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THE ASYMPTOTIC BEHAVIOR OF THE RENEWAL PROCESS CONSTRUCTED FROM A RANDOM WALK WITH A RESTRICTED MULTIDIMENSIONAL TIME DOMAIN

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Dedicated to Academician Imre Kátai
on the occasion of his 65th birthday

Abstract. We consider the renewal process defined for a random walk whose time domain is a subset of the d-dimensional time space. Our main result provides the asymptotic behavior of the renewal process as $t \to \infty$. In general, the problem considered in this paper corresponds to the classical setting where the asymptotic behavior of the renewal process is studied for a random walk considered only at a subsequence of indices.

1. Some facts about the classical renewal theory

Let $\{X_n, n \geq 1\}$ be independent identically distributed random variables, and put $S_n = \sum_{k=1}^{n} X_k$. The sequence $\{S_n, n \geq 1\}$ is called a *random walk*. The renewal process N is defined by the random walk as follows:

$$N(t) = \max\{n: S_n < t\}, \qquad t > 0. \tag{1}$$

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The process $N(t)$ is well defined if $\mathbf{P}(\Omega') = 1$, where the random event Ω' is defined by

$$\Omega' = \left\{\omega \in \Omega\colon \lim_{n\to\infty} S_n(\omega) = \infty\right\}. \tag{2}$$

In this case $N(t)$ is finite for $\omega \in \Omega'$, and one can put, for example, $N(t) = 0$ for $\omega \notin \Omega'$. Note that condition (2) is satisfied if, for instance, random variables X_n are nonnegative and their expectation $\mu = \mathbf{E}X_n$ is positive. The latter assumption is common for the classical renewal theory whose classical problems are to find the asymptotic behavior of both the renewal process $N(t)$ and its expectation $U(t) = \mathbf{E}N(t)$ (called the renewal function) as $t \to \infty$. One can easily show that $U(t)$ is finite for all $t > 0$, if the random variables $\{X_n, n \ge 1\}$ are nonnegative and nondegenerate.

The so called *renewal theorem* asserts that if the random variables X_n are nonnegative and

$$0 < \mu = \mathbf{E}X_n < \infty, \tag{3}$$

then

$$\lim_{t\to\infty} \frac{U(t)}{t} = \frac{1}{\mu}.$$

The dual result for the renewal process holds under the same condition (3), namely

$$\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu} \qquad a.s.$$

(here and in what follows "a.s." stands for "almost surely").

Another representation for the renewal process and renewal function is often useful. It is clear that

$$N(t) = \sum_{n=1}^{\infty} \sharp\{S_n < t\} \tag{4}$$

(here and in what follows $\sharp$ denotes the indicator function of a random event written in brackets after it). Taking its expectation we prove that

$$U(t) = \sum_{n=1}^{\infty} \mathbf{P}(S_n < t).$$

Along with the function U, its "density" u is considered in the renewal theory. Assume that the random variables $\{X_n, n \geq 1\}$ are integer valued. Then the following function

$$u(t) = \sum_{n=1}^{\infty} \mathbf{P}(S_n = t)$$

is well defined for integer values of t (for other t it vanishes). The well known renewal theorem of Erdős, Feller, and Pollard [2] asserts that

$$\lim_{t\to\infty} u(t) = \frac{1}{\mu}$$

provided random variables $\{X_n, n \geq 1\}$ are aperiodic and condition (3) holds.

In general, results for u imply their counterparts for U. However the conditions are often non-optimal when one derives results for U from those from u. Moreover the case of non-integer valued random variables cannot be treated in this way.

2. Some results for multiple sums

There are various generalizations of these three results above. We are concerned with the case where multiple sums (instead of cumulative sums S_n) are used to define the process N. First we present some notation and results for multiple sums of random variables.

Let $\mathbb{N}^d$ be the space of vectors with d positive integer coordinates. Elements of $\mathbb{N}^d$ are denoted by $\overline{k}$, $\overline{n}$, etc. Consider a family $\{X(\overline{n}), \overline{n} \in \mathbb{N}^d\}$ of independent, identically distributed random variables and their multiple (rectangular) sums

$$S(\overline{n}) = \sum_{\overline{k} \prec \overline{n}} X(\overline{k}),$$

where "$\prec$" is the coordinate-wise (partial) ordering, meaning $k_1 \leq n_1, \ldots, k_d \leq$ $\leq n_d$ for $\overline{k} = (k_1, \ldots, k_d)$ and $\overline{n} = (n_1, \ldots, n_d)$. By analogy with the case $d = 1$, the family $\{S(\overline{n}), \overline{n} \in \mathbb{N}^d\}$ is called a *random walk in multidimensional time*.

Some properties of random walks in multidimensional time are immediate consequences of their counterparts for the classical random walk. For example, if the expectation $\mu = \mathbf{E}X_n$ exists, then

$$\frac{S(\overline{n})}{|\overline{n}|} \xleftarrow{\mathbf{P}} \mu \quad \text{as} \quad |\overline{n}| \to \infty,$$

where $\overline{n} = (n_1, \ldots, n_d)$ and $|\overline{n}| = n_1 \cdots n_d$. The proof is obvious and makes use of the classical law of large numbers.

Some other properties of random walks in multidimensional time are not so clear and immediate. Below we will apply the following Smythe [13] strong law of large numbers. In what follows we use a random variable with the same distribution as other random variables $X(\overline{n})$, and put $\ln^+ z = \ln(1+z)$ for $z \geq 0$.

Theorem 1. *Let $\{X(\overline{n}), \overline{n} \in \mathbb{N}^d\}$ be independent identically distributed random variables such that*

$$\mathbf{E}X = \mu \qquad \textit{exists} \tag{5}$$

and

$$\mathbf{E}|X| \left(\ln^+ |X|\right)^{d-1} < \infty. \tag{6}$$

Then

$$\mathbf{P}\left(\left|\frac{S(\overline{n})}{|\overline{n}|} - \mu\right| \geq \varepsilon \ i.o.\right) = 0 \tag{7}$$

for all $\varepsilon > 0$ ("i.o." is the abbreviation of "infinitely often").

Note that (7) $\Longleftrightarrow$ (5)–(6) (see [13]).

Several extensions of this result are known in the literature. We mention the following one because it is heavily related to the subject of this paper (see Indlekofer and Klesov [7]) .

Theorem 2. *Let $\{X(\overline{n}), \overline{n} \in \mathbb{N}^d\}$ be independent identically distributed random variables and let $D \subseteq \mathbb{N}^d$. The relation*

$$\mathbf{P}\left(\left|\frac{S(\overline{n})}{|\overline{n}|} - \mu\right| \geq \varepsilon \ i.o. \ for \ \overline{n} \in D\right) = 0 \tag{8}$$

holds if and only if (5) holds and

$$\sum_{\overline{n} \in D} \mathbf{P}(|X| \geq |\overline{n}|) < \infty. \tag{9}$$

Theorems 2 and 1 coincide in the case of $D = \mathbb{N}^d$, since (9) $\Longleftrightarrow$ (6) in this case.

We should also like to mention that Theorem 2 is proved in [7] only for the case of $d = 2$ and a special choice of the domains D described in Conjecture 1. In this respect its formulation above can be treated as a conjecture in the case of a general domain D. The authors hope that it holds for all $d \geq 2$ and for the obvious generalization of domains D to the case of $d \geq 3$.

3. Renewal function in multidimensional time

In the case of $d = 2$, Ney and Wainger [12] consider the function

$$u_d(t) = \sum_{\overline{n} \in \mathbb{N}^d} \mathbf{P}(S(\overline{n}) = t)$$

for integer t and for integer valued random variables $X(\overline{n})$. They called it the *renewal sequence constructed by a random walk in a multidimensional time*. The name for u_d is clear in view of the analogy with the case $d = 1$, although there is no "renewal" process behind it if $d > 1$. The problem for $d > 1$ mimics the one for $d = 1$, namely it is to investigate the asymptotic behavior of $u_d(t)$ as $t \to \infty$ (t integer).

Ney and Wainger [12] realized that the behavior of u_d for $d > 1$ is different of what is seen in the case of $d = 1$. Say, u_d is no more bounded. They also mention that the classical method of a difference equation satisfied by $u(t)$ does not work for $d > 1$, since "... there does not appear to be a natural analog of this equation in dimension two, mainly because the lattice points of the plain are not linearly ordered under the natural order". The same, of course, is true for higher dimensions. Using Tauberian methods Ney and Wainger [12] nevertheless were able to give the asymptotics for both u_d and

$$U_d(t) = \sum_{\overline{n} \in \mathbb{N}^d} \mathbf{P}(S(\overline{n}) < t)$$

for $d = 2$ and under some additional conditions (as it became clear after further investigations their conditions are too restrictive).

The conditions of Ney and Wainger [12] are weakened in Maejima and Mori [11]. Moreover Maejima and Mori [11] consider the case of general d. It is also true that their conditions work effectively only for $d = 2, 3$. However they mention that "... our results might be true for $d \geq 4$ if an order estimate in the divisor problem is improved for such d". Note also that the methods of the proof in Maejima and Mori [11] are the same as in Ney and Wainger [12], except for better estimates of the rate of convergence in the local central limit theorem which allow them to relax conditions of Ney and Wainger [12].

Maejima and Mori [11] were perhaps the first to mention explicitly the relationship between the Dirichlet divisors problem in number theory and limit theorems for multiple sums in probability theory (see [6] for other examples of such relationships). One can say even more, namely that any improvement in the Dirichlet divisors problem will result in an improvement in the renewal theorem for multiple sums.

Further developments of the topic are due to Galambos and Kátai [4]–[5]. They used a better estimate in the local central limit theorem and obtained a further sharpening of results in Maejima and Mori [11], however they understand that "... lack of knowledge in number theory imposes limitations on our results ...".

The authors of all the papers mentioned above tried to present an explicit asymptotics of u_d in the form

$$\lim_{t\to\infty} \frac{u_d(t)}{(\ln t)^{d-1}} = \frac{1}{\mu(d-1)!}$$

(t is integer). However this result can be achieved only for $d \le 3$ at the present time. We stress once more that this is because the "expected" rate of decay of the remainder term in the Dirichlet divisors problem is not proved yet.

It is clear that the behavior of u_d depends on the distribution of terms $X(\overline{n})$, although asymptotically it becomes the same irrespective of a distribution from a certain (wide) class. To highlight the dependence of u_d on the distribution F of terms $X(\overline{n})$ we even will sometimes write $u_{d,F}$ rather than u_d.

In Galambos, Indlekofer and Kátai [3] a successful attempt was made to describe the asymptotics of $u_{d,F}$ via the asymptotics of $u_{d,\Phi}$, where Φ is the standard Gaussian distribution:

$$u_{d,F}(t) = u_{d,\Phi}(t) + o\left((\ln t)^{d-1}\right), \qquad t \to \infty$$

(t is integer). The idea of approximation in terms of the Gaussian distribution is common in probability theory. In the context of the renewal theorem in multidimensional time, it not only made it possible to relax conditions up to the existence of the second moment of $X(\overline{n})$ but also worked for all $d \ge 1$. It is also worthwhile to mention that the asymptotics of $u_{d,\Phi}$ still depends on the asymptotics of the remainder term in the Dirichlet divisors problem.

In contrast to u_d, the asymptotic behavior of U_d can be obtained for all $d \ge 1$. It is shown in Klesov [8] that

$$\lim_{t\to\infty} \frac{U_d(t)}{(\ln t)^{d-1}} = \frac{1}{\mu(d-1)!}$$

provided the first moment exists and is positive. The direct probabilistic methods developed in Klesov [8] allow one to prove the result for all $d \ge 1$. A sharpening of this result (also proved in Klesov [8]) reads as follows: *there is a polynomial $\mathcal{P}$ of degree $d-1$ such that*

$$\lim_{t\to\infty} \left[\frac{U_d(t)}{t} - \frac{1}{\mu}\mathcal{P}\left(\ln\frac{t}{\mu}\right)\right] = 0. \tag{10}$$

The polynomial $\mathcal{P}$ is strongly related to the polynomial in the decomposition of the number of divisors in the Dirichlet problem; its leading coefficient obviously is $\dfrac{1}{(d-1)!}$. The condition on terms $X(\overline{n})$ imposed in [8] is

$$t(\log t)^{2(d-1)}\mathbf{P}(X \ge t) \to 0, \qquad t\to\infty,$$

which is much weaker than the existence of the second moment. Moreover, in the case of $d=1$ the latter condition is even weaker that the main assumption (3).

The approach in Klesov [8] differs from those in the preceding papers. It is based on direct probabilistic methods that allow one to reduce the problem to the law of large numbers for original sums $S(\overline{n})$.

A closed problem is considered in Klesov and Steinebach [10]. Namely let $D \subseteq \mathbb{N}^d$ and

$$U_D(t) = \sum_{\overline{n}\in D} \mathbf{P}(S(\overline{n}) < t).$$

Put

$$A_D(t) = \operatorname{card}\{\overline{n}\in D\colon\ |\overline{n}| \le t\}. \tag{11}$$

Then under appropriate conditions on $X(\overline{n})$ and on the function A

$$\lim_{t\to\infty} \frac{U_D(t)}{A_D(t/\mu)} = 1. \tag{12}$$

The proof in Klesov and Steinebach [10] is completely "probabilistic" in the sense that (12) is derived from the law of large numbers for sums $S(\overline{n})$ (similarly to Klesov [8]) and some ideas from Indlekofer and Klesov [7].

4. Renewal process in multidimensional time

Up to now we were talking about the renewal function U_d and its density u_d. Nothing was said about the renewal process itself. The reason is that relation (1) is meaningless for $d > 1$, since the set $\mathbb{N}^d$ is not linearly ordered with respect to the natural order, and thus the definition of the renewal process is not straightforward if $d > 1$. Relation (4) is helpful in this respect and the representation of N via the sum of indicator functions serves as the definition for all $d \geq 1$:

$$N_d(t) = \operatorname{card}\{\overline{n}: S(\overline{n}) < t\} = \sum_{\overline{n}\in\mathbb{N}^d} \sharp\{S(\overline{n}) < t\}.$$

The asymptotics of N_d defined in this way is studied in Klesov and Steinebach [9]. Again the proof there is based on an idea of reducing the problem to known results in probability theory, namely to the strong law of large numbers for multiple sums (Theorem 1 above). The expansion like (10) is also obtained in Klesov and Steinebach [9], however it holds for $d = 2$ and $d = 3$, while for $d > 3$ it holds only under a conjecture on a underlying rate of approximation in the Dirichlet divisors problem.

In this paper we consider the renewal process in the setting similar to Theorem 2 and relation (12). Namely let $D \subseteq \mathbb{N}^d$ and put

$$N_D(t) = \operatorname{card}\{\overline{n} \in D: S(\overline{n}) < t\} = \sum_{\overline{n}\in D} \sharp\{S(\overline{n}) < t\}. \tag{13}$$

Since there will be no confusion in notation we will omit the subscript D and write $N(t)$ rather than $N_D(t)$. The result below holds for all $d \geq 1$ and all domains D for which the function A_D defined by (11) is *pseudo regularly varying* (see defining property below) The subscript D is also omitted for the function A.

5. Main result

Now we are ready to state the main result.

Theorem 3. *Assume that* $\{X(\overline{n}), \overline{n} \in \mathbb{N}^d\}$ *are nonnegative independent identically distributed random variables such that relations (5) and (6) hold and* $\mu > 0$. *Let* $D \subseteq \mathbb{N}^d$ *be an infinite domain of* $\mathbb{N}^d$ *for which*

$$\lim_{c \downarrow 1} \limsup_{t \to \infty} \frac{A(ct)}{A(t)} = 1, \tag{14}$$

where the function A *is defined by (11) (we omit the subscript* D*). Then*

$$\lim_{t \to \infty} \frac{N(t)}{A(t/\mu)} = 1 \qquad a.s.$$

where the process N *is defined by (13) (we omit the subscript* D*).*

The following result is obtained in [9].

Corollary. *Assume that* $\{X(\overline{n}), \overline{n} \in \mathbb{N}^d\}$ *are nonnegative independent identically distributed random variables such that relations (5) and (6) hold and* $\mu > 0$. *If* $D = \mathbb{N}^d$, *then*

$$\lim_{t \to \infty} \frac{N(t)}{t(\ln t)^{d-1}} = \frac{1}{\mu(d-1)!} \qquad a.s.$$

The corollary follows immediately from Theorem 3, since

$$A(t) \sim \frac{1}{(d-1)!} t(\ln t)^{d-1}, \qquad t \to \infty.$$

The latter is a rough estimate in the Dirichlet divisors problem. Indeed, $A(t)$ represents in this case the number of solutions of the inequality $|\overline{n}| \le t$ in positive integer numbers. Several proofs of this results are known in the literature; an elementary proof is given in [10].

6. Proof

First we consider the case of $\mu = 1$. Fix $0 < \varepsilon < 1$ and define three processes

$$N_1(t) = \sum_{\substack{\overline{n} \in D; \\ |\overline{n}| < (1-\varepsilon)t}} \sharp\{S(\overline{n}) < t\},$$

$$N_2(t) = \sum_{\substack{\overline{n} \in D; \\ (1-\varepsilon)t \le |\overline{n}| < (1+\varepsilon)t}} \sharp\{S(\overline{n}) < t\},$$

$$N_3(t) = \sum_{\substack{\overline{n} \in D; \\ |\overline{n}| \ge (1+\varepsilon)t}} \sharp\{S(\overline{n}) < t\}.$$

Below we obtain necessary estimates for the processes N_1, N_2, and N_3.

Case of N_1. We have (recall that $\mu = 1$)

$$|N_1(t) - A((1-\varepsilon)t)| = \sum_{\substack{\overline{n}\in D;\\ |\overline{n}|<(1-\varepsilon)t}} \sharp\{S(\overline{n}) \geq t\} \leq \sum_{\substack{\overline{n}\in D;\\ |\overline{n}|<(1-\varepsilon)t}} \sharp\left\{\frac{S(\overline{n})}{|\overline{n}|} - \mu \geq \varepsilon_1\right\},$$

where

$$\varepsilon_1 = \frac{\varepsilon}{1-\varepsilon}.$$

Let

$$\xi(\varepsilon,\omega) = \operatorname{card}\left\{\overline{n} \in D\colon \left|\frac{S(\overline{n},\omega)}{|\overline{n}|} - \mu\right| \geq \varepsilon\right\}$$

for $\varepsilon > 0$. We finally obtain

$$|N_1(t) - A((1-\varepsilon)t)| \leq \xi(\varepsilon_1). \tag{15}$$

Case of N_2. Since the indicator function does not exceed 1,

$$N_2(t) \leq A((1+\varepsilon)t - A((1-\varepsilon)t). \tag{16}$$

Case of N_3. Put

$$\varepsilon_2 = \frac{\varepsilon}{1+\varepsilon}.$$

Then

$$N_3(t) \geq \sum_{\substack{\overline{n}\in D\\ |\overline{n}|\geq(1+\varepsilon)t}} \left\{\frac{S(\overline{n})}{|\overline{n}|} - \mu \leq -\varepsilon_2\right\} \leq \xi(\varepsilon_2). \tag{17}$$

Now we combine estimates (15), (16) and (17):

$$N(t) \leq |N_1(t) - A((1-\varepsilon)t)| + A((1-\varepsilon)t) + N_2(t) + N_3(t) \leq$$

$$\leq \xi(\varepsilon_1) + A((1+\varepsilon)t) + \xi(\varepsilon_2).$$

By Theorem 1 there are random events Ω_1 and Ω_2 such that $\mathbf{P}(\Omega_1) = \mathbf{P}(\Omega_2) = 1$ and $\xi(\varepsilon_1,\omega)$ is finite for $\omega \in \Omega_1$, while $\xi(\varepsilon_2,\omega)$ is finite for $\omega \in \Omega_2$. Therefore for all $\omega \in \Omega_1 \cap \Omega_2$

$$\limsup_{t\to\infty} \frac{N(t,\omega)}{A(t/\mu)} \leq \limsup_{t\to\infty} \frac{A((1+\varepsilon)t/\mu)}{A(t/\mu)},$$

whence for all $\omega \in \Omega_1 \cap \Omega_2$

$$\limsup_{t\to\infty} \frac{N(t,\omega)}{A(t/\mu)} \le \lim_{\varepsilon\downarrow 0} \limsup_{t\to\infty} \frac{A((1+\varepsilon)t/\mu)}{A(t/\mu)} = 1 \tag{18}$$

by condition (14).

Similarly we obtain the lower bound

$$N(t) \ge$$

$$\ge -|N_1(t) - A((1-\varepsilon)t)| + A((1-\varepsilon)t) + N_2(t) + N_3(t) \ge -\xi(\varepsilon_1) + A((1-\varepsilon)t).$$

Again for all $\omega \in \Omega_1$

$$\liminf_{t\to\infty} \frac{N(t,\omega)}{A(t/\mu)} \ge \lim_{\varepsilon\downarrow 0} \liminf_{t\to\infty} \frac{A((1-\varepsilon)t/\mu)}{A(t/\mu)} = 1$$

by condition (14). The latter estimate together with (18) completes the proof of the theorem in the case of $\mu = 1$.

In the case of a general μ, we introduce random variables $X_1(\overline{n}) = X(\overline{n})/\mu$, their rectangular sums $S_1(\overline{n})$, and the corresponding renewal process $N_1(t)$. Obviously $\mathbf{E}X_1(\overline{n}) = 1$ and $N_1(t) = N(t\mu)$. Thus applying the part of the theorem proved in the case of the unit expectation we get

$$\lim_{t\to\infty} \frac{N(t\mu)}{A(t)} = 1 \qquad a.s.$$

which is equivalent to what had to be proved.

7. Concluding remarks

Remark 1. We used condition (6) in the proof of the theorem. It is seen from the proof that this condition is optimal if $D = \mathbb{N}^d$, that is in the case of the above corollary. For other sets D it may be too restrictive, however one can improve it in many "regular" cases (see [7] for more details).

Remark 2. Functions satisfying condition (14) are called pseudo regularly varying in [1], where one can also find other properties of those functions similar to those that regularly varying functions have. An example of pseudo regularly varying functions is presented by Karamata's regularly varying functions. On the other hand, there are non-regularly varying functions that satisfy condition (14), say

$$A(t) = \begin{cases} 0, & \text{for } t = 0, \\ t \exp\{\sin(\ln t)\}, & \text{for } t > 0. \end{cases}$$

A less "exotic" example of a non-regularly varying function being nevertheless a pseudo regularly varying function is given by

$$A(t) = \begin{cases} 1, & \text{for } t \in [0,1); \\ 2^k, & \text{for } t \in \left[2^{2k}, 2^{2k+1}\right), \ k \geq 0; \\ t/2^{k+1}, & \text{for } t \in \left[2^{2k+1}, 2^{2(k+1)}\right), \ k \geq 0. \end{cases}$$

An open problem is to describe domains D for which condition (14) holds. A partial case of this problem for $d = 2$ is as follows.

Conjecture 1. Let f and g be two functions such that

$$f(x) \leq x \leq g(x), \qquad x \geq 1.$$

Let $D = \{(i,j) \colon f(i) \leq j \leq g(i)\}$. Is it true that condition (14) holds?

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