ON THE CLASS OF GENERALIZING DIFFERENTIAL OPERATORS IN CLIFFORD ALGEBRA

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Dedicated to Professor Imre Kátaı
on his 65th birthday

Abstract. Let $\mathcal{A}$ be a universal Clifford algebra induced by an $m$-dimensional real linear space. It is well-known that the differential operator $\mu = \sum_{k=0}^{m} \frac{\partial}{\partial x_k} e_k$ satisfies the relations $\mu \overline{\mu} = \overline{\mu} \mu = \Delta_{m+1}$, where $\overline{\mu}$ is the conjugate operator of $\mu$ and $\Delta_{m+1} = \sum_{k=0}^{m} \frac{\partial^2}{\partial x_k^2}$ (see [1]). Let $m \equiv 2 \pmod{4}$ and $L(e_0, e_{A_1}, \ldots, e_{A_{m+1}})$ be the invertible subspace in $\mathcal{A}$ (see [3]). In this paper we give the some conditions for the generalizing differential $D^* = \sum_{k=0}^{m+1} \alpha_k \frac{\partial}{\partial x_k}$, where $\alpha_k \in L(e_0, e_{A_1}, \ldots, e_{A_{m+1}})$ such that any solution of a differential equation $D^* u = 0$ is always a solution of Laplace’s equation $\Delta_{m+2} u = 0$, where $\Delta_{m+2} = \sum_{k=0}^{m+1} \frac{\partial^2}{\partial x_k^2}$.

1. Preliminaries

Consider the $2^m$-dimensional real space $\mathcal{A}$ with basis $E = \{e_0, e_1, \ldots, e_m, e_{12}, \ldots, e_{12m}\}$. 

[This is a mathematical paper discussing the properties of differential operators in the context of Clifford algebra, with a focus on generalizing these operators to higher dimensions. The abstract highlights the conditions under which solutions of certain differential equations remain solutions of a related Laplace equation. The section 1 provides background information necessary for understanding the subsequent discussion.]
The product of two elements $e_A, e_B \in E$ is given by

$$e_A e_B = (-1)^{2(A \cap B)} (-1)^{P(A, B)} e_{A \Delta B} \quad A, B \subseteq \{1, 2, ..., m\},$$

where

$$
\begin{align*}
P(A, B) &= \sum_{j \in B} P(A, j), \\
P(A, j) &= 2 \{ i \in A : i > j \}, \\
A \Delta B &= (A \setminus B) \cup (B \setminus A),
\end{align*}
$$

and $2A$ denotes the number of elements of $A$.

Each element $a = \sum_A a_A e_A \in A \quad (a_A \in I \mathbb{R})$ is called a Clifford number. The product of two Clifford numbers $a = \sum_A a_A e_A; \quad b = \sum_B b_B e_B$ is defined by the formula

$$ab = \sum_A \sum_B a_A b_B e_A e_B.$$

It is easy to check that in such way $A$ is turned into an associative non-commutative algebra over $I \mathbb{R}$. It is called the Clifford algebra.

The involution for basic vector $e_{k_1, k_2, ..., k_t} \in E; \quad k_1, k_2, ..., k_t \in \{1, 2, ..., m\}$ is given by $e_{k_1, k_2, ..., k_t} = (-1)^{t(t+1)/2} e_{k_1 k_2 ... k_t}$.

For any $a = \sum_A a_A e_A \in A$, we write $\overline{a} = \sum_A a_A \overline{e}_A$ and $|a| = \left( \sum_A a_A^2 \right)^{1/2}$.

2. Generalizing differential operators

**Definition 1** (see [3]). i) An element $a \in A$ is said to be invertible if there exists an element $a^{-1} \in A$ such that $a a^{-1} = a^{-1} a = e_0$; $a^{-1}$ is said to be the inverse of $a$.

ii) A subspace $X \subseteq A$ is said to be invertible if every non-zero element in $X$ is invertible in $A$.

iii) $L(u_1, u_2, \ldots, u_n) = \text{lin}\{u_1, u_2, \ldots, u_n\}, \quad u_i \in A \quad (i = 1, 2, \ldots, n)$.

Let $m \equiv 2 \pmod{4}$ and $L(e_0, e_{A_1}, \ldots, e_{A_{m+1}})$ be the invertible subspace in $A$ (see [3]).
Definition 2. i) The operator \( D^* = \sum_{k=0}^{m+1} \alpha_k \frac{\partial}{\partial x_k} \), where 
\[ \alpha_k \in L(e_0, e_{A_1}, \ldots, e_{A_{m+1}}) \quad (k = 0, 1, \ldots, m+1) \]
is called a generalizing operator in \( \mathcal{A} \).

ii) Let \( \alpha_k = \sum_{i=0}^{m+1} a_{ik} e_{A_i} \) \( k = 0, 1, \ldots, m+1 \), where \( e_{A_0} = e_{A_0} \). The matrix \( A = (a_{ij})_{m+2} \) is called the symbol of \( D^* \).

iii) The operator \( D^* = \sum_{k=0}^{m+1} \alpha_k \frac{\partial}{\partial x_k} \) is called the conjugate operator of \( D \).

iv) The matrix \( \bar{A} = (a'_{ij})_{m+2} \) defined by
\[
\begin{align*}
a'_{0j} &= a_{0j}, \\
a'_{ij} &= -a_{ij}
\end{align*}
\]
for \( i = 1, \ldots, m+1 \) and \( j = 0, \ldots, m+1 \)
is called the conjugate of the matrix \( A \).

Remark. From these notations we can write
\[ D^* = (e_0, e_{A_1}, \ldots, e_{A_{m+1}})A \left( \frac{\partial}{\partial x_0} \frac{\partial}{\partial x_1} \ldots \frac{\partial}{\partial x_{m+1}} \right)^T, \]
where \( M^T \) denotes the transpose of the matrix \( M \).

Lemma 1 (see [2]). If \( L(e_0, e_{A_1}, \ldots, e_{A_l}) \), \( e_{A_i} \in E \), \( e_{A_i} \neq e_{A_j} \) for all \( i \neq j \), \( i, j \in \{0, 1, \ldots, l\} \), is invertible then

\[ \text{either} \quad \#A_j = 4p_j + 1 \quad \text{or} \quad \#A_j = 4p_j + 2 \quad (p_j \in \mathbb{N}, j = 1, \ldots, l). \]

Proof. Suppose that there exits \( e_{A_i} \in \{e_{A_1}, e_{A_2}, \ldots, e_{A_l}\} \) such that
\[ \sharp A_j = 4p_j \quad \text{or} \quad \sharp A_j = 4p_j + 3. \]
Hence
\[ (e_0 + e_{A_1})(e_0 - e_{A_1}) = e_0 + e_{A_1} - e_{A_1} + e_{A_1} = 0. \]
So \( e_0 + e_{A_i} \) is not invertible.

Lemma 2 (see [2]). \( L(e_0, e_{A_1}, \ldots, e_{A_l}) \), \( e_{A_i} \in E \), \( e_{A_i} \neq e_{A_j} \) for all \( i \neq j \), is invertible if and only if
\[ e_{A_i} e_{A_j} + e_{A_j} e_{A_i} = 0 \quad \text{for all} \quad i \neq j; \quad i, j \in \{0, 1, \ldots, l\}, \]
where \(e_{A_0} = e_0\).

**Proof.** Sufficiency. Suppose that \(e_{A_i} \bar{r}_{A_j} + e_{A_0} \bar{r}_{A_i} = 0\) for \(i \neq j,\) \(i, j \in \{1, 2, \ldots , l\}\). From \(e_0 \bar{r}_{A_j} + e_{A_i} \bar{r}_{A_i} = 0,\) we have \(\bar{r}_{A_j} + e_{A_i} = 0,\) \(j \in \{1, 2, \ldots , l\}\).

Take \(a = a_0 e_0 + \sum_{i=1}^{l} a_i e_{A_i} \in L(e_0, e_{A_1}, e_{A_2}, \ldots, e_{A_l}),\) \((a \neq 0)\). Write
\[
a^{-1} = \frac{1}{|a|^2} \left( a_0 e_0 + \sum_{i=1}^{l} a_i e_{A_i} \right).
\]

Then
\[
a a^{-1} = \frac{1}{|a|^2} \left( a_0 e_0 + \sum_{i=1}^{l} a_i e_{A_i} \right) \left( a_0 e_0 + \sum_{j=1}^{l} a_j e_{A_j} \right) = \frac{1}{|a|^2} \left[ a_0^2 e_0 + a_0 \left( \sum_{i=1}^{l} a_i e_{A_i} + \sum_{j=1}^{l} a_j e_{A_j} \right) + \sum_{i=1}^{l} a_i^2 e_{A_i} + \sum_{i<j} a_i a_j (e_{A_i} \bar{r}_{A_j} + e_{A_j} \bar{r}_{A_i}) \right] = \frac{1}{|a|^2} \left( \sum_{i=0}^{l} a_i^2 \right) e_0 = e_0.
\]

Similarly, one can check the equality \(a^{-1} a = e_0\).

Necessity. Suppose that \(L(e_0, e_{A_1}, e_{A_2}, \ldots, e_{A_l})\) is invertible. By Lemma 1 we have
\[\sharp A_j = 4p_j + 1\] or \[\sharp A_j = 4p_j + 2,\] \(p_j \in \mathbb{N}, j \in \{1, 2, \ldots , l\}\). Hence
\[e_0 \bar{r}_{A_j} + e_{A_i} \bar{r}_{A_i} = \bar{r}_{A_j} + e_{A_i} = 0\] for \(j \in \{1, 2, \ldots , l\}\).

Suppose that there exists \(e_{A_i}, e_{A_j} \in \{e_{A_1}, e_{A_2}, \ldots, e_{A_l}\}\) such that \(e_{A_i} \bar{r}_{A_j} + e_{A_j} \bar{r}_{A_i} \neq 0\).

By Lemma 1 we have \(-e_{A_i} e_{A_j} - e_{A_j} e_{A_i} \neq 0\). It is easy to see that either \(e_{A_i} e_{A_i} = e_{A_i} e_{A_i}\) or \(e_{A_i} e_{A_j} = -e_{A_i} e_{A_j},\) \(\forall e_{A_i}, e_{A_j} \in E.\)
Hence $e_{A_i}e_{A_j} = e_{A_j}e_{A_i}$. Write $a = e_{A_i} + e_{A_j}$ and $b = e_{A_i} - e_{A_j}$. Then we get

$$ab = (e_{A_i} + e_{A_j})(e_{A_i} - e_{A_j}) = e_0 + e_{A_j}e_{A_i} - e_{A_i}e_{A_j} - e_0 = -e_{A_i}e_{A_j} + e_{A_j}e_{A_i} = 0.$$  

So $a$ is not invertible.

**Lemma 3.** The generalized differential operator $D = \sum_{k=0}^{m+1} \frac{\partial}{\partial x_k} e_{A_k}$ with $e_{A_0} = e_0$ satisfies $D^*D = \overline{D}D = \Delta_{m+2}$, where $\overline{D} = \sum_{k=0}^{m+1} \frac{\partial}{\partial x_k} e_{A_k}$ is the conjugate operator of $D$ and $\Delta_{m+2} = \sum_{k=0}^{m+1} \frac{\partial^2}{\partial x_k^2}$.

**Proof.** By Lemma 1 and Lemma 2 we get

$$D^*D = \sum_{k=0}^{m+1} \frac{\partial}{\partial x_k} e_{A_k} \sum_{l=0}^{m+1} \frac{\partial}{\partial x_l} e_{A_l} =$$

$$= \sum_{k=0}^{m+1} \frac{\partial^2}{\partial x_k^2} e_{A_k} e_{A_k} + \sum_{k \neq l} \frac{\partial^2}{\partial x_k \partial x_l} (e_{A_j} e_{A_i} + e_{A_i} e_{A_k}) = \Delta_{m+2}.$$  

Similarly, one can check that $\overline{D}D = \Delta_{m+2}$.

**Lemma 4.** If the matrix $A$ is a symbol of the generalizing differential operator $D^*$ then $\overline{A}$ is the symbol of $\overline{D}^*$.

**Proof.** By Lemma 1 we have $\sharp A_j = 4p_j + 1$ or $\# A_j = 4p_j + 2 (p_j \in \mathbb{N} ; j = 1, m + 1)$. Hence $e_{A_i} = -e_{A_j}$, $i = 1, m + 1$.

Suppose that $\alpha_k = \sum_{i=0}^{m+1} a_{ik} e_{A_i}$ $(k = 0, 1, ..., m + 1)$, then

$$\overline{\alpha}_k = \sum_{i=0}^{m+1} a_{ik} e_{A_i} = a_{0k} e_0 + \sum_{i=1}^{m+1} a_{ik} e_{A_i} = a_{0k} e_0 - \sum_{i=1}^{m+1} a_{ik} e_{A_i}.$$  

Since

$$\overline{D}^* = \sum_{k=0}^{m+1} \overline{\alpha}_k \frac{\partial}{\partial x_k} = (e_0, e_{A_1}, ..., e_{A_{m+1}}) \overline{A} \left( \frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_{m+1}} \right)^T.$$  

The Lemma is proved.
Definition 3. i) Let \( \Omega \) be a certain open domain in \( \mathbb{R}^{m+2} \). A function \( f \in C^1(\Omega;A) \) is said to be left monogenic in \( \Omega \) if and only if \( D^* f = 0 \) in \( \Omega \).

ii) The set of left monogenic functions in \( \Omega \) is denoted by \( H(\Omega;A) \), and the set of orthogonal matrices of order \( m+2 \) is denoted by \( O(m+2) \).

Lemma 5. Let \( A \) be the symbol of the generalizing differential operator \( D^* = \sum_{k=0}^{m+1} \alpha_k \frac{\partial}{\partial x_k} \), where \( \alpha_k \in L(e_{A_0}, e_{A_1}, ..., e_{A_{m+1}}) \). Then \( A = \lambda O \), where \( \lambda > 0 \) and \( O \in O(m+2) \) if and only if

\[
\begin{align*}
\alpha_i \alpha_j + \alpha_j \alpha_i &= 0 & \text{for } i \neq j \\
\alpha_i &= \lambda & i, j = 0, 1, \ldots, m+1; i \neq j.
\end{align*}
\]

Proof. By Lemma 1 we have \( \#A_j = 4p_j + 1 \) or \( \#A_j = 4p_j + 2 \) (\( p_j \in \mathbb{N}, j = 1, m+1 \)). Hence it is easy to check that \( e_{A_k} \bar{e}_{A_k} = 1 \) and \( \alpha_k \bar{\alpha}_k = |\alpha_k|^2 \) for \( k = 0, m+1 \).

Let \( B = A^T A \). Then \( b_{ij} = \sum_{k=0}^{m+1} a_{ki} a_{kj} \) \( i, j = 0, ..., m+1 \). By Lemma 2 we get

\[
\alpha_i \alpha_j + \alpha_j \alpha_i = \left( \sum_{l=0}^{m+1} a_{li} e_{A_l} \right) \left( \sum_{k=0}^{m+1} a_{kj} e_{A_k} \right) + \left( \sum_{k=0}^{m+1} a_{kj} e_{A_k} \right) \left( \sum_{l=0}^{m+1} a_{li} e_{A_l} \right) =
\]

\[
= \sum_{l=0}^{m+1} \sum_{k=0}^{m+1} a_{li} a_{kj} (e_{A_l} \bar{e}_{A_k} + e_{A_k} \bar{e}_{A_l}) =
\]

\[
= 2 \sum_{k=0}^{m+1} a_{ki} a_{kj} (e_{A_k} \bar{A_k} - e_{A_k} \bar{A_k}) = 2 \sum_{k=0}^{m+1} a_{ki} a_{kj} = 2b_{ij}.
\]

Thus

\[
A = \lambda O, \text{ where } O \in O(m+2) \iff \begin{cases} b_{ij} = 0 \text{ for } i \neq j \\ b_{ii} = \lambda^2 \end{cases} i, j = 0, ..., m+1
\]

\[
\iff \begin{cases} \alpha_i \alpha_j + \alpha_j \alpha_i = 0 \text{ for } i \neq j \\ 2\alpha_i \alpha_i = 2|\alpha_i|^2 = 2b_{ij} = 2\lambda^2 \end{cases} i, j = 0, 1, ..., m+1.
\]

Lemma 5 is proved.

Theorem. Let \( D^* = \sum_{k=0}^{m+1} \alpha_k \frac{\partial}{\partial x_k} \), where \( \alpha_k \in L(e_{A_0}, e_{A_1}, ..., e_{A_{m+1}}) \) and \( A \) be the symbol of \( D^* \). Then \( D^* D^* = D^* D^* = \lambda^2 \Delta_{m+2} \) if and only if \( A = \lambda O \), where \( O \in O(m+2) \) and \( \Delta_{m+2} = \sum_{k=0}^{m+1} \frac{\partial^2}{\partial x_k^2}, \lambda > 0 \).
Proof. By Lemma 5 we have
\[ D^\ast D^\ast = \left( \sum_{i=0}^{m+1} \alpha_i \frac{\partial}{\partial x_i} \right) \left( \sum_{j=0}^{m+1} \alpha_j \frac{\partial}{\partial x_j} \right) = \sum_{i,j=0}^{m+1} \alpha_i \alpha_j = \lambda^2 \Delta_{m+2} \quad \iff \]
\[ \left\{ \begin{array}{l}
\alpha_i \alpha_j + \alpha_j \alpha_i = 0 \\
\alpha_i^2 = \lambda^2 \quad \text{for } i, j = 0, 1, \ldots, m + 1 \text{ and } i \neq j
\end{array} \right. \iff \]
\[ A = \lambda O. \]

Corollary 1. Let \( D^\ast \) be a generalizing differential operator in \( \mathcal{A}, f \in H(\Omega; \mathcal{A}) \). If \( A = \lambda O, \) where \( \lambda > 0, O \in O(m + 1), A \) is the symbol of \( D^\ast, \) then \( f \) is the solution of Laplace's equation \( \Delta_{m+2} f = 0. \)

Proof. From \( A = \lambda O \) and \( D^\ast f = 0 \) we get \( \mathcal{D}^\ast D^\ast f = \lambda^2 \Delta_{m+2} f = 0. \) So \( \Delta_{m+2} f = 0. \)

Corollary 2. If the generalizing operator \( D^\ast = \sum_{k=0}^{m+1} \alpha_k \frac{\partial}{\partial x_k} \) satisfies the relation \( D^\ast \mathcal{D}^\ast = \mathcal{D}^\ast D^\ast = \lambda^2 \Delta_{m+2}, \) where \( \Delta_{m+2} = \sum_{k=0}^{m+1} \frac{\partial^2}{\partial x_k^2}, \) then exits the linear transformation \( y_i = \sum_{j=0}^{m+1} p_{ij} x_j \) such that \( D^\ast = \sum_{k=0}^{m+1} \frac{\partial}{\partial y_k} e_{A_k}. \)

Proof. Let \( A \) be the symbol of \( D^\ast. \) By Theorem we have \( A = \lambda O, \) where \( \lambda > 0 \) and \( O \in O(m + 2). \) If we choose \( P = (A^T)^{-1} = (\lambda O^T)^{-1} = \frac{1}{\lambda} O = \frac{1}{\lambda^2} A^T, \) then we get
\[
D^\ast = \left( e_0 \ e_{A_1} \cdots e_{A_{m+1}} \right) A \left( \frac{\partial}{\partial x_0} \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_{m+1}} \right)^T =
\]
\[
= \left( e_0 \ e_{A_1} \cdots e_{A_{m+1}} \right) \left( \frac{\partial}{\partial y_0} \frac{\partial}{\partial y_1} \cdots \frac{\partial}{\partial y_{m+1}} \right)^T =
\]
\[
= \left( e_0 \ e_{A_1} \cdots e_{A_{m+1}} \right) \left( \frac{\partial}{\partial y_0} \frac{\partial}{\partial y_1} \cdots \frac{\partial}{\partial y_{m+1}} \right)^T.
\]
References


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