Annales Univ. Sci. Budapest., Sect. Comp. 24 (2004) 253-273

WEIGHTED (0,2)–INTERPOLATION WITH INTERPOLATORY BOUNDARY CONDITIONS

M. Lénárd (Kuwait)

Dedicated to Professor Imre Kátai,
on the occasion of his 65th birthday

Abstract. The weighted (0,2)-interpolation is studied in a unified way with two additional interpolatory conditions. The question is how to choose the nodal points and the weight function w so that the problem is regular. We formulate sufficient conditions on the nodal points and on the weight function. In the regular cases we find simple explicit forms of the interpolational polynomial. Special cases are presented when the nodes are the zeros of the classical orthogonal polynomials.

1. Introduction

P. Turán initiated the study of (0,2)-interpolation in order to get an approximate solution to the differential equation

$$y'' + f \cdot y = 0.$$

The first results were published by J. Surányi and P. Turán [12] in 1955. In 1961 J. Balázs [2] introduced a generalization of this problem, the *weighted (0,2)-interpolation problem*: Let the system of nodes

$$(1) \qquad -\infty \leq a < x_{n,n} < x_{n-1,n} < \ldots < x_{1,n} < b \leq \infty$$

Research work supported by Kuwait University Grant SM 01/03.
AMS Subject Classification: 41A05

be given in the finite (or infinite) open (or closed) interval (a,b) and let $w \in C^2(a,b)$ be a weight function. Find a polynomial R_n of minimal degree satisfying the conditions

$$(2) \qquad R_n(x_{k,n}) = y_{k,n}; \quad (wR_n)''(x_{k,n}) = y''_{k,n} \quad (k=1,\dots,n; n\in N),$$

where $y_{k,n}$, $y''_{k,n}$ are arbitrary given real numbers.

The questions are how to choose the nodal points and the weight function w, so that the problem is regular (it has a unique solution) and in the regular case to find simple explicit form of R_n in order to prove convergence.

J. Balázs [2] investigated the above problem on the interval $[-1,1]$, when the nodes are the roots of the ultraspherical polynomial $P_n^{(\alpha)}$ $(\alpha > -1)$, and the weight function is $w(x) = (1-x^2)^{(\alpha+1)/2}$. He showed, that in this case there does not exist a polynomial of degree $\leq 2n-1$ satisfying the requirements (2). He proved, that if n is even, then under the condition

$$(3) \qquad R_n(0) = \sum_{k=1}^{n} y_{k,n} l_{k,n}^2(0)$$

there exists a unique polynomial of degree $\leq 2n$ which satisfies (2) (if n is odd, then the uniqueness fails). (Here $l_{k,n}(x)$ represent the Lagrange-fundamental polynomials corresponding to the nodal points $x_{k,n}$.) He gave the explicit form of this polynomial and proved convergence theorem.

Several authors investigated the weighted (0,2)-interpolation with the additional Balázs-type condition (3) on the roots of the classical orthogonal polynomials (I. Joó [5], I. Joó and L. Szili [6], J. Prasad [7], [8], [9], [10], L. Szili [14]). Then L. Szili [15] treated the weighted (0,2)-interpolational problem with Balázs-type condition in a unified way on the roots of all classical orthogonal polynomials with respect to the existence, uniqueness and representation (explicit formulae).

In special cases J. Bajpai [1], S. Eneduanya [4], and J. Balázs [3] substituted the additional condition (3) with interpolatory type conditions. For more results on (0,2) interpolation we refer to the survey paper of L. Szili [16].

In this paper we study the weighted (0,2)-interpolation problem in a unified way with different interpolatory conditions. In these cases we determine sufficient conditions on the nodes and the weight function, for the problem to be regular. In the corollaries we give examples, when the nodes are the zeros of the classical orthogonal polynomials.

The problem: On the finite or infinite interval $[a,b]$ let $x_{i,n}$, $i=0,\dots,n$ be distinct points (the *nodal points of interpolation*), and let $w \in C^2(a,b)$ be a *weight function* on (a,b). Find a polynomial Q_n of minimal degree satisfying the weighted (0,2)-interpolational conditions

$$Q_n(x_{i,n}) = y_{i,n}, \quad (wQ_n)''(x_{i,n}) = y''_{i,n} \qquad (i = 1, \dots, n-1),$$

with the additional interpolatory conditions

$$Q_n(x_{0,n}) = y_{0,n}, \quad Q_n(x_{n,n}) = y_{n,n}, \qquad (\text{boundary} - \text{type})$$

or

$$Q_n(x_{0,n}) = y_{0,n}, \quad Q'_n(x_{0,n}) = y'_{0,n}, \qquad (\text{initial} - \text{type})$$

or

$$Q'_n(x_{0,n}) = y'_{0,n}, \quad Q'_n(x_{n,n}) = y'_{n,n},$$

where $y_{i,n}$, $y'_{0,n}$, $y'_{n,n}$ and $y''_{i,n}$ are arbitrary real numbers.

In what follows, let n be a fixed positive integer and for the sake of simplicity we will use x_i instead of the double indexed $x_{i,n}$. Let $[a,b]$ be a finite or infinite interval, and let

(4)
$$x_0, x_1, \dots, x_{n-1}, x_n \in [a,b]$$

be distinct nodes. Let p_{n-1} be a polynomial of degree n-1, for which

(5)
$$p_{n-1}(x_i) = 0 \qquad (i = 1, \dots, n-1),$$

and let

(6)
$$l_j(x) = \frac{p_{n-1}(x)}{p'_{n-1}(x_j)(x - x_j)} \qquad (j = 1, \dots, n-1)$$

be the fundamental polynomials of Lagrange interpolation corresponding to the nodal points $x_1, \dots, x_{n-1}$. Hence

(7)
$$l_j(x_i) = \delta_{i,j} \qquad (i, j = 1, \dots, n-1),$$

and

(8)
$$l'_j(x_j) = \frac{p''_{n-1}(x_j)}{2p'_{n-1}(x_j)} \qquad (j = 1, \dots, n-1).$$

2. Preliminaries

2.1. The classical orthogonal polynomials

Let us consider the homogeneous differential equation

$$u'' + f \cdot u = 0. \tag{9}$$

Lemma 1. *If $P_n^{(\alpha,\beta)}$ $(\alpha, \beta > -1)$ denotes the Jacobi polynomial of degree n, and*

$$w(x) = (1-x)^{\frac{\alpha+1}{2}}(1+x)^{\frac{\beta+1}{2}},$$

then $u = wP_n^{(\alpha,\beta)}$ satisfies the differential equation (9) with

$$f(x) = \frac{1}{4}\frac{1-\alpha^2}{(1-x)^2} + \frac{1}{4}\frac{1-\beta^2}{(1+x)^2} + \frac{2n(n+\alpha+\beta+1)+(\alpha+1)(\beta+1)}{2(1-x^2)}.$$

Proof. Cf. (4.24.1) in [13].

Lemma 2. *If $L_n^{(\alpha)}$ $(\alpha > -1)$ denotes the Laguerre polynomial of degree n, and*

$$w(x) = e^{-\frac{x}{2}} x^{\frac{\alpha+1}{2}},$$

then $u = wL_n^{(\alpha)}$ satisfies the differential equation (9) with

$$f(x) = \frac{2n+\alpha+1}{2x} + \frac{1-\alpha^2}{4x^2} - \frac{1}{4}.$$

Proof. Cf. (5.1.2) in [13].

Lemma 3. *If H_n denotes the Hermite polynomial of degree n, and*

$$w(x) = e^{-\frac{x^2}{2}},$$

then $u = wH_n$ satisfies the differential equation (9) with

$$f(x) = 2n + 1 - x^2.$$

Proof. Cf. (5.5.2) in [13].

2.2. The fundamental polynomials

Let us introduce the notations

$$r(x) = (x - x_0)^{\varepsilon_1}(x - x_n)^{\varepsilon_2},$$

$$q(x) = (x - x_0)^{\delta_1}(x - x_n)^{\delta_2},$$

where $\varepsilon_i, \delta_i \in \{0, 1, 2, \dots\}$ and $\varepsilon_i \geq \delta_i$ for $i = 1, 2$.

Lemma 4. *If on the system of nodes* (4) *the weight function* w *satisfies the conditions*

(10) $$w(x_i) \neq 0, \quad (qwp_{n-1})''(x_i) = 0 \qquad (i = 1, \dots, n-1),$$

then for $k = 1, \dots, n-1$ *the polynomials*

(11)
$$A_k(x) = \frac{r(x)}{r(x_k)}\, l_k^2(x) + \frac{q(x)p_{n-1}(x)}{r(x_k)p_{n-1}'(x_k)} \times \\ \times \left\{ c_k + \int_{x_0}^{x} \left[\frac{l_k'(x_k)l_k(t) - l_k'(t)}{t - x_k} \cdot \frac{r(t)}{q(t)} + a_k l_k(t) + b_k p_{n-1}(t) \right] dt \right\}$$

satisfy the weighted (0,2)-interpolational conditions

(12) $$A_k(x_i) = \delta_{i,k}, \qquad (wA_k)''(x_i) = 0 \qquad (i = 1, \dots, n-1),$$

where

(13) $$a_k = -\frac{(rw)''(x_k)}{2(qw)(x_k)} - 2l_k'(x_k)\left(\frac{r}{q}\right)'(x_k),$$

and b_k, c_k *are arbitrary constants. Furthermore, for* $k = 1, \dots, n-1$ *the polynomials*

(14) $$B_k(x) = \frac{q(x)p_{n-1}(x)}{2q(x_k)w(x_k)p_{n-1}'(x_k)} \left\{ \tilde{a}_k + \int_{x_0}^{x} \left[l_k(t) + \tilde{b}_k p_{n-1}(t) \right] dt \right\}$$

satisfy the weighted (0,2)-interpolational conditions

(15) $$B_k(x_i) = 0, \qquad (wB_k)''(x_i) = \delta_{i,k} \qquad (i = 1, \dots, n-1),$$

where $\tilde{a}_k$ *and* $\tilde{b}_k$ *are arbitrary constants.*

Proof. We are looking for $A_k \quad (k = 1, \dots, n-1)$ in the form of

$$A_k(x) = \frac{1}{r(x_k)}\Big\{r(x)l_k^2(x) + q(x)p_{n-1}(x)g_k(x)\Big\}, \tag{16}$$

where g_k is a polynomial. From (7) it is obvious, that $A_k(x_i) = \delta_{i,k}$ for $i, k = 1, \dots, n-1$. For $i \neq k$, the condition $(wA_k)''(x_i) = 0$ is equivalent to the equation

$$2{l'_k}^2(x_i)w(x_i)r(x_i) + 2w(x_i)q(x_i)p'_{n-1}(x_i)g'_k(x_i) = 0,$$

and because of $w(x_i) \neq 0$, $q(x_i) \neq 0$, and $p'_{n-1}(x_i) \neq 0$, we have

$$g'_k(x_i) = -\frac{{l'_k}^2(x_i)r(x_i)}{p'_{n-1}(x_i)q(x_i)} = \frac{-1}{p'_{n-1}(x_k)} \cdot \frac{l'_k(x_i)}{x_i - x_k} \cdot \frac{r(x_i)}{q(x_i)}.$$

This inspires us to define g'_k as

$$g'_k(x) = \frac{1}{p'_{n-1}(x_k)} \left\{ \frac{l'_k(x_k)l_k(x) - l'_k(x)}{x - x_k} \cdot \frac{r(x)}{q(x)} + a_k l_k(x) + b_k p_{n-1}(x) \right\}.$$

It is clear, that g'_k is a polynomial, and

$$g_k(x) =$$

$$= \frac{1}{p'_{n-1}(x_k)} \left\{ c_k + \int_{x_0}^{x} \left[\frac{l'_k(x_k)l_k(t) - l'_k(t)}{t - x_k} \cdot \frac{r(t)}{q(t)} + a_k l_k(t) + b_k p_{n-1}(t) \right] dt \right\},$$

furthermore

$$g'_k(x_k) = \frac{1}{p'_{n-1}(x_k)} \left\{ \frac{r(x_k)}{q(x_k)} \left[{l'_k}^2(x_k) - l''_k(x_k)\right] + a_k \right\}. \tag{17}$$

The coefficient a_k we determine from the condition $(wA_k)''(x_k) = 0$, which is equivalent to

$$\begin{aligned} (wr)''(x_k) &+ 4r(x_k)l'_k(x_k)\big[w'(x_k) + w(x_k)l'_k(x_k)\big] + \\ &+ 4w(x_k)r'(x_k)l'_k(x_k) + 2w(x_k)q(x_k)a_k = 0, \end{aligned} \tag{18}$$

where we substituted (17). On using $p_{n-1}(x_k) = 0$, from the condition (10)

$$(wp_{n-1})''(x_k) = -\frac{2q'(x_k)w(x_k)p'_{n-1}(x_k)}{q(x_k)},$$

and hence, by (8)

$$w'(x_k) + w(x_k)l_k'(x_k) = \frac{1}{2p_{n-1}'(x_k)}\Big[2p_{n-1}'(x_k)w'(x_k) + w(x_k)p_{n-1}''(x_k)\Big] =$$

$$= \frac{(wp_{n-1})''(x_k)}{2p_{n-1}'(x_k)} = -\frac{q'(x_k)}{q(x_k)}w(x_k).$$

Thus the equation (18) can be written in the form

$$(wr)''(x_k) + 4l_k'(x_k)w(x_k)\left[-r(x_k)\frac{q'(x_k)}{q(x_k)} + r'(x_k)\right] + 2(wq)(x_k)a_k = 0,$$

and we obtain (13) for a_k $(k = 1, \ldots, n-1)$.

Finally, applying (10) it is easy to verify that

$$(wB_k)''(x_i) = \frac{1}{2q(x_k)w(x_k)p_{n-1}'(x_k)} \times 2q(x_i)w(x_i)p_{n-1}'(x_i)l_k(x_i) = \delta_{i,k},$$

and $B_k(x_i) = 0$ for $i, k = 1, \ldots, n-1$.

In the next section we will determine the constants b_k, c_k, $\tilde{a}_k$ and $\tilde{b}_k$, such that, the polynomials A_k and B_k are of minimal degree and fulfil the additional interpolational conditions for different choices of q.

3. Results

Theorem 1. *For $n \geq 2$ let $\{x_i\}_{i=0}^{n}$ be a set of distinct nodes in $[a, b]$, and $p_{n-1}(x) = c(x - x_1)\ldots(x - x_{n-1})$. Let $w \in C^2(a, b)$ be a weight function. If*

$$\int_{x_0}^{x_n} p_{n-1}(t)dt \neq 0, \tag{19}$$

and

$$w(x_i) \neq 0, \qquad (wp_{n-1})''(x_i) = 0 \qquad (i = 1, \ldots, n-1), \tag{20}$$

then there exists a unique polynomial Q_n of degree at most $2n-1$, which fulfils weighted (0,2)-interpolational conditions at $x_1,\dots,x_{n-1}$ with boundary-type conditions at x_0 and x_n, that is

$$\begin{aligned} Q_n(x_i) &= y_i \qquad (i=0,1,\dots,n-1,n), \\ (wQ_n)''(x_i) &= y_i'' \qquad (i=1,\dots,n-1), \end{aligned} \tag{21}$$

where y_i, y_i'' are arbitrary real numbers.

Proof. Applying Lemma 4 with $r(x)=(x-x_0)(x-x_n)$ and $q(x)=1$, and from the condition $A_k(x_0)=0$ we obtain for $k=1,\dots,n-1$

$$\begin{aligned} A_k(x) &= \frac{(x-x_0)(x-x_n)}{(x_k-x_0)(x_k-x_n)} l_k^2(x) + \frac{p_{n-1}(x)}{(x_k-x_0)(x_k-x_n)p_{n-1}'(x_k)} \times \\ &\times \int_{x_0}^{x} \left[\frac{l_k'(x_k)l_k(t)-l_k'(t)}{t-x_k}(t-x_0)(t-x_n) + a_k l_k(t)dt + b_k p_{n-1}(t) \right] dt, \end{aligned} \tag{22}$$

where

$$a_k = -\frac{\big((x-x_0)(x-x_n)w\big)''(x_k)}{2w(x_k)} - 2l_k'(x_k)(2x_k-x_0-x_n). \tag{23}$$

From the condition $A_k(x_n)=0$ we get

$$b_k = \frac{-1}{\int_{x_0}^{x_n} p_{n-1}(t)dt} \left\{ \int_{x_0}^{x_n} \frac{l_k'(x_k)l_k(t)-l_k'(t)}{t-x_k}(t-x_0)(t-x_n)dt + a_k \int_{x_0}^{x_n} l_k(t)dt \right\}. \tag{24}$$

Now we are looking for A_0 in the form of

$$A_0(x) = p_{n-1}(x) g_0(x),$$

where g_0 is a polynomial of degree at most n. It is obvious that $A_0(x_i)=0$ $(i=1,\dots,n-1)$. For the weighted second derivative at x_i $(i=1,\dots,n-1)$ we have

$$(wA_0)''(x_i) = 2w(x_i)p_{n-1}'(x_i)g_0'(x_i) = 0,$$

hence

$$g_0'(x_i) = 0 \qquad i=1,\dots,n-1,$$

that is

$$g_0'(x) = a_0 p_{n-1}(x),$$

and it follows

$$g_0(x) = a_0 \int\limits_{x_n}^{x} p_{n-1}(t)dt + c_0.$$

From the condition $A_0(x_n) = 0$ we have $c_0 = 0$, and from $A_0(x_0) = 1$

$$a_0 = \frac{1}{p_{n-1}(x_0) \int\limits_{x_n}^{x_0} p_{n-1}(t)dt},$$

and hence

$$A_0(x) = \frac{p_{n-1}(x)}{p_{n-1}(x_0) \int\limits_{x_0}^{x_n} p_{n-1}(t)dt} \int\limits_{x}^{x_n} p_{n-1}(t)dt. \tag{25}$$

In a similar way we construct

$$A_n(x) = \frac{p_{n-1}(x)}{p_{n-1}(x_n) \int\limits_{x_0}^{x_n} p_{n-1}(t)dt} \int\limits_{x_0}^{x} p_{n-1}(t)dt. \tag{26}$$

It is obvious, that the polynomials A_k $(k = 0, 1, \ldots, n)$ are of degree at most $2n - 1$, and $A_k(x_i) = \delta_{i,k}$ for $i = 0, 1, \ldots, n$, and $(wA_k)''(x_i) = 0$ for $i = 1, \ldots, n - 1$.

Now applying (14), from the condition $B_k(x_0) = 0$ we get $\tilde{a}_k = 0$, and

$$B_k(x) = \frac{p_{n-1}(x)}{2w(x_k)p'_{n-1}(x_k)} \int\limits_{x_0}^{x} \left[l_k(t) + \tilde{b}_k p_{n-1}(t) \right] dt, \tag{27}$$

where

$$\tilde{b}_k = - \frac{\int\limits_{x_0}^{x_n} l_k(t)dt}{\int\limits_{x_0}^{x_n} p_{n-1}(t)dt} \tag{28}$$

is determined by the condition $B_k(x_n) = 0$. Hence B_k is a polynomial of degree $\leq 2n - 1$, furthermore $B_k(x_i) = 0$ $(i = 0, \ldots, n)$, and $(wB_k)''(x_i) = \delta_{i,k}$ $(i = 1, \ldots, n - 1)$.

As the polynomials A_k $(k=0,1,\dots,n)$ and B_k $(k=1,\dots,n-1)$ defined by (22)-(28) are the basis polynomials of the interpolational problem (21), the polynomial

$$Q_n(x) = \sum_{k=0}^{n} y_k\ A_k(x) + \sum_{k=1}^{n-1} y_k''\ B_k(x) \tag{29}$$

is of degree at most $2n-1$ and fulfils the equations (21).

For the proof of the uniqueness we study the homogeneous problem: Find a polynomial $\bar{R}_n$ of degree at most $2n-1$ such that $\bar{R}_n(x_i)=0$ $(i=0,1,\dots,n)$, and $(w\bar{R}_n)''(x_i)=0$ $(i=1,\dots,n-1)$. From these conditions it is obvious, that

$$\bar{R}_n(x) = (x-x_0)(x-x_n)p_{n-1}(x)\bar{g}_{n-2}(x),$$

where $\bar{g}_{n-2}$ is a polynomial of degree at most $n-2$. As for $i=1,\dots,n-1$

$$(w\bar{R}_n)''(x_i) = 2w(x_i)p'_{n-1}(x_i)[(x-x_0)(x-x_n)\bar{g}_{n-2}]'(x_i) = 0,$$

and $w(x_i)\neq 0$, $p'_{n-1}(x_i)\neq 0$, therefore with a constant $\bar{c}$

$$(x-x_0)(x-x_n)\bar{g}_{n-2}(x) = \bar{c}\int_{x_0}^{x} p_{n-1}(t)dt.$$

Substituting $x=x_n$ we get $\bar{c}\int_{x_0}^{x_n} p_{n-1}(t)dt = 0$, and hence $\bar{c}=0$, that is $R_n(x)\equiv 0$, which completes the proof.

Corollary 1. *Let the set of nodes be*

$$-1 = x_n < x_{n-1} < \dots < x_1 < x_0 = 1$$

where $\{x_i\}_{i=1}^{n-1}$ *are the roots of the Jacobi polynomial* $P_{n-1}^{(\alpha,\beta)}$ *of degree* $n-1$ $(\alpha,\beta>-1; n\geq 2)$*, and let the weight function be*

$$w(x) = (1-x)^{\frac{\alpha+1}{2}}(1+x)^{\frac{\beta+1}{2}}.$$

If

$$\int_{-1}^{1} P_{n-1}^{(\alpha,\beta)}(t)dt \neq 0,$$

then there exists a unique polynomial Q_n *of degree at most* $2n-1$*, which fulfils weighted (0,2)-interpolational conditions at the zeros of* $P_{n-1}^{(\alpha,\beta)}$ *with boundary-type conditions at* $x_0 = 1$ *and* $x_n = -1$*.*

Proof. Now $p_{n-1}(x) = P_{n-1}^{(\alpha,\beta)}(x)$. By Lemma 1 the condition (20) is satisfied.

Remark. Corollary 1 was stated and proved by L. Szili [15] in 1993.

In the special case $\alpha = \beta = -1/2$, when the inner nodes are the zeros of the Tchebyscheff polynomials of of first kind, the explicit form of the interpolational polynomial was given by S. Eneduanya [4] in 1985.

In 1994 P. Bajpai [1] studied the special case $\alpha = \beta = 1/2$, when the inner nodes are the zeros of the Tchebyscheff polynomials of second kind. He also proved convergence theorem.

In 1969 J. Prasad and A. Verma [11] studied the special case $\alpha = \beta$, they also proved convergence theorem.

Theorem 2. *For* $n \geq 2$ *let* $\{x_i\}_{i=0}^n$ *be a set of distinct nodes in* $[a,b]$*, and* $p_{n-1}(x) = c(x-x_1)\dots(x-x_{n-1})$*. Let* $w \in C^2(a,b)$ *be a weight function. If*

$$w(x_i) \neq 0, \qquad \big((x-x_n)wp_{n-1}\big)''(x_i) = 0 \qquad (i = 1,\dots,n-1), \tag{30}$$

then there exists a unique polynomial Q_n *of degree at most* $2n-1$*, which fulfils weighted (0,2)-interpolational conditions at* $x_1,\dots,x_{n-1}$ *with boundary-type conditions at* x_0 *and* x_n*.*

Proof. We apply Lemma 4 with $r(x) = (x-x_0)(x-x_n)$ and $q(x) = (x-x_n)$. In order to get the minimal degree $2n-1$ for A_k, let $b_k = 0$, and $c_k = 0$ due to the condition $A_k(x_0) = 0$. Hence we obtain for $k = 1,\dots,n-1$

$$\begin{aligned} A_k(x) =& \frac{(x-x_0)(x-x_n)}{(x_k-x_0)(x_k-x_n)} l_k^2(x) + \frac{(x-x_n)p_{n-1}(x)}{(x_k-x_0)(x_k-x_n)p_{n-1}'(x_k)} \times \\ & \times \int\limits_{x_0}^{x} \left[\frac{l_k'(x_k)l_k(t) - l_k'(t)}{t-x_k}(t-x_0) + a_k l_k(t) \right] dt, \end{aligned} \tag{31}$$

where

$$a_k = -\frac{\big((x-x_0)(x-x_n)w\big)''(x_k)}{2(x_k-x_n)w(x_k)} - 2l_k'(x_k). \tag{32}$$

Furthermore let

$$A_0(x) = \frac{(x-x_n)p_{n-1}(x)}{(x_0-x_n)p_{n-1}(x_0)}, \tag{33}$$

and

$$A_n(x) = \frac{p_{n-1}^2(x)}{p_{n-1}^2(x_n)} - \frac{(x-x_n)p_{n-1}(x)}{p_{n-1}^2(x_n)} \times \tag{34}$$
$$\times \left\{ \frac{p_{n-1}(x_0)}{x_0 - x_n} + \frac{1}{p_{n-1}(x_n)} \int_{x_0}^{x} \frac{p_{n-1}(x_n)p'_{n-1}(t) - p'_{n-1}(x_n)p_{n-1}(t)}{t - x_n} dt \right\}.$$

It is obvious that the polynomials A_k $(k = 0, 1, \ldots, n)$ are of degree at most $2n-1$, and $A_k(x_i) = \delta_{i,k}$ for $i = 0, 1, \ldots, n$, and $(wA_k)''(x_i) = 0$ for $i = 1, \ldots, n-1$.

Now applying (14), for $k = 1, \ldots, n-1$ we obtain

$$B_k(x) = \frac{(x-x_n)p_{n-1}(x)}{2w(x_k)(x_k - x_n)p'_{n-1}(x_k)} \int_{x_0}^{x} l_k(t)dt, \tag{35}$$

which is a polynomial of degree $\leq 2n-1$, and also $B_k(x_i) = 0$ $(i = 0, \ldots, n)$, and $(wB_k)''(x_i) = \delta_{i,k}$ $(i = 1, \ldots, n-1)$.

As the polynomials A_k $(k = 0, 1, \ldots, n)$ and B_k $(k = 1, \ldots, n-1)$ defined by (31) - (35) are the basis polynomials of the interpolational problem (21), the polynomial

$$Q_n(x) = \sum_{k=0}^{n} y_k \ A_k(x) + \sum_{k=1}^{n-1} y''_k \ B_k(x) \tag{36}$$

is of degree at most $2n-1$, and fulfils the equations (21). The uniqueness can be proved in a similar way as in Theorem 1.

Corollary 2. *If the nodes are*

$$-1 = x_n < x_{n-1} < ... < x_1 < x_0 = 1$$

where $\{x_i\}_{i=1}^{n-1}$ *are the roots of the Jacobi polynomial* $P_{n-1}^{(\alpha,\beta)}$ *of degree* $n-1$ $(\alpha, \beta > -1; n \geq 2)$*, and*

$$w(x) = (1-x)^{\frac{\alpha+1}{2}}(1+x)^{\frac{\beta-1}{2}}$$

is the weight function, then there exists a unique polynomial Q_n *of degree at most* $2n-1$*, which fulfils weighted (0,2)-interpolational conditions at the zeros of* $P_{n-1}^{(\alpha,\beta)}$ *with boundary-type conditions at* $x_0 = 1$ *and* $x_n = -1$.

Proof. Now $p_{n-1}(x) = P_{n-1}^{(\alpha,\beta)}(x)$. By Lemma 1 the function $(1+x)wp_{n-1}$ fulfils the conditions (30).

Theorem 3. *For $n \geq 2$ let $\{x_i\}_{i=0}^{n-1}$ be a set of distinct nodes in $[a,b]$, and $p_{n-1}(x) = c(x-x_1)\dots(x-x_{n-1})$. Let $w \in C^2(a,b)$ be a weight function. If*

$$w(x_i) \neq 0, \qquad (wp_{n-1})''(x_i) = 0 \qquad (i = 1, \dots, n-1), \tag{37}$$

then there exists a unique polynomial Q_n of degree at most $2n-1$, which fulfils weighted (0,2)-interpolational conditions at $x_1, \dots, x_{n-1}$ with initial-type conditions at x_0, that is

$$\begin{aligned} Q_n(x_i) &= y_i \qquad (i = 0, 1, \dots, n-1), \\ Q_n'(x_0) &= y_0' \\ (wQ_n)''(x_i) &= y_i'' \qquad (i = 1, \dots, n-1), \end{aligned} \tag{38}$$

where y_i, y_i'', y_0' are arbitrary real numbers.

Proof. Applying Lemma 4 with $r(x) = (x-x_0)^2$ and $q(x) = 1$, and using the condition $A_k(x_0) = 0$, we obtain for $k = 1, \dots, n-1$

$$\begin{aligned} A_k(x) = {} & \frac{(x-x_0)^2}{(x_k-x_0)^2} l_k^2(x) + \frac{p_{n-1}(x)}{(x_k-x_0)^2 p_{n-1}'(x_k)} \times \\ & \times \int_{x_0}^{x} \left[\frac{l_k'(x_k) l_k(t) - l_k'(t)}{t - x_k} (t-x_0)^2 + a_k l_k(t) + b_k p_{n-1}(t) \right] dt, \end{aligned} \tag{39}$$

where

$$a_k = -\frac{\left((x-x_0)^2 w\right)''(x_k)}{2w(x_k)} - 4l_k'(x_k)(x_k - x_0). \tag{40}$$

From the condition $A_k'(x_0) = 0$ we get

$$b_k = -\frac{a_k l_k(x_0)}{p_{n-1}(x_0)}. \tag{41}$$

Furthermore, let

$$A_0(x) = \frac{p_{n-1}(x)}{p_{n-1}(x_0)} \left\{ 1 - \frac{p_{n-1}'(x_0)}{p_{n-1}^2(x_0)} \int_{x_0}^{x} p_{n-1}(t) dt \right\}. \tag{42}$$

It is obvious that the polynomials A_k $(k = 0, 1, \ldots, n-1)$ are of degree at most $2n-1$, and $A_k(x_i) = \delta_{i,k}$ for $i = 0, 1, \ldots, n-1$, $A'_k(x_0) = 0$ and $(wA_k)''(x_i) = 0$ for $i = 1, \ldots, n-1$.

The polynomial

$$C_0(x) = \frac{p_{n-1}(x)}{p^2_{n-1}(x_0)} \int\limits_{x_0}^{x} p_{n-1}(t)dt \tag{43}$$

is also of degree at most $2n-1$, and $C_0(x_i) = 0$ for $i = 0, 1, \ldots, n-1$, $C'_0(x_0) = 1$ and $(wC_0)''(x_i) = 0$ for $i = 1, \ldots, n-1$.

Now applying (14), we have

$$B_k(x) = \frac{p_{n-1}(x)}{2w(x_k)p'_{n-1}(x_k)} \int\limits_{x_0}^{x} \Big[l_k(t) + \tilde{b}_k p_{n-1}(t) \Big] dt, \tag{44}$$

where

$$\tilde{b}_k = -\frac{l_k(x_0)}{p_{n-1}(x_0)}. \tag{45}$$

It is easy to verify, that B_k is a polynomial of degree $\leq 2n-1$, furthermore $B_k(x_i) = 0$ $(i = 0, \ldots, n-1)$, $B'_k(x_0) = 0$, and $(wB_k)''(x_i) = \delta_{i,k}$ $(i = = 1, \ldots, n-1)$.

As the polynomials A_k $(k = 0, 1, \ldots, n-1)$, B_k $(k = 1, \ldots, n-1)$ and C_0 defined by (39)-(45) are the basis polynomials of the interpolational problem (38), the polynomial

$$Q_n(x) = \sum_{k=0}^{n-1} y_k \; A_k(x) + \sum_{k=1}^{n-1} y''_k \; B_k(x) + y'_0 C_0(x) \tag{46}$$

is of degree at most $2n-1$ and fulfils the equations (38). The uniqueness can be proved in a similar way as in Theorem 1.

Remark. Theorem 3 was stated and proved by J. Balázs [3] in 1998. In [3] the basis polynomials A_k are derived in a different form.

Theorem 4. *For $n \geq 2$ let $\{x_i\}_{i=0}^{n-1}$ be a set of distinct nodes in $[a, b]$, and $p_{n-1}(x) = c(x - x_1) \ldots (x - x_{n-1})$. Let $w \in C^2(a, b)$ be a weight function. If*

$$w(x_i) \neq 0, \qquad \big((x - x_0) w p_{n-1}\big)''(x_i) = 0 \qquad (i = 1, \ldots, n-1), \tag{47}$$

then there exists a unique polynomial Q_n of degree at most $2n-1$, which fulfils weighted (0,2)-interpolational conditions at $x_1,\dots,x_{n-1}$ with initial-type conditions at x_0.

Proof. We apply Lemma 4 with $r(x)=(x-x_0)^2$ and $q(x)=(x-x_0)$. Following the steps of the proof of Theorem 3, we obtain for $k=1,\dots,n-1$

$$A_k(x)=\frac{(x-x_0)^2}{(x_k-x_0)^2}l_k^2(x)+\frac{(x-x_0)p_{n-1}(x)}{(x_k-x_0)^2p'_{n-1}(x_k)}\times \times\int_{x_0}^{x}\left[\frac{l'_k(x_k)l_k(t)-l'_k(t)}{t-x_k}(t-x_0)+a_kl_k(t)\right]dt, \tag{48}$$

where

$$a_k=-\frac{\big((x-x_0)^2w\big)''(x_k)}{2(x_k-x_0)w(x_k)}-2l'_k(x_k); \tag{49}$$

$$A_0(x)=\frac{p_{n-1}^2(x)}{p_{n-1}^2(x_0)}-\frac{(x-x_0)p_{n-1}(x)}{p_{n-1}^2(x_0)}\times \times\left\{2p'_{n-1}(x_0)+\frac{1}{p_{n-1}(x_0)}\int_{x_0}^{x}\frac{p_{n-1}(x_0)p'_{n-1}(t)-p'_{n-1}(x_0)p_{n-1}(t)}{t-x_0}dt\right\}, \tag{50}$$

$$C_0(x)=\frac{(x-x_0)p_{n-1}(x)}{p_{n-1}(x_0)}, \tag{51}$$

and

$$B_k(x)=\frac{(x-x_0)p_{n-1}(x)}{2w(x_k)(x_k-x_0)p'_{n-1}(x_k)}\int_{x_0}^{x}l_k(t)dt. \tag{52}$$

As the polynomials A_k $(k=0,1,\dots,n)$, B_k $(k=1,\dots,n-1)$ and C_0, defined by (48)-(52) are the basis polynomials of the interpolational problem (38), the polynomial

$$Q_n(x)=\sum_{k=0}^{n-1}y_k\ A_k(x)+\sum_{k=1}^{n-1}y''_k\ B_k(x)+y'_0C_0(x) \tag{53}$$

is of degree at most $2n-1$ and fulfils the equations (38). The uniqueness can be proved in a similar way as in Theorem 1.

Corollary 3. *If the nodes are*

$$0 = x_0 < x_1 < \ldots < x_{n-1},$$

where $\{x_i\}_{i=1}^{n-1}$ *are the roots of the Laguerre polynomial* $L_{n-1}^{(\alpha)}$ *of degree* $n-1$ *(*$\alpha > -1; n \geq 2$*), and*

$$w_1(x) = e^{-\frac{x}{2}} x^{\frac{\alpha+1}{2}}$$

or

$$w_2(x) = e^{-\frac{x}{2}} x^{\frac{\alpha-1}{2}}$$

are weight functions, then there exists a unique polynomial Q_n *of degree at most* $2n-1$*, which fulfils weighted (0,2)-interpolational conditions at the zeros of* $L_{n-1}^{(\alpha)}$ *with initial-type conditions at* $x_0 = 0$*.*

Proof. Let $p_{n-1}(x) = L_{n-1}^{(\alpha,\beta)}(x)$. By Lemma 2 the conditions (37) and (47) are satisfied with the weight functions w_1 and w_2, respectively.

Theorem 5. *For* $n \geq 2$ *let* $\{x_i\}_{i=0}^{n}$ *be a set of distinct nodes in* $[a,b]$*, and* $p_{n-1}(x) = c(x-x_1)\ldots(x-x_{n-1})$*. Let* $w \in C^2(a,b)$ *be a weight function. If*

$$w(x_i) \neq 0, \qquad (wp_{n-1})''(x_i) = 0 \qquad (i = 1, \ldots, n-1), \tag{54}$$

and

$$\begin{aligned} &p'_{n-1}(x_0)p'_{n-1}(x_n)\int_{x_0}^{x_n} p_{n-1}(t)dt+ \\ &\qquad +p'_{n-1}(x_0)p^2_{n-1}(x_n) - p'_{n-1}(x_n)p^2_{n-1}(x_0) \neq 0, \end{aligned} \tag{55}$$

then there exists a unique polynomial Q_n *of degree at most* $2n-1$*, which fulfils weighted (0,2)-interpolational conditions at* $x_1, \ldots, x_{n-1}$ *with additional interpolatory conditions at* x_0 *and* x_n*, that is*

$$\begin{aligned} &Q_n(x_i) = y_i, \qquad (wQ_n)''(x_i) = y_i'', \qquad (i = 1, \ldots, n-1), \\ &Q'_n(x_0) = y'_0, \qquad Q'_n(x_n) = y'_n, \end{aligned} \tag{56}$$

where y_i, y_i'', y_0' *and* y_n' *are arbitrary real numbers.*

Proof. Applying Lemma 4 with $r(x) = (x - x_0)^2(x - x_n)$ and $q(x) = 1$, we obtain for $k = 1, \dots, n-1$

(57)
$$A_k(x) = \frac{(x - x_0)^2(x - x_n)}{(x_k - x_0)^2(x_k - x_n)} l_k^2(x) + \frac{p_{n-1}(x)}{(x_k - x_0)^2(x_k - x_n)p'_{n-1}(x_k)} \times$$
$$\times \left\{ c_k + \int_{x_0}^{x} \left[\frac{l'_k(x_k)l_k(t) - l'_k(t)}{t - x_k}(t - x_0)^2(t - x_n) + a_k l_k(t) + b_k p_{n-1}(t) \right] dt \right\},$$

where

$$a_k = -\frac{\left((x - x_0)^2(x - x_n)w\right)''(x_k)}{2w(x_k)} - 2l'_k(x_k)(x_k - x_0)(3x_k - 2x_n - x_0). \tag{58}$$

The equations $A'_k(x_0) = 0$ and $A'_k(x_n) = 0$ are equivalent to the linear system

(59)
$$p'_{n-1}(x_0)c_k + p^2_{n-1}(x_0)b_k = -a_k p'_{n-1}(x_0)l_k(x_0),$$
$$p'_{n-1}(x_n)c_k + \left[p'_{n-1}(x_n) \int_{x_0}^{x_n} p_{n-1}(t)dt + p^2_{n-1}(x_n) \right] b_k =$$
$$= -p'_{n-1}(x_n) \left\{ (x_n - x_0)^2 l_k^2(x_0) + \right.$$
$$\left. + \int_{x_0}^{x_n} \left[\frac{l'_k(x_k)l_k(t) - l'_k(t)}{t - x_k}(t - x_0)^2(t - x_n) + a_k l_k(t) \right] dt \right\} -$$
$$-a_k p_{n-1}(x_n)l_k(x_n),$$

which has unique solution for b_k and c_k if and only if its determinant is not 0, that is the condition (55) is fulfilled.

From (14) we have

$$B_k(x) = \frac{p_{n-1}(x)}{2w(x_k)p'_{n-1}(x_k)} \left\{ \tilde{a}_k + \int_{x_0}^{x} \left[l_k(t) + \tilde{b}_k p_{n-1}(t) \right] dt \right\}, \tag{60}$$

where the constants $\tilde{a}_k$ and $\tilde{b}_k$ are determined from the equations

$$B'_k(x_0) = 0, \qquad B'_k(x_n) = 0,$$

that is, from the linear system

(61)
$$\begin{aligned} p'_{n-1}(x_0)\tilde{a}_k + p^2_{n-1}(x_0)\tilde{b}_k &= -p'_{n-1}(x_0)l_k(x_0), \\ p'_{n-1}(x_n)\tilde{a}_k + \Big[p'_{n-1}(x_n)\int\limits_{x_0}^{x_n} p_{n-1}(t)dt + p^2_{n-1}(x_n)\Big]\tilde{b}_k &= -p_{n-1}(x_n)l_k(x_n) - \\ &\quad - \int\limits_{x_0}^{x_n} l_k(t)dt, \end{aligned}$$

which has unique solution for $\tilde{a}_k$ and $\tilde{b}_k$ if and only if the condition (55) is fulfilled.

Furthermore, the polynomial

$$C_0(x) = p_{n-1}(x)\left\{c_0\int\limits_{x_0}^{x} p_{n-1}(t)dt + d_0\right\} \tag{62}$$

fulfils the conditions

$$\begin{aligned} &C_0(x_i) = 0, \qquad (wC_0)''(x_i) = 0, \qquad (i = 1,\dots,n-1), \\ &C'_0(x_0) = 1, \qquad C'_0(x_n) = 0, \end{aligned}$$

where the constants c_0 and d_0 are the unique solutions of the linear system

$$\begin{aligned} p'_{n-1}(x_0)d_0 + p^2_{n-1}(x_0)c_0 &= 1, \\ p'_{n-1}(x_n)d_0 + \left[p'_{n-1}(x_n)\int\limits_{x_0}^{x_n} p_{n-1}(t)dt + p^2_{n-1}(x_n)\right]c_0 &= 0, \end{aligned} \tag{63}$$

if and only if the condition (55) is fulfilled.

In a similar way we obtain, that the polynomial

$$C_n(x) = p_{n-1}(x)\left\{c_n\int\limits_{x_0}^{x} p_{n-1}(t)dt + d_n\right\} \tag{64}$$

fulfils the conditions

$$\begin{aligned} &C_n(x_i) = 0, \qquad (wC_n)''(x_i) = 0 \qquad (i = 1,\dots,n-1), \\ &C'_n(x_0) = 0, \qquad C'_n(x_n) = 1, \end{aligned}$$

where the constants c_n and d_n are the unique solutions of the linear system

$$p'_{n-1}(x_0)d_n + p^2_{n-1}(x_0)c_n = 0,$$

(65)
$$p'_{n-1}(x_n)d_n + \left[p'_{n-1}(x_n)\int_{x_0}^{x_n} p_{n-1}(t)dt + p^2_{n-1}(x_n)\right] c_n = 1,$$

if and only if the condition (55) is fulfilled.

As the polynomials A_k, B_k $(k = 1, \dots, n-1)$, C_0 and C_n defined by (57)-(65) are the basis polynomials of the interpolational problem (56), the polynomial

(66)
$$Q_n(x) = \sum_{k=1}^{n-1} y_k \; A_k(x) + \sum_{k=1}^{n-1} y''_k \; B_k(x) + y'_0 C_0(x) + y'_n C_n(x)$$

is of degree at most $2n-1$ and fulfils the equations (56). The uniqueness can be proved in a similar way as in Theorem 1.

Corollary 4. *On* $[-1, 1]$, *if the weight function is*

$$w(x) = (1-x^2)^{\frac{\alpha+1}{2}},$$

then for odd n *there exists a unique polynomial* Q_n *of degree at most* $2n-1$, *which fulfils weighted (0,2)-interpolational conditions at the zeros of* $P^{(\alpha,\beta)}_{n-1}$ *with additional interpolatory conditions for the first derivative at* $x_0 = 1$ *and* $x_n = -1$.

Proof. Let $p_{n-1}(x) = P^{(\alpha)}_{n-1}(x)$. By Lemma 1 the function wp_{n-1} fulfils the conditions (54). For odd n the polynomial $P^{(\alpha)}_{n-1}$ is even function, and using $P^{(\alpha)}_{n-1}(1) = \binom{n-1+\alpha}{n-1}$ and $P^{(\alpha)'}_{n-1}(x) = \frac{1}{2}(n+2\alpha)P^{(\alpha+1)}_{n-2}(x)$, one can verify (55).

References

[1] **Bajpai P.,** Weighted (0,2) interpolation on the extended Tchebycheff nodes of second kind, *Acta Math. Hung.,* **63** (2) (1994), 167-181.

[2] **Balázs J.,** Súlyozott $(0,2)$-interpoláció ultraszférikus polinom gyökein, *MTA III. Oszt. Közl.,* **11** (1961), 305-338. (Weighted $(0,2)$-interpolation on the zeros of the ultraspherical polynomials (in Hungarian))

[3] **Balázs J.,** Modified weighted (0,2) interpolation, *Approximation Theory. In memory of A. K. Varma,* Pure Appl. Math. **212**, ed. N.K. Govil et al., Marcel Dekker, New York, 1998, 61-85.

[4] **Eneduanya S.A.N.,** The weighted (0,2) interpolation I, *Demonstracio Mathematica*, **18** (1) (1985), 9-13.

[5] **Joó I.,** On weighted $(0,2)$ interpolation, *Annales Univ. Sci. Budapest., Sect. Math.*, **38** (1995), 185-222.

[6] **Joó I. and Szili L.,** On weighted $(0,2)$-interpolation on the roots of Jacobi polynomials, *Acta Math. Hung.*, **66** (1-2) (1995), 25-50.

[7] **Prasad J.,** Some interpolatory polynomials on Hermite abscissas, *Math. Japonica*, **12** (1967), 73-80.

[8] **Prasad J.,** Balázs-type interpolation on Laguerre abscissas, *Math. Japonica,* **13** (1967), 47-53.

[9] **Prasad J.,** On the weighted $(0,2)$-interpolation, *SIAM J. Numer. Anal.,* **7** (1970), 428-446.

[10] **Prasad J.,** On the representation of functions by interpolatory polynomials, *Mathematica (Cluj),* **15** (1973), 289-305.

[11] **Prasad J. and Verma A.,** An analogue problem of J. Balázs and P. Turán III., *Math. Japonica*, **14** (1969), 85-99.

[12] **Surányi J. and Turán P.,** Notes on interpolation I. On some interpolatorical properties of the ultraspherical polynomials, *Acta Math. Acad. Sci. Hung.,* **6** (1955), 66-79.

[13] **Szegő G.,** *Orthogonal polynomials,* Amer. Math. Soc. Coll. Publ. **23**, New York, 1939. (Fourth edition in 1975)

[14] **Szili L.,** Weighted $(0,2)$-interpolation on the roots of Hermite polynomials, *Annales Univ. Sci. Budapest., Sect. Math.,* **27** (1985), 153-166.

[15] **Szili L.,** Weighted $(0,2)$ interpolation on the roots of the classical orthogonal polynomials, *Bull. Allahabad Math. Soc.,* **8-9** (1993-94), 111-120.

[16] **Szili L.,** A survey on (0,2) interpolation, *Annales Univ. Sci. Budapest., Sect. Comp.,* **16** (1996), 377-390.

(Received May 21, 2004)

M. Lénárd
Department of Mathematics and Computer Science
P.O.B. 5969
13060 Safat, Kuwait
lenard@mcc.sci.kuniv.edu.kw