Annales Univ. Sci. Budapest., Sect. Comp. 24 (2004) 275-284

$\mathbf{H}^p$ MULTIPLIERS ON THE DYADIC FIELD

J.E. Daly (Colorado Springs, CO, USA)

S. Fridli (Budapest, Hungary)

Dedicated to Professor I. Kátai on the occasion of his 65th birthday

Abstract. In this paper we consider a classical multiplier condition, the Hörmander-Mihlin condition, originally introduced for the trigonometric case. It implies that the multiplier operator is bounded on $\mathbf{L^p}$, $1 < p < \infty$. Here we study the corresponding problem with respect to the Walsh transform and the noncompact dyadic Hardy spaces $\mathbf{H^p[0,\infty)}$, $0 < p < 1$. We also show that our result is sharp. We note that a similar program was carried out for the trigonometric case and the classical Hardy spaces, and for the Walsh system and the dyadic Hardy spaces on $[0,1]$ in our previous papers [1] and [2].

1. Introduction

Set $\mathbb{R}^+ = [0,\infty)$. The binary expansion of $x \in \mathbb{R}^+$ is $x = \sum\limits_{j=-\infty}^{\infty} x_j 2^{-j-1}$, where $x_j = 0$ or 1. In case of dyadic rationals, i.e. when there are two expansions of this form, we take the one that terminates in 0's. Then the Walsh functions are defined as

$$w_x(y) = (-1)^{\sum\limits_{k=-\infty}^{\infty} x_k y_{-k-1}} \qquad (x, y \in \mathbb{R}^+). \tag{1.1}$$

This research was supported by OTKA under grant T047128.

AMS Subject Classification: Primary 42C10, Secondary 42A45, 44A35.

We note that if $x = 2^k$ $(k \in \mathbb{Z})$ then $w_x(y) = w_{2^k}(y) = (-1)^{y_{-k-1}}$. Consequently, w_{2^k} is equal to the k-th Rademacher function.

Let the Walsh-Dirichlet kernels be denoted by D_t :

$$D_t(y) = \int_0^t w_x(y)\,dx \qquad (t, y \in \mathbb{R}^+).$$

It is known (see [5] or [12]) that

$$D_{2^n}(y) = \begin{cases} 2^n & 0 \le y < 2^{-n}, \\ 0 & 2^{-n} \le y < \infty \qquad (n \in \mathbb{Z}). \end{cases} \tag{1.2}$$

It is known that the Walsh system can be considered as the dual group of a locally compact Vilenkin group, the dyadic group. Taibleson ([13]) has developed a distribution theory for local fields. Following his concept of distributions we will consider the dyadic Hardy spaces $\mathbf{H}^p(\mathbb{R}^+)$ $(0 < p < 1)$ as subspaces of the space of dyadic distributions. More precisely, $\mathbf{H}^p(\mathbb{R}^+)$ will be defined by means of atomic decomposition of distributions. To this order let the intervals of the form $[k2^{-n}, (k+1)2^{-n})$ $(k \in \mathbb{N}, n \in \mathbb{Z})$ be called dyadic intervals. The Lebesgue measure of a measurable set A will be denoted by $|A|$. Then a function $\mathbf{a} : \mathbb{R}^+ \mapsto \mathbb{R}$ is a p-atom if there exists a dyadic interval I such that

$$\text{i) } \operatorname{supp} \mathbf{a} \subset I, \qquad \text{ii) } \|\mathbf{a}\|_{\mathbf{L}^\infty(\mathbb{R}^+)} \le |I|^{-1/p}, \qquad \text{iii) } \int_I \mathbf{a} = 0.$$

We say that a dyadic distribution f belongs to $\mathbf{H}^p(\mathbb{R}^+)$ $(0 < p < 1)$ if there exist α_k real numbers with $\sum_{k=0}^{\infty} |\alpha_k|^p < \infty$ and $\mathbf{a}_k$ p-atoms such that

$$f = \sum_{k=0}^{\infty} \alpha_k \mathbf{a}_k. \tag{1.3}$$

The decomposition is understood in the sense of distributions. The $\mathbf{H}^p(\mathbb{R}^+)$ norm is defined by

$$\|f\|_{\mathbf{H}^p(\mathbb{R}^+)} = \inf \left(\sum_{k=0}^{\infty} |\alpha_k|^p \right)^{1/p}$$

with taking the infimum over all decompositions of the form (1.3).

Let $\phi : \mathbb{R}^+ \to \mathbb{R}$, then the Walsh multiplier operator T_ϕ is said to be bounded on $\mathbf{H}^p(\mathbb{R}^+)$ $(0 < p < 1)$ if for every $f \in \mathbf{H}^p(\mathbb{R}^+)$ there exists a $T_\phi \in \mathbf{H}^p(\mathbb{R}^+)$ such that

$$\widehat{T_\phi f}(x) = \phi_k \hat{f}(x) \qquad (0 \le x < \infty),$$

where $\hat{f}$ stands for the Walsh-Fourier transform. Throughout the paper C will denote an absolute positive constant not necessarily the same in different occurrences.

2. Results

In our first theorem we consider a Hörmander-Mihlin ([7], [9]) type condition. We prove that it is sufficient to give boundedness on certain $\mathbf{H}^p(\mathbb{R}^+)$ spaces.

Theorem 2.1. *Let* $1 < r \le 2$ *and* $\dfrac{r}{2r-1} < p < 1$. *Suppose that* $\varphi \in$ $\in L^\infty(\mathbb{R}^+)$ *is differentiable and the inclusion* $\varphi' \in L^r_{\text{loc}}(\mathbb{R}^+)$ *holds. If*

$$\left(\int_{2^j}^{2^{j+1}} |\varphi'(t)|^r \, dt \right)^{1/r} \le C 2^{-j(1-1/r)} \qquad (j \in \mathbb{Z}) \tag{2.1}$$

then T_φ *is bounded on* $\mathbf{H}^p(\mathbb{R}^+)$.

In our next theorem we show that *Theorem 2.1* is sharp in the sense that the condition on p can not be relaxed by replacing the right side by any number smaller than $r/(2r-1)$.

Theorem 2.2. *Let* $1 \le r \le 2$. *If* $p < r/(2r-1)$ *then there exists a differentiable* $\varphi \in L^\infty(\mathbb{R}^+)$ *that satisfies (2.1), but* T_ϕ *is not bounded from* $H^p(\mathbb{R}^+)$ *to* $L^p(\mathbb{R}^+)$.

For previous results on multipliers on the dyadic Hardy spaces, and Hardy spaces over locally compact Vilenkin groups we refer the reader to the papers [1], [3], [4] and [11].

3. Proofs

For the proof of *Theorem 2.1* we need the following lemma which is a Sidon type inequality. The trigonometric version of it was proved by Móricz [10].

Lemma 3.1. *Let $n, N \in \mathbb{Z}$, and $1 < q \le 2$. Then for any $\gamma \in L^1_{\text{loc}}(\mathbb{R}^+)$ we have*

$$\int_{2^N}^{\infty} \left| \int_0^{2^n} \gamma(t) D_t(x)\, dt \right| dx \le C_q 2^{-N(1-1/q)} \left(\int_0^{2^n} |\gamma(t)|^q\, dt \right)^{1/q} . \tag{3.1}$$

Proof. Without loss of generality we may assume $n > N$. Let us start with the following decomposition formula ([6]) for the Dirichlet kernels

$$D_t(x) = w_t(x) \sum_{j=-\infty}^{\infty} t_j w_{2^{-j-1}}(x) D_{2^{-j-1}}(x) \qquad (t, x \in \mathbb{R}^+).$$

Before using this in the left side of (3.1) note that the integration with respect to x is over the interval $[2^N, \infty)$. By (1.2) we have that $D_{2^{-j-1}}(x) = 0$ holds for any $x \ge 2^N$ if $j \le N-1$. Hence

$$\int_{2^N}^{\infty} \left| \int_0^{2^n} \gamma(t) D_t(x)\, dt \right| dx = \int_{2^N}^{\infty} \left| \sum_{j=N}^{\infty} w_{2^{-j-1}}(x) D_{2^{-j-1}}(x) \int_0^{2^n} t_j \gamma(t) w_t(x)\, dt \right| dx\, .$$

After changing the order of integration and summation we obtain

$$\int_{2^N}^{\infty} \left| \int_0^{2^n} \gamma(t) D_t(x)\, dt \right| dx \le \sum_{j=N}^{\infty} \int_{2^N}^{\infty} \left| w_{2^{-j-1}}(x) D_{2^{-j-1}}(x) \int_0^{2^n} t_j \gamma(t) w_t(x)\, dt \right| dx\, .$$

We proceed by introducing the notation $g_j(x) = \operatorname{sgn} \int_0^{2^n} t_j \gamma(t) w_t(x)\, dt$, and rewriting $D_{2^{-j-1}}$ as $2^{-(j+1)} \chi_{[0,2^{j+1}]}$, where $\chi_{[0,2^{j+1}]}$ is the characteristic function of $[0, 2^{j+1}]$. Then, after performing a change in the order of integration, our estimate takes the form

$$\int_{2^N}^{\infty} \left| \int_0^{2^n} \gamma(t) D_t(x) dt \right| dx \le \sum_{j=N}^{\infty} 2^{-(j+1)} \int_0^{2^n} t_j \gamma(t) \int_{2^N}^{\infty} \chi_{[0,2^{j+1}]}(x) g_j(x) w_t(x) dx dt.$$

The inner integral will be considered as the Walsh-Fourier transform, in notation $(\widehat{g_j\chi_{[0,2^{j+1}]}})(t)$, of $g_j\chi_{[0,2^{j+1}]}$ at t. By using Hölder's inequality for the outer integral and then the Hausdorff-Young inequality for the Walsh-Fourier transform we obtain

$$\int_{2^N}^{\infty}\left|\int_0^{2^n}\gamma(t)D_t(x)\,dt\right|\,dx \le \sum_{j=N}^{\infty}2^{-(j+1)}\|\chi_{[0,2^n]}\gamma\|_{L^q(\mathbb{R}^+)}\|(\widehat{g_j\chi_{[0,2^{j+1}]}})\|_{L^p(\mathbb{R}^+)} \le$$

$$\le C_q\left(\int_0^{2^n}|\gamma(t)|^q\,dt\right)^{1/q}\sum_{j=-N}^{\infty}2^{-(j+1)}\|\chi_{[0,2^{j+1}]}g_j\|_{L^q(\mathbb{R}^+)},$$

where $1/p+1/q=1$.

By the definition of g_j we have $\|\chi_{[0,2^{j+1}]}g_j\|_{L^q(\mathbb{R}^+)}\le 2^{(j+1)/q}$. Therefore

$$\sum_{j=-N}^{\infty}2^{-(j+1)}\|\chi_{[0,2^{j+1}]}g_j\|_{L^q(\mathbb{R}^+)} \le \sum_{j=N}^{\infty}2^{-(j+1)(1-1/q)} \le$$
$$\le C_q 2^{-N(1-1/q)}$$

which is the desired estimate.

Proof of Theorem 2.1. We will show that (2.1) implies that φ satisfies the following condition:

$$\sum_{n=-\infty}^{\infty}2^{n(p-1)}\left(\int_{2^{-(n+1)}}^{2^{-n}}\left|\int_{2^{j-1}}^{2^j}\varphi(t)w_t(x)\,dt\right|\,dx\right)^p \le C2^{j(p-1)} \qquad (j\in\mathbb{Z}). \tag{3.2}$$

It was proved by Kitada [8] that (3.2) is sufficient for T_ϕ be bounded on $\mathbf{H}^p(\mathbb{R}^+)$, $0<p<1$. Let us split the sum in (3.2) at $n=j$ and consider the case $n\ge j$ first

$$I_2=\sum_{n=j}^{\infty}2^{n(p-1)}\left(\int_{2^{-(n+1)}}^{2^{-n}}\left|\int_{2^{j-1}}^{2^j}\varphi(t)w_t(x)\,dt\right|\,dx\right)^p.$$

If $x<2^{-n}$ then $x_k=0$ for every $k<n$. Similarly, $t<2^j$ means $t_k=0$ for every $k<-j$. Since $j\le n$ we have by definition (1.1) that $w_t(x)=1$. Therefore,

$$I_2=\sum_{n=j}^{\infty}2^{n(p-1)}\left(2^{-(n+1)}\left|\int_{2^{j-1}}^{2^j}\varphi(t)\,dt\right|\right)^p.$$

Making use of the fact that φ is bounded, we obtain

$$I_2 \le \sum_{n=j}^{\infty} 2^{n(p-1)} \left(2^{-(n+1)} 2^j C\right)^p \le C 2^{j(p-1)},$$

which is corresponds to (3.2).

Let us take the $n < j$ part:

$$I_1 = \sum_{n=-\infty}^{j-1} 2^{n(p-1)} \left(\int_{2^{-(n+1)}}^{2^{-n}} \left| \int_{2^{j-1}}^{2^j} \varphi(t) w_t(x)\, dt \right| dx \right)^p .$$

We start with using integration by parts for the integral with respect to t

$$\int_{2^{j-1}}^{2^j} \varphi(t) w_t(x)\, dt = \varphi(t) D_t(x)\Big]_{2^{j-1}}^{2^j} - \int_{2^{j-1}}^{2^j} \varphi'(t) D_t(x)\, dt\,.$$

Hence

$$\left| \int_{2^{j-1}}^{2^j} \varphi(t) w_t(x)\, dt \right| \le |\varphi(2^j)| D_{2^j}(x) + |\varphi(2^{j-1})| D_{2^{j-1}}(x) + \left| \int_{2^{j-1}}^{2^j} \varphi'(t) D_t(x)\, dt \right| .$$

Then we have

$$I_1 \le \sum_{n=-\infty}^{j-1} 2^{n(p-1)} \left(\int_{2^{-(n+1)}}^{2^{-n}} |\varphi(2^{j-1})| D_{2^{j-1}}(x) + |\varphi(2^j)| D_{2^j}(x)\, dx \right)^p +$$

$$+ \sum_{n=-\infty}^{j-1} 2^{n(p-1)} \left(\int_{2^{-(n+1)}}^{2^{-n}} \left| \int_{2^{j-1}}^{2^j} \varphi'(t) D_t(x)\, dt \right| dx \right)^p = I_{11} + I_{12}.$$

If $n < j-1$ then $[2^{-(n+1)}, 2^{-n}] \subset [2^{-j+1}, 1]$. Recall that $D_{2^j} = 2^j \chi_{[0,2^{-j})}$, and $D_{2^{j-1}} = 2^{j-1} \chi_{[0,2^{-j+1})}$. This means that the sum in I_{11} reduces to a single term

$$I_{11} = 2^{(j-1)(p-1)} \big(2^{-j} |\varphi(2^{j-1})| 2^{j-1}\big)^p.$$

Again, it follows from the boundedness of φ that $I_{11} \le C 2^{j(p-1)}$.

Applying *Lemma 3.1* to the integral in I_{12} we obtain

$$\int_{2^{-(n+1)}}^{2^{-n}} \left| \int_{2^{j-1}}^{2^j} \varphi'(t) D_t(x)\, dt \right| dx \le C 2^{(n+1)(1-1/r)} \left(\int_{2^{j-1}}^{2^j} |\varphi'(t)|^r\, dt \right)^{1/r} .$$

Hence we have by (2.1)

$$I_{22} \le C \sum_{n=-\infty}^{j-1} 2^{n(p-1)} \left(2^{(n+1)(1-1/r)} 2^{j(1/r-1)} \right)^p =$$

$$= C 2^{jp(1/r-1)} \sum_{n=-\infty}^{j-1} 2^{n(2p-1-p/r)} .$$

It follows from the assumption $p > \dfrac{r}{2r-1}$ that $2p - 1 - \dfrac{p}{r} > 0$. Consequently,

$$I_{12} \le C 2^{jp(1/r-1)} 2^{j(2p-1-p/r)} = C 2^{j(p-1)}.$$

Combining the estimates for I_1, and I_2 we obtain the claimed estimate.

Proof of Theorem 2.2. Set

$$\sigma(t) = \begin{cases} \dfrac{1}{2}(1 - \cos 2\pi t) & \text{if} \quad 0 \le t \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Define $\varphi \in L^\infty(\mathbb{R}^+)$ as follows

$$\varphi(t) = \sum_{k=0}^{\infty} 2^{-k(1-1/r)} \tau_{2^k} \sigma(t) \qquad (t \in \mathbb{R}^+),$$

where $\tau_x \sigma(t) = \sigma(t-x)$, $x \in \mathbb{R}$. Then $\varphi \in L^\infty(\mathbb{R}^+)$, $\operatorname{supp} \varphi = \bigcup_{k=0}^{\infty} [2^k, 2^k + 1]$, and φ is differentiable. Moreover

$$\left(\int_{2^k}^{2^{k+1}} |\varphi'(t)|^r\, dt \right)^{1/r} = \left(\int_{2^k}^{2^k+1} 2^{-k(1-1/r)} |2\pi \sin 2\pi t|^r\, dt \right)^{1/r} < 2\pi 2^{-k(1-1/r)}.$$

Consequently, φ satisfies condition (2.1).

We will define the function $f \in H^p(\mathbb{R}^+)$ by means of the p-atoms

$$a_k = 2^{k(1/p-1)}(D_{2^{k+1}} - D_{2^k}) \quad (k \in \mathbb{N}).$$

Let us choose the coefficients λ_k as

$$\lambda_k = 2^{-k(1/p+1/r-2)} \qquad (k \in \mathbb{N}).$$

Then it follows from the condition $p < r/(2r-1)$ that $1/p + 1/r - 2 > 0$. Thus $\sum_{k=0}^{\infty} |\lambda_k|^p < \infty$, i.e. $f = \sum_{k=0}^{\infty} \lambda_k a_k \in H^p(\mathbb{R}^+)$.

The action of the multiplier φ on f can be calculated as follows

$$T_\varphi f(x) = \sum_{k=0}^{\infty} \lambda_k 2^{k(1/p-1)} 2^{-k(1-1/r)} \int_{2^k}^{2^{k+1}} \tau_{2^k}\sigma(t) w_t(x)\, dt \qquad (x \in \mathbb{R}^+).$$

We will show that $\chi_{[0,1)} T_\varphi \notin L^p[0,1)$. To this order let us calculate

$$\int_{2^k}^{2^{k+1}} \tau_{2^k}\sigma(t) w_t(x)\, dt, \qquad 0 \le x < 1.$$

Since $w_t(x) = w_{[t]}(x)$ $(x \in [0,1),\ t \in \mathbb{R}^+)$ (see e.g. [12]) we have

$$\int_{2^k}^{2^{k+1}} \tau_{2^k}\sigma(t) w_t(x)\, dt = w_{2^k}(x) \int_0^1 \frac{1}{2}(1 - \cos 2\pi t)\, dt = w_{2^k}(x) \qquad (x \in [0,1))\,.$$

Consequently, $\chi_{[0,1)} T_\varphi$ takes the form of a Rademacher series. i.e.

$$T_\varphi(x) = \sum_{k=0}^{\infty} r_k(x) \qquad (x \in [0,1))\,.$$

By the Khintchin inequality, $\left\| \sum_{k=0}^{\infty} c_k r_k \right\|_{L^p([0,1)} \approx \left(\sum_{k=0}^{\infty} c_k^2 \right)^{1/2}$. In particular, $\int_0^1 |T_\varphi(x)|^p\, dx = \infty$, i.e. $T_\varphi \notin L^p(\mathbb{R}^+)$.

References

[1] **Daly J. and Fridli S.,** Walsh multipliers for dyadic Hardy spaces, *Appl. Anal.,* **82** (7) (2003), 689-700.

[2] **Daly J. and Fridli S.,** Trigonometric multipliers on $H_{2\pi}$, *Can. Math. Bull.* (to appear)

[3] **Daly J. and Phillips K.,** Walsh multipliers and square functions for the Hardy space H^1, *Acta Math. Hungar.,* **79** (4) (1998), 311-327.

[4] **Daly J. and Phillips K.,** A note on H^1 multipliers for locally compact Vilenkin groups, *Canad. Math. Bull.,* **41** (4) (1998), 392-397.

[5] **Fine N.J.,** The generalized Walsh functions, *Trans. Amer. Math. Soc.,* **69** (1950), 66-77.

[6] **Fridli S.,** On the integrability of Walsh-Fourier transform, *Mathematica Pannonica,* **10** (1) (1999), 93-102.

[7] **Hörmander L.,** Estimates for translation invariant operators in L^p spaces, *Acta Math.,* **104** (1960), 93-139.

[8] **Kitada K.,** H^p multiplier theorems on certain totally disconnected groups, *Sci. Rep. Hirosaki Univ.,* **34** (1987), 1-7.

[9] **Михлин С.Г.,** О мультипликаторах интегралов Фурье, *Доклады АН СССР,* **109** (1956), 701-703. (Mihlin S.G., On the multipliers of Fourier integrals, *Dokl. Akad. Nauk SSSR,* **109** (1956), 701-703.)

[10] **Móricz F.,** Sidon type inequalities, *Publ. de Inst. Math. (Beograd),* **48** (62) (1990), 101-109.

[11] **Onneweer C.W. and Quek T.S.,** H^p multiplier results on locally compact Vilenkin groups, *Quart. J. Math. Oxford,* **40** (2) (1989), 313-323.

[12] **Schipp F. and Wade W.R. and Simon P.,** *Walsh series*, Adam Hilger, Bristol, New York, 1990.

[13] **Taibleson M.,** *Fourier analysis on local fields,* Mathematical Notes **15**, Princeton University Press, 1975.

(Received September 9, 2004)

J.E. Daly
Department of Mathematics
University of Colorado
Colorado Springs
CO 80933-7150, USA
jedaly@math.uccs.edu

S. Fridli
Department of Numerical Analysis
Eötvös Loránd University
Pázmány Péter s. 1/C
H-1117 Budapest, Hungary
fridli@numanal.inf.elte.hu