H^p MULTIPLIERS ON THE DYADIC FIELD

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Dedicated to Professor I. Kátai on the occasion of his 65th birthday

Abstract. In this paper we consider a classical multiplier condition, the Hörmander-Mihlin condition, originally introduced for the trigonometric case. It implies that the multiplier operator is bounded on $L^p$, $1 < p < \infty$. Here we study the corresponding problem with respect to the Walsh transform and the noncompact dyadic Hardy spaces $H^p[0, \infty)$, $0 < p < 1$. We also show that our result is sharp. We note that a similar program was carried out for the trigonometric case and the classical Hardy spaces, and for the Walsh system and the dyadic Hardy spaces on $[0, 1]$ in our previous papers [1] and [2].

1. Introduction

Set $\mathbb{R}^+ = [0, \infty)$. The binary expansion of $x \in \mathbb{R}^+$ is $x = \sum_{j=-\infty}^{\infty} x_j 2^{-j-1}$, where $x_j = 0$ or $1$. In case of dyadic rationals, i.e. when there are two expansions of this form, we take the one that terminates in 0’s. Then the Walsh functions are defined as

\begin{equation}
(1.1) \quad w_x(y) = (-1)^{x+y-1} \sum_{y \in \mathbb{R}^+} x_k y^{-k-1} \quad (x, y \in \mathbb{R}^+).
\end{equation}

This research was supported by OTKA under grant T047128.

AMS Subject Classification: Primary 42C10, Secondary 42A45, 44A35.
We note that if $x = 2^k (k \in \mathbb{Z})$ then $w_x(y) = w_{2k}(y) = (-1)^{y-k-1}$. Consequently, $w_{2k}$ is equal to the $k$-th Rademacher function.

Let the Walsh-Dirichlet kernels be denoted by $D_t$:

$$D_t(y) = \int_0^t w_x(y) \, dx \quad (t, y \in \mathbb{R}^+)$$

It is known (see [5] or [12]) that

$$D_{2^n}(y) = \begin{cases} 2^n & 0 \leq y < 2^{-n}, \\ 0 & 2^{-n} \leq y < \infty \end{cases} (n \in \mathbb{Z}).$$

It is known that the Walsh system can be considered as the dual group of a locally compact Vilenkin group, the dyadic group. Taibleson ([13]) has developed a distribution theory for local fields. Following his concept of distributions we will consider the dyadic Hardy spaces $H^p(\mathbb{R}^+) (0 < p < 1)$ as subspaces of the space of dyadic distributions. More precisely, $H^p(\mathbb{R}^+)$ will be defined by means of atomic decomposition of distributions. To this order let the intervals of the form $[k 2^{-n}, (k + 1)2^{-n}) \ (k \in \mathbb{N}, n \in \mathbb{Z})$ be called dyadic intervals. The Lebesgue measure of a measurable set $A$ will be denoted by $|A|$. Then a function $a : \mathbb{R}^+ \mapsto \mathbb{R}$ is a $p$-atom if there exists a dyadic interval $I$ such that

i) $\text{supp } a \subset I,$

ii) $\|a\|_{L^\infty(\mathbb{R}^+)} \leq |I|^{-1/p},$

iii) $\int_I a = 0.$

We say that a dyadic distribution $f$ belongs to $H^p(\mathbb{R}^+) (0 < p < 1)$ if there exist $\alpha_k$ real numbers with $\sum_{k=0}^{\infty} |\alpha_k|^p < \infty$ and $a_k$ $p$-atoms such that

$$f = \sum_{k=0}^{\infty} \alpha_k a_k.$$  \hspace{1cm} \text{(1.3)}

The decomposition is understood in the sense of distributions. The $H^p(\mathbb{R}^+)$ norm is defined by

$$\|f\|_{H^p(\mathbb{R}^+)} = \inf \left( \sum_{k=0}^{\infty} |\alpha_k|^p \right)^{1/p}$$

with taking the infimum over all decompositions of the form (1.3).
Let $\phi : \mathbb{R}^+ \to \mathbb{R}$, then the Walsh multiplier operator $T_\phi$ is said to be bounded on $H^p(\mathbb{R}^+) \ (0 < p < 1)$ if for every $f \in H^p(\mathbb{R}^+)$ there exists a $T_\phi \in H^p(\mathbb{R}^+)$ such that
\[
\hat{T_\phi f}(x) = \phi_k \hat{f}(x) \quad (0 \leq x < \infty),
\]
where $\hat{f}$ stands for the Walsh-Fourier transform. Throughout the paper $C$ will denote an absolute positive constant not necessarily the same in different occurrences.

2. Results

In our first theorem we consider a Hörmander-Mihlin ([7], [9]) type condition. We prove that it is sufficient to give boundedness on certain $H^p(\mathbb{R}^+)$ spaces.

**Theorem 2.1.** Let $1 < r \leq 2$ and $\frac{r}{2r-1} < p < 1$. Suppose that $\varphi \in L^\infty(\mathbb{R}^+)$ is differentiable and the inclusion $\varphi' \in L^r_{\text{loc}}(\mathbb{R}^+)$ holds. If

\[
\left(\int_{2^j}^{2^{j+1}} |\varphi'(t)|^r \, dt\right)^{1/r} \leq C 2^{-j(1-1/r)} \quad (j \in \mathbb{Z})
\]

then $T_\varphi$ is bounded on $H^p(\mathbb{R}^+)$. 

In our next theorem we show that Theorem 2.1 is sharp in the sense that the condition on $p$ can not be relaxed by replacing the right side by any number smaller than $r/(2r-1)$.

**Theorem 2.2.** Let $1 \leq r \leq 2$. If $p < r/(2r-1)$ then there exists a differentiable $\varphi \in L^\infty(\mathbb{R}^+)$ that satisfies (2.1), but $T_\varphi$ is not bounded from $H^p(\mathbb{R}^+)$ to $L^p(\mathbb{R}^+)$. 

For previous results on multipliers on the dyadic Hardy spaces, and Hardy spaces over locally compact Vilenkin groups we refer the reader to the papers [1], [3], [4] and [11].
3. Proofs

For the proof of Theorem 2.1 we need the following lemma which is a Sidon type inequality. The trigonometric version of it was proved by Móricz [10].

**Lemma 3.1.** Let \( n, N \in \mathbb{Z} \), and \( 1 < q \leq 2 \). Then for any \( \gamma \in L^1_{\text{loc}}(\mathbb{R}^+) \) we have

\[
\left( 2^N \right) \int_0^{2^N} \left| \int_0^t \gamma(t) D_t(x) \, dt \right| \, dx \leq C_q 2^{-N(1-1/q)} \left( \int_0^{2^N} |\gamma(t)|^q \, dt \right)^{1/q}.
\]

**Proof.** Without loss of generality we may assume \( n > N \). Let us start with the following decomposition formula ([6]) for the Dirichlet kernels

\[
D_t(x) = w_t(x) \sum_{j=-\infty}^{\infty} t_j w_{2^{-j-1}}(x) D_{2^{-j-1}}(x) \quad (t, x \in \mathbb{R}^+).
\]

Before using this in the left side of (3.1) note that the integration with respect to \( x \) is over the interval \([2^N, \infty)\). By (1.2) we have that \( D_{2^{-j-1}}(x) = 0 \) holds for any \( x \geq 2^N \) if \( j \leq N - 1 \). Hence

\[
\left( 2^N \right) \int_0^{2^N} \left| \int_0^t \gamma(t) D_t(x) \, dt \right| \, dx = \left( 2^N \right) \int_0^{2^N} \left| \sum_{j=1}^{\infty} w_{2^{-j-1}}(x) D_{2^{-j-1}}(x) \int_0^{2^N} t_j \gamma(t) w_t(x) \, dt \right| \, dx.
\]

After changing the order of integration and summation we obtain

\[
\left( 2^N \right) \int_0^{2^N} \left| \int_0^t \gamma(t) D_t(x) \, dt \right| \, dx \leq \sum_{j=N}^{\infty} \left( 2^N \right) \int_0^{2^N} w_{2^{-j-1}}(x) D_{2^{-j-1}}(x) \int_0^{2^N} t_j \gamma(t) w_t(x) \, dt \right| \, dx.
\]

We proceed by introducing the notation \( g_j(x) = \text{sgn} \int_0^{2^N} t_j \gamma(t) w_t(x) \, dt \), and rewriting \( D_{2^{-j-1}} \) as \( 2^{-(j+1)} \chi_{[0,2^{j+1}]} \), where \( \chi_{[0,2^{j+1}]} \) is the characteristic function of \([0,2^{j+1}]\). Then, after performing a change in the order of integration, our estimate takes the form

\[
\left( 2^N \right) \int_0^{2^N} \gamma(t) D_t(x) \, dt \, dx \leq \sum_{j=N}^{\infty} 2^{-(j+1)} \int_0^{2^N} t_j \gamma(t) \int_0^{\infty} \chi_{[0,2^{j+1}]}(x) g_j(x) w_t(x) \, dx \, dt.
\]
The inner integral will be considered as the Walsh-Fourier transform, in notation \((g_j \chi_{[0,2^{j+1}]}) (t)\), of \(g_j \chi_{[0,2^{j+1}]}\) at \(t\). By using Hölder’s inequality for the outer integral and then the Hausdorff-Young inequality for the Walsh-Fourier transform we obtain

\[
\int_2^{2^N} \int_0^{2^n} \gamma(t) D_t(x) \, dx \, dt \leq \sum_{j=N}^{\infty} 2^{-(j+1)} \|\chi_{[0,2^n]} \gamma\|_{L^q(\mathbb{R}^+)} \|\widehat{g_j \chi_{[0,2^{j+1}]}\}}\|_{L^p(\mathbb{R}^+)} \leq \]

\[
\leq C_q \left( \int_0^{2^n} |\gamma(t)|^q \, dt \right)^{1/q} \sum_{j=-N}^{\infty} 2^{-(j+1)} \|\chi_{[0,2^{j+1}]} g_j\|_{L^q(\mathbb{R}^+)},
\]

where \(1/p + 1/q = 1\).

By the definition of \(g_j\) we have \(\|\chi_{[0,2^{j+1}]} g_j\|_{L^q(\mathbb{R}^+)} \leq 2^{(j+1)/q}\). Therefore

\[
\sum_{j=-N}^{\infty} 2^{-(j+1)} \|\chi_{[0,2^{j+1}]} g_j\|_{L^q(\mathbb{R}^+)} \leq \sum_{j=N}^{\infty} 2^{-(j+1)(1-1/q)} \leq C_q 2^{-N(1-1/q)}
\]

which is the desired estimate.

**Proof of Theorem 2.1.** We will show that (2.1) implies that \(\varphi\) satisfies the following condition:

\[
(3.2) \quad \sum_{n=-\infty}^{\infty} 2^{n(p-1)} \left( \int_{2^{-n+1}}^{2^n} \left( \int_{2^{-j-1}}^{2^j} \varphi(t) w_t(x) \, dt \right) \, dx \right)^p \leq C 2^{j(p-1)} \quad (j \in \mathbb{Z}).
\]

It was proved by Kitada [8] that (3.2) is sufficient for \(T_\varphi\) be bounded on \(H^p(\mathbb{R}^+)\), \(0 < p < 1\). Let us split the sum in (3.2) at \(n = j\) and consider the case \(n \geq j\) first

\[
I_2 = \sum_{n=j}^{\infty} 2^{n(p-1)} \left( \int_{2^{-n+1}}^{2^n} \left( \int_{2^{-j-1}}^{2^j} \varphi(t) w_t(x) \, dt \right) \, dx \right)^p.
\]

If \(x < 2^{-n}\) then \(x_k = 0\) for every \(k < n\). Similarly, \(t < 2^j\) means \(t_k = 0\) for every \(k < -j\). Since \(j \leq n\) we have by definition (1.1) that \(w_t(x) = 1\). Therefore,

\[
I_2 = \sum_{n=j}^{\infty} 2^{n(p-1)} \left( \int_{2^{-j-1}}^{2^j} \varphi(t) \, dt \right)^p.
\]
Making use of the fact that $\varphi$ is bounded, we obtain

$$I_2 \leq \sum_{n=0}^{\infty} 2^{n(p-1)} \left( 2^{-(n+1)} 2^j C \right)^p \leq C 2^{j(p-1)},$$

which is corresponds to (3.2).

Let us take the $n < j$ part:

$$I_1 = \sum_{n=-\infty}^{j-1} 2^{n(p-1)} \left( \int_{2^{-(n+1)}}^{2^{-n}} \int_{2^{-j-1}}^{2^j} \varphi(t) w_1(x) dt \right)^p.$$  

We start with using integration by parts for the integral with respect to $t$

$$\int_{2^{-j-1}}^{2^j} \varphi(t) w_1(x) dt = \varphi(t) D_t(x) \bigg|_{2^{-j-1}}^{2^j} - \int_{2^{-j-1}}^{2^j} \varphi'(t) D_t(x) dt.$$  

Hence

$$\left| \int_{2^{-j-1}}^{2^j} \varphi(t) w_1(x) dt \right| \leq |\varphi(2^j)| D_{2^j}(x) + |\varphi(2^{j-1})| D_{2^{j-1}}(x) + \left| \int_{2^{-j-1}}^{2^j} \varphi'(t) D_t(x) dt \right|.$$  

Then we have

$$I_1 \leq \sum_{n=-\infty}^{j-1} 2^{n(p-1)} \left( \int_{2^{-(n+1)}}^{2^{-n}} |\varphi(2^{j-1})| D_{2^{j-1}}(x) + |\varphi(2^j)| D_{2^j}(x) dx \right)^p +$$

$$+ \sum_{n=-\infty}^{j-1} 2^{n(p-1)} \left( \int_{2^{-(n+1)}}^{2^{-n}} \int_{2^{-j-1}}^{2^j} \varphi'(t) D_t(x) dt \right)^p = I_{11} + I_{12}.$$  

If $n < j - 1$ then $[2^{-(n+1)}, 2^{-n}] \subset [2^{-j+1}, 1]$. Recall that $D_{2^j} = 2^j \chi_{[0, 2^{-j+1}]}$, and $D_{2^{j-1}} = 2^{j-1} \chi_{[0, 2^{-j+1}]}$. This means that the sum in $I_{11}$ reduces to a single term

$$I_{11} = 2^{(j-1)(p-1)} (2^{-j} |\varphi(2^{j-1})| 2^{j-1})^p.$$  

Again, it follows from the boundedness of $\varphi$ that $I_{11} \leq C 2^{j(p-1)}$.  

Applying Lemma 3.1 to the integral in $I_{12}$ we obtain

$$
\int_{2^{-(n+1)}}^{2^{-n}} \int_{2^{j-1}}^{2^j} \varphi'(t)D_x(x) \, dt \, dx \leq C2^{(n+1)(1-1/r)} \left( \int_{2^{j-1}}^{2^j} |\varphi'(t)|^r \, dt \right)^{1/r}.
$$

Hence we have by (2.1)

$$
I_{22} \leq C \sum_{n=-\infty}^{j-1} 2^n(2^{n+1}(1-1/r)2^{(1/r-1)})^p = \frac{C2^{(1/r-1)}}{2^{j+1}} \sum_{n=-\infty}^{j-1} 2^{n(2p-1-p/r)},
$$

It follows from the assumption $p > \frac{r}{2r-1}$ that $2p-1 - \frac{p}{r} > 0$. Consequently,

$$
I_{12} \leq C2^{(1/r-1)}2^{(2p-1-p/r)} = C2^{(p-1)}.
$$

Combining the estimates for $I_1$ and $I_2$ we obtain the claimed estimate.

**Proof of Theorem 2.2.** Set

$$
\sigma(t) = \begin{cases} 
\frac{1}{2}(1 - \cos 2\pi t) & \text{if } 0 \leq t \leq 1, \\
0 & \text{otherwise.}
\end{cases}
$$

Define $\varphi \in L^\infty(\mathbb{R}^+)$ as follows

$$
\varphi(t) = \sum_{k=0}^{\infty} 2^{-k(1-1/r)}\tau_{2^k}\sigma(t) \quad (t \in \mathbb{R}^+),
$$

where $\tau_x\sigma(t) = \sigma(t-x)$, $x \in \mathbb{R}$. Then $\varphi \in L^\infty(\mathbb{R}^+)$, $\text{supp} \varphi = \bigcup_{k=0}^{\infty} [2^k, 2^{k+1}]$, and $\varphi$ is differentiable. Moreover

$$
\left( \int_{2^k}^{2^{k+1}} |\varphi'(t)|^r \, dt \right)^{1/r} = \left( \int_{2^k}^{2^{k+1}} \tau_{2^k} \varphi'(t) \, dt \right)^{1/r} < 2\pi 2^{-k(1-1/r)}.
$$

Consequently, $\varphi$ satisfies condition (2.1).
We will define the function $f \in H^p(\mathbb{R}^+)$ by means of the $p$-atoms

$$a_k = 2^{k(1/p-1)}(D_{2^{k+1}} - D_{2^k}) \quad (k \in \mathbb{N}).$$

Let us choose the coefficients $\lambda_k$ as

$$\lambda_k = 2^{-k(1/p+1/r-2)} \quad (k \in \mathbb{N}).$$

Then it follows from the condition $p < r/(2r - 1)$ that $1/p + 1/r - 2 > 0$. Thus

$$\sum_{k=0}^{\infty} |\lambda_k|^p < \infty,$$

i.e. $f = \sum_{k=0}^{\infty} \lambda_k a_k \in H^p(\mathbb{R}^+)$. The action of the multiplier $\varphi$ on $f$ can be calculated as follows

$$T_\varphi f(x) = \sum_{k=0}^{\infty} \lambda_k 2^{k(1/p-1)}2^{-k(1-1/r)} \int_{2^k}^{2^{k+1}} \tau_{2^k} \sigma(t)w_t(x) dt \quad (x \in \mathbb{R}^+).$$

We will show that $\chi_{[0,1]}T_\varphi \not\in L^p[0,1]$. To this order let us calculate

$$\int_{2^k}^{2^{k+1}} \tau_{2^k} \sigma(t)w_t(x) dt, \quad 0 \leq x < 1.$$

Since $w_t(x) = w_t(x) \quad (x \in [0,1], t \in \mathbb{R}^+)$ (see e.g. [12]) we have

$$\int_{2^k}^{2^{k+1}} \tau_{2^k} \sigma(t)w_t(x) dt = w_{2^k}(x) \int_{0}^{1} (1 - \cos 2\pi t) dt = w_{2^k}(x) \quad (x \in [0,1]).$$

Consequently, $\chi_{[0,1]}T_\varphi$ takes the form of a Rademacher series. i.e.

$$T_\varphi(x) = \sum_{k=0}^{\infty} r_k(x) \quad (x \in [0,1]).$$

By the Khintchin inequality, $\left\| \sum_{k=0}^{\infty} c_k r_k \right\|_{L^p([0,1])} \approx \left( \sum_{k=0}^{\infty} c_k^p \right)^{1/2}$. In particular,

$$\int_{0}^{1} |T_\varphi(x)|^p dx = \infty,$$

i.e. $T_\varphi \not\in L^p(\mathbb{R}^+)$. 

References


(Received September 9, 2004)
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