wu [12].

A. Smati and J. Wu [8] deduced some interesting analogues of known results on the divisor function $\tau(n)$ in case of $\tau^{(e)}(n)$. They remarked that their results can be stated also for certain other prime independent multiplicative functions f if $f(n)$ depends only on the squarefull kernel of n.

We point out two such results in case of $\phi^{(e)}(n)$. Note that, since $\phi(1) = \phi(2) = 1$, $\phi^{(e)}(n)$ depends only on the cubfull kernel of n. These results are contained in Section 4. Here some open problems are also stated.

2. Main results

Regarding the average orders of the functions $\phi^{(e)}(n),\ \tilde{\sigma}(n)$ and $\tilde{P}(n)$ we prove the following results.

Theorem 1.

$$\sum_{n\le x} \phi^{(e)}(n) = C_1 x + C_2 x^{1/3} + O\left(x^{1/5+\varepsilon}\right),$$

for every $\varepsilon > 0$, *where* C_1, C_2 *are constants given by*

$$C_1 = \prod_p \left(1 + \sum_{a=3}^{\infty} \frac{\phi(a) - \phi(a-1)}{p^a}\right),$$

$$C_2 = \zeta(1/3)\left(1 + \sum_{a=5}^{\infty} \frac{\phi(a) - \phi(a-1) - \phi(a-3) - \phi(a-4)}{p^{a/3}}\right).$$

Theorem 2. *Let* $u > 1/3$ *be fixed real number. Then*

$$\sum_{n\le x} (\tilde{\sigma}(n))^u = C_3 x^{u+1} + O\left(x^{u+1/2}\delta(x)\right),$$

where C_3 *is given by*

$$C_3 = \frac{1}{u+1}\prod_p \left(1 + \sum_{a=2}^{\infty} \frac{(\tilde{\sigma}(p^a))^u - p^u(\tilde{\sigma}(p^{a-1}))^u}{p^{a(u+1)}}\right)$$

and

$$\delta(x) = \exp\left(-A(\log x)^{3/5}(\log\log x)^{-1/5}\right),$$

A *being a positive constant.*

Theroem 3.

$$\sum_{n\le x} \tilde{P}(n) = C_4 x^2 + O\left(x(\log x)^{5/3}\right),$$

where the constant C_4 *is given by*

$$C_4 = \frac{1}{2}\prod_p \left(1 + \sum_{a=2}^{\infty} \frac{\tilde{P}(p^a) - p\tilde{P}(p^{a-1})}{p^{2a}}\right).$$

Concerning the maximal order of the function $\tilde{P}(n)$ we have

Theorem 4.

$$\limsup_{n\to\infty} \frac{\tilde{P}(n)}{n\log\log n} = \frac{6}{\pi^2}e^{\gamma},$$

where γ is Euler's constant.

Theorem 5. *If $k \geq 2$ is a fixed integer, then*

$$\sum_{n\leq x} q_k^{(e)}(n) = D_k x + O\left(x^{1/2^k}\delta(x)\right),$$

where

$$D_k = \prod_p \left(1 + \sum_{a=2^k}^{\infty} \frac{q_k(a) - q_k(a-1)}{p^a}\right),$$

$q_k(n)$ denoting the characteristic function of k-free integers.

In the special case $k = 2$ case this formula is due to J. Wu [12], improving an earlier result of M.V. Subbarao [9].

3. Proofs

The proof of Theorem 1 is based on the following lemma.

Lemma 1. *The Dirichlet series of $\phi^{(e)}$ is absolutely convergent for $Re\, s > 1$ and it is of form*

$$\sum_{n=1}^{\infty} \frac{\phi^{(e)}(n)}{n^s} = \zeta(s)\zeta(3s)V(s),$$

where the Dirichlet series $V(s) = \sum_{n=1}^{\infty} \frac{v(n)}{n^s}$ is absolutely convergent for $Re\, s > 1/5$.

Proof of Lemma 1. Let $\mu_3(n) = \mu(m)$ or 0, according as $n = m^3$ or not, where μ is the Möbius function, and let $f = \mu_3 * \mu$ in terms of the Dirichlet convolution. Then we can formally obtain the desired expression by taking $v = \phi^{(e)} * f$. Both f and v are multiplicative and easy computations show that $f(p) = f(p^3) = -1$, $f(p^4) = 1$, $f(p^2) = f(p^a) = 0$ for each $a \geq 5$, and $v(p^a) = 0$ for $1 \leq a \leq 4$, $v(p^a) = \phi(a) - \phi(a-1) - \phi(a-3) - \phi(a-4)$ for $a \geq 5$.

Since $|v(p^a)| < 4a$ for $a \geq 5$, we obtain that $V(s)$ is absolutely convergent for $Re\, s > 1/5$.

Proof of Theorem 1. Lemma 1 shows that $\phi^{(e)} = v * \tau(1,3,\cdot)$, where $\tau(1,3,n) = \sum\limits_{ab^3=n} 1$ for which

$$\sum_{n\le x} \tau(1,3,n) = \zeta(3)x + \zeta(1/3)x^{1/3} + O\left(x^{1/5}\right),$$

cf. [4], pp. 196-199. Therefore,

$$\sum_{n\le x} \phi^{(e)}(n) = \sum_{d\le x} v(d) \sum_{e\le x/d} \tau(1,3,e) =$$

$$= \zeta(3)x \sum_{d\le x} \frac{v(d)}{d} + \zeta(1/3)x^{1/3} \sum_{d\le x} \frac{v(d)}{d^{1/3}} + O\left(x^{1/5+\varepsilon} \sum_{d\le x} \frac{|v(d)|}{d^{1/5+\varepsilon}}\right),$$

and obtain the desired by usual estimates.

For the proof of Theorem 2 we use the following general result due to V. Sita Ramaiah and D. Suryanarayana [7], Theorem 1.

Lemma 2. *Let $k \ge 2$ be a fixed integer, $\beta > (k+1)^{-1}$ be a fixed real number and g be a multiplicative arithmetic function such that $|g(n)| \le 1$ for all $n \ge 1$. Suppose that either*

(i) $|g(p^j) - 1| \le p^{-1}$ for $1 \le j \le k-1$, $g(p^k) = 0$ for all primes p, or
(ii) $g(p^j) = 1$ for $1 \le j \le k-1$, $g(p^k) = p^{-\beta}$ for all primes p.

Then

$$\sum_{n\le x} g(n) = x \sum_{n=1}^{\infty} \frac{(g*u)(n)}{n} + O\left(x^{1/k}\delta(x)\right).$$

Proof of Theorem 2. This is a direct consequence of Lemma 2 of above. Take $g(n) = (\tilde{\sigma}(n)/n)^u$. Here $g(p) = 1$, $g(p^2) = p^{-u}$, $g(p^a) \le p^{-au}(p + p^2 + + \ldots + p^{a-1})^u < (p-1)^{-u} \le 1$ for every $a \ge 3$, hence $0 < g(n) \le 1$ for all $n \ge 1$. Choosing $k = 2$, $\beta = u$, we obtain the given result by partial summation.

Lemma 3. *The Dirichlet series of $\tilde{P}(n)$ is absolutely convergent for $Re\, s > 2$ and it is of form*

$$\sum_{n=1}^{\infty} \frac{\tilde{P}(n)}{n^s} = \frac{\zeta(s-1)\zeta(2s-1)}{\zeta(3s-2)} W(s),$$

where the Dirichlet series $W(s) = \sum\limits_{n=1}^{\infty} \frac{w(n)}{n^s}$ is absolutely convergent for $Re\, s > > 3/4$.

Proof of Lemma 3.

$$\sum_{n=1}^{\infty} \frac{\tilde{P}(n)}{n^s} = \prod_p \left(1 + \sum_{a=1}^{\infty} \sum_{d|a} \frac{p^d \phi(a/d)}{p^{as}}\right) =$$

$$= \prod_p \left(1 + \sum_{j=1}^{\infty} \phi(j) \sum_{d=1}^{\infty} \frac{1}{p^d(js-1)}\right) = \prod_p \left(1 + \sum_{j=1}^{\infty} \frac{\phi(j)}{p^{js-1}-1}\right) =$$

$$= \frac{\zeta(s-1)\zeta(2s-1)}{\zeta(3s-2)} W(s),$$

where

$$W(s) := \prod_p \left(1 + \frac{(p^{s-1}-1)(p^{2s-1}-1)}{p^{3s-2}-1} \sum_{j=3}^{\infty} \frac{\phi(j)}{p^{js-1}-1}\right),$$

which is absolutely convergent for $Re\, s > 3/4$.

Proof of Theorem 3. By Lemma 3, $\tilde{P} = h * w$, where

$$h(n) = \sum_{ab^2c^3=n} abc^2\mu(c),$$

and obtain the desired result, exactly like in proof of Theorem 2 of [5], using the estimate

$$\sum_{mn^2 \le x} mn = \frac{1}{2}\zeta(3)x^2 + O(x(\log x)^{2/3})$$

due to Y.-F.S. Pétermann and J. Wu [5], Theorem 1.

Theorem 4 is a direct consequence of the following general result of L. Tóth and E. Wirsing [10], Corollary 1.

Lemma 4. *Let f be a nonnegative real-valued multiplicative function. Suppose that for all primes p we have $\rho(p) := \sup_{\nu \ge 0} f(p^\nu) \le (1-1/p)^{-1}$ and that for all primes p there is an exponent $e_p = p^{o(1)}$ such that $f(p^{e_p}) \ge 1 + 1/p$. Then*

$$\limsup_{n \to \infty} \frac{f(n)}{\log \log n} = e^\gamma \prod_p \left(1 - \frac{1}{p}\right) \rho(p).$$

Proof of Theorem 4. Apply Lemma 4 for $f(n) = \tilde{P}(n)/n$, where $f(p^a) \le$ $\le (p + p^2 + \ldots + p^a)p^{-a} < (1-1/p)^{-1}$ for every $a \ge 1$ and $f(p^2) = 1 + 1/p$, hence we can choose $e_p = 2$ for all p. Moreover, $\rho(p) = 1 + 1/p$ for all p and obtain the desired result.

Proof of Theorem 5. This follows from Lemma 2 by taking 2^k instead of k, where $q_k^{(e)}(p) = q_k^{(e)}(p^2) = \ldots = q_k^{(e)}(p^{2^k-1}) = 1,\ q_k^{(e)}(p^{2^k}) = 0$.

4. Further results and problems

The next result in an analogue of the exponential divisor problem of Titchmarsh, see Theorem 1 of [8]. The proof is the same using that $\phi^{(e)}(n)$ is a prime independent multiplicative function depending only on the squarefull (cubfull) kernel of n and that $\phi^{(e)}(p^a) = \phi(a) \le a$ for every $a \ge 1$.

Theorem 6. *For every fixed $B > 0$*

$$\sum_{p \le x} \phi^{(e)}(p-1) = C_5 \ \mathrm{li}\, x + O(x/(\log x)^B),$$

where

$$C_5 = \prod_p \left(1 + \sum_{k=3}^{\infty} \frac{\phi(k)-1}{p^k}\right).$$

Let $\omega(n)$ and $\Omega(n)$ denote, as usual, the number of prime factors of n and the number of prime power factors of n, respectively.

Theorem 7. *A maximal order of $\Omega(\phi^{(e)}(n))$ is $2(\log n)/5 \log\log n$.*

This can be obtained by the same arguments as those given in the proof of Theorem 3 (i) of [8]. Here the upper bound is attained for $n_k = (p_1 \ldots p_k)^5$, where p_k is the k-th prime.

Problem 1. *Determine a maximal order of $\omega(\phi^{(e)}(n))$.*

Since $\tilde{\sigma}(n) \le n$ for all $n \ge 1$ and $\tilde{\sigma}(p) = p$ for all primes p, it is clear that a maximal order of $\tilde{\sigma}(n)$ is n.

Problem 2. *Determine a minimal order of $\tilde{\sigma}(n)$.*

J. Sándor [6] considered in fact the function $\varphi_e(n)$ defined as the number of integers $1 < a < n$ for which a and n are exponentially coprime ($n > 1$) and $\varphi_e(1) = 1$. Although $\varphi_e(p^a) = \phi^{(e)}(p^a) = \phi(a)$ for any prime power p^a, functions φ_e and $\phi^{(e)}$ are not the same. Take for example $n = 2^3 \cdot 3^2$, then numbers $a < n$ exponentially coprime to n are $a = 2 \cdot 3,\ 2^2 \cdot 3,\ 2^4 \cdot 3$, hence $\varphi_e(2^3 \cdot 3^2) = 3 \neq 2 \cdot 1 = \phi(3)\phi(2) = \varphi_e(2^3) \cdot \varphi_e(3^2)$.

Therefore, φ_e is not multiplicative and $\varphi_e(n) \geq \phi^{(e)}(n)$ for every $n \geq 1$.

Problem 3. *What can be said on the order of the function $\varphi_e(n)$?*

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(Received September 18, 2004)

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