CM SOLUTIONS OF SOME FUNCTIONAL EQUATIONS OF ASSOCIATIVE TYPE

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Dedicated to Professor Imre Kátai on his 65th birthday

Abstract. In this note we give the continuous solutions of some functional equations of associative type that are strictly monotonic in each variable. Neither solvability nor differentiability conditions will be assumed.

1. Introduction

Forty-fifty years ago Hosszú [3], [4] investigated the following functional equations

$$F(F(x,u),v) = F(x,G(u,v)) \qquad \text{(equation of transformation)}, \tag{1.1}$$

$$F(F(x,u),v) = F(x,u+v) \qquad \text{(equation of tanslation)}, \tag{1.2}$$

$$F(F(x,y),z) = F(x,F(z,y)) \qquad \text{(Tarski's associative law)}, \tag{1.3}$$

$$F(x,F(y,z)) = F(z,F(y,x)) \qquad \text{(Grassman's associative law)}, \tag{1.4}$$

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(1.5) $F(x, F(y, z)) = F(F(z, x), y)$ (cyclic associative law),

(1.6) $F(x, F(y, z)) = F(y, F(x, z))$,

where the real-valued unknown functions were defined on the Cartesian product of two real intervals of positive length. The basic tools he applied were Aczél's theorems and ideas on the ordinary associativity equation

$$F(F(x, y), z) = F(x, F(y, z)) \tag{1.7}$$

(see [1]), on the bisymmetry equation

$$F(F(x, y), F(u, v)) = F(F(x, u), F(y, v)) \tag{1.8}$$

(see [1]), and his own results on the continuously differentiable (local) solutions of the generalized equation of associativity

$$F(G(x, y), z) = H(x, K(y, z)). \tag{1.9}$$

In some cases he assumed solvability conditions, too (see e.g. equations (1.1), (1.5), (1.6) in [3] and [4]). In this note we use a unified method to find the continuous solutions of equations (1.1) - (1.6) that are strictly monotonic in each variable. Further conditions, however , will not be supposed on the unknown functions.

In the following $\mathbb{R}$ denotes the set of all real numbers. By an interval we mean a subinterval of positive length of $\mathbb{R}$ and a real-valued function is said to be CM function if it is defined on an interval or on the Cartesian product of two intervals, continuous and strictly monotonic in each variable.

2. Basic tools

The following theorem which provides the CM solutions (solutions that are CM functions) of (1.9) is proved in Maksa [5]. The proof is based on a result of von Stengel [7]. (For a self-contained proof see Maksa [4]).

Theorem 1. *Let X, Y and Z be intervals, $G : X \times Y \to \mathbb{R}$, $K : Y \times Z \to \to \mathbb{R}$, $F : G(X, Y) \times Z \to \mathbb{R}$ and $H : X \times K(Y, Z) \to \mathbb{R}$ be CM functions. Then the generalized associativity equation*

$$F(G(x, y), z) = H(x, K(y, z)) \tag{1.9}$$

holds for all $(x, y, z) \in X \times Y \times Z$, *if and only if there exist CM functions* $\alpha : X \to \mathbb{R}$, $\beta : Y \to \mathbb{R}$, $\gamma : Z \to \mathbb{R}$, $\delta_1 : \alpha(X)+\beta(Y) \to \mathbb{R}$, $\delta_2 : \beta(Y)+\gamma(Z) \to \mathbb{R}$ *and* $\varphi : \alpha(X) + \beta(Y) + \gamma(Z) \to \mathbb{R}$ *such that*

$$\begin{aligned} F(\xi, z) &= \varphi(\delta_1^{-1}(\xi) + \gamma(z)), \\ G(x, y) &= \delta_1(\alpha(x) + \beta(y)), \\ H(x, \eta) &= \varphi(\alpha(x) + \delta_2^{-1}(\eta)) \qquad \text{and} \\ K(z, y) &= \delta_2(\beta(y) + \gamma(z)) \end{aligned}$$

hold for all $x \in X$, $y \in Y$, $z \in Z$, $\xi \in G(X, Y)$ *and* $\eta \in K(Y, Z)$.

It is obvious that equations (1.1) - (1.7) are particular cases of (1.9). However it is not easy to get their solutions directly from Theorem 1. The following obvious consequence turns out to be more useful for this purpose.

Theorem 2. *If the CM functions F, G, H and K in Theorem 1 satisfy (1.9) for all* $x \in X$, $y \in Y$, $z \in Z$, *then there exist CM functions* $\alpha : X \to \mathbb{R}$, $\beta : Y \to \mathbb{R}$, $Z \to \mathbb{R}$, *and* $\varphi : \alpha(X) + \beta(Y) + \gamma(Z) \to \mathbb{R}$ *such that*

$$F(G(x, y), z) = H(x, K(y, z)) = \varphi(\alpha(x) + \beta(y) + \gamma(z)) \tag{2.1}$$

hold for all $x \in X$, $y \in Y$, $z \in Z$.

It should be noted that the two theorems above are equivalent. (See [5]). An other tool, we shall use here, is the following.

Lemma. *Let P and Q be intervals,* $f : P + Q \to \mathbb{R}$, $q : P \to \mathbb{R}$ *and* $h : Q \to \mathbb{R}$ *be CM functions satisfying the Pexider equation*

$$f(p + q) = g(p) + h(q) \qquad (p \in P,\ q \in Q).$$

Then

$$\begin{aligned} f(t) &= b_0 t + b_1 + b_2, && t \in P + Q \\ g(p) &= b_0 p + b_1, && p \in P \quad \text{and} \\ h(q) &= b_0 q + b_2, && q \in Q \end{aligned}$$

for some $b_0, b_1, b_2 \in \mathbb{R}$, $b_0 \neq 0$.

This statement follows easily from Corollary 3 of Radó-Baker [6].

3. The results

First we give the CM solutions of the equation of transformation (1.1).

Theorem 3. *Let X and U be intervals, $F : X\times U \to X$ and $G : U\times U \to U$ be CM functions. Then*

$$F(F(x,u),v) = F(x,G(u,v)) \tag{1.1}$$

holds for all $x \in X,\ u,v \in U$, if and only if there exist CM functions $\alpha : X \to \mathbb{R}$ and $\beta : U \to \mathbb{R}$ such that

$$F(x,u) = \alpha^{-1}(\alpha(x) + \beta(u)), \qquad (x,u) \in X \times U \tag{3.1}$$

and

$$G(u,v) = \beta^{-1}(\beta(u) + \beta(v)), \qquad (u,v) \in U \times U. \tag{3.2}$$

Proof. The "if" part is obvious. For the proof of the "only if" part apply Theorem 2 to get that

$$F(F(x,u),v) = F(x,G(u,v)) = \varphi(\alpha(x) + \beta_1(u) + \beta_2(v)) \tag{3.3}$$

for some CM functions $\alpha : X \to \mathbb{R},\ \beta_1,\beta_2 : U \to \mathbb{R}$ and $\varphi : \alpha(X) + \beta_1(U) + {}+\beta_2(U) \to \mathbb{R}$ and for all $x \in X,\ u,v \in U$. Let $v = v_0 \in U$ be fixed in (3.3) and define the functions f_1 and Ψ by

$$f_1(x) = F(x,v_0),\ x \in X \quad \text{and} \quad \Psi(t) = f_1^{-1} \circ \varphi(t + \beta_2(v_0)),\ t \in \alpha(X) + \beta_1(U).$$

Then f_1 and Ψ are CM functions and (3.3), with $v = v_0$, implies that

$$F(x,u) = \Psi(\alpha(x) + \beta_1(u)), \qquad (x,u) \in X \times U. \tag{3.4}$$

Therefore, again by (3.3),

$$\alpha(x) + \beta_1 \circ G(u,v) = \alpha \circ \Psi(\alpha(x) + \beta_1(u)) + \beta_1(v)$$

for all $x \in X,\ u,v \in U$. Thus, for all $p \in \alpha(X),\ q,r \in \beta_1(U)$, we obtain that

$$\alpha \circ \Psi(p+q) = p + \beta_1 \circ G\big(\beta_a^{-1}(q), \beta_1^{-1}(r)\big) - \beta_1^{-1}(r).$$

For each fixed $r \in \beta_1(U)$ this is a Pexider equation with CM functions. Hence, by our Lemma,

$$\alpha \circ \Psi(t) = t + b_2(r), \qquad t \in \alpha(X) + \beta_1(U) \tag{3.5}$$

and

$$\beta_1 \circ G(\beta_1^{-1}(q), \beta_1^{-1}(r)) - \beta_1^{-1}(r) = q + b_2(r), \qquad q, r \in \beta_1(U) \tag{3.6}$$

for some function $b_2 : \beta_1(U) \to \mathbb{R}$. (The constant b_2 in the Lemma may depend on the fixed r.) However (3.5) implies immediately that $b_2 = b$ for all $r \in \beta_1(U)$ and for some $b \in \mathbb{R}$. Thus, by (3.5) and (3.6),

$$\Psi(t) = \alpha^{-1}(t + b), \qquad t \in \alpha(X) + \beta_1(U)$$

and

$$G(u, v) = \beta_1^{-1}(\beta_1(u) + \beta_1(v) + b), \qquad u, v \in U,$$

respectively. Finally, (3.1) and (3.2) follow from these equations and from (3.4) with the definition $\beta(u) = \beta_1(u) + b, u \in U$.

Remark. In case $U = X$ and $G = F$ equation (1.1) becomes the ordinary associativity equation (1.7) and Theorem 1 implies Aczél's classical theorem on (1.7) (see [1].)

Corollary 1. *Let X and U be intervals such that $(U, +)$ is a semigroup. Suppose that the CM function $F : X \times U \to X$ satisfies the equation of translation*

$$F(F(x, u), v) = F(x, u + v) \tag{1.2}$$

for all $x \in X$ and $u, v \in U$. Then there exists a CM function $\gamma : X \to \mathbb{R}$ such that

$$F(x, u) = \gamma^{-1}(\gamma(x) + u), \qquad (x, u) \in X \times U. \tag{3.7}$$

Proof. Apply Theorem 1 in case $G(u, v) = u + v$ $(u, v \in U)$. Then (3.2) yields that β satisfies the Cauchy equation

$$\beta(u + v) = \beta(u) + \beta(v) \qquad (u, v \in U).$$

Therefore, by our Lemma, $\beta(u) = cu,\ u \in U$ for some $0 \neq c \in \mathbb{R}$. Thus (3.7) follows from (3.1) with the definition $\gamma(x) = \frac{1}{c}\alpha(x),\ x \in X$.

Corollary 2. *Let X be an interval and suppose that the CM function $F : X \times X \to X$ satisfies the Tarski's associative law*

$$F(F(x,y),z) = F(x,F(z,y)) \qquad (x,y,z \in X). \tag{1.3}$$

Then there exists a CM function $\beta : X \to \mathbb{R}$ such that

$$F(x,y) = \beta^{-1}(\beta(x) + \beta(y))$$

for all $x, y \in X$.

Proof. Let $U = X$ and $G(y,z) = F(z,y)$, $(y,z) \in X \times X$ in Theorem 1. Then the statement follows from (3.2).

Among the remaining and listed associative laws we deal with only one in details.

Theorem 4. *Let X be an interval and $F : X \times X \to X$ be a CM function. Suppose that F satisfies the Grassmann's associative law*

$$F(x,F(y,z)) = F(z,F(y,x)). \qquad (x,y,z \in X). \tag{1.4}$$

Then there exist a CM function $\beta : X \to \mathbb{R}$ and real numbers $\lambda \neq 0$ and μ such that

$$F(x,y) = \beta^{-1}(\lambda^2\beta(x) + \lambda\beta(y) + \mu) \tag{3.8}$$

for all $x, y \in X$.

Proof. Applying Theorem 2 again, as in the proof of Theorem 3 we find that F has the form

$$F(x,y) = \Psi(\alpha(x) + \beta(y)) \qquad (x,y \in X) \tag{3.9}$$

with CM functions $\alpha, \beta : X \to \mathbb{R}$, $\Psi : \alpha(X) + \beta(X) \to \mathbb{R}$. This and (1.4) imply that

$$\alpha(x) + \beta \circ \Psi(\alpha(y) + \beta(z)) = \alpha(z) + \beta \circ \Psi(\alpha(y) + \beta(x))$$

for all $x, y, z \in X$. Therefore

$$\beta \circ \Psi(q + r) = \beta \circ \Psi(q + \beta \circ \alpha^{-1}(p)) - p + \alpha \circ \beta^{-1}(r)$$

for all $p, q \in \alpha(X)$ and $r \in \beta(X)$. For each fixed $p \in \alpha(X)$ this is a Pexider equation with CM functions, therefore, by our Lemma, we obtain that

$$\beta \circ \Psi(t) = b_0(p)t + b_1(p) + b_2(p), \qquad t \in \alpha(X) + \beta(X),\ p \in \alpha(X) \tag{3.10}$$

and

$$\alpha \circ \beta^{-1}(r) = b_0(p)r + b_2(p), \qquad r \in \beta(X),\ p \in \alpha(X) \tag{3.11}$$

with some functions $b_0 : \alpha(X) \to \mathbb{R} \setminus \{0\}, b_1, b_2 : \alpha(X) \to \mathbb{R}$. (The constants b_0, b_1, b_2 in the Lemma may depend on the temporarily fixed $p \in \alpha(X)$).

However (3.10) immediately implies that b_0 and $b_1 + b_2$ are constant functions. Hence, by (3.11), b_2 is constant and thus so is b_1. Consequently, by (3.10) and (3.11),

$$\beta \circ \Psi(t) = \lambda t + \mu_1, \qquad t \in \alpha(X) + \beta(X)$$

and

$$\alpha \circ \beta^{-1}(w) = \lambda w + \mu_2, \qquad w \in \beta(X)$$

with some real numbers μ_1 and μ_2. Finally, these equations and (3.9) imply that

$$\begin{aligned} F(x, y) &= \beta^{-1}(\lambda(\alpha(x) + \beta(y)) + \mu_1) = \\ &= \beta^{-1}(\lambda^2\beta(x) + \lambda\beta(y) + \lambda\mu_2 + \mu_1) \end{aligned}$$

whence (3.8) follows with $\mu = \lambda\mu_2 + \mu_1$.

Equations (1.5) and (1.6) can similarly be handled: first apply Theorem 2 to get the possible form of F and next use the Lemma on Pexider equation to have the exact form of F. The result is the following.

Theorem 5. *Let X be an interval and $F : X \times X \to X$ be a CM function. Then*

(a) F satisfies the cyclic associative law

$$F(x, F(y, z)) = F(F(z, x), y) \qquad (x, y, z \in X),$$

if and only if,

$$F(x, y) = \beta^{-1}(\beta(x) + \beta(y)) \qquad (x, y \in X)$$

for some CM function $\beta : X \to \mathbb{R}$ and

(b) F satisfies the associative law

$$F(x, F(y, z)) = F(y, F(x, z)) \qquad (x, y, z \in X),$$

if and only if,

$$F(x, y) = \beta^{-1}(\alpha(x) + \beta(y)) \qquad (x, y \in X)$$

for some CM functions $\alpha, \beta : X \to \mathbb{R}$.

Remark. Starting from the ordinary associative equation (1.7) we can derive 15 further equation of associatice type from it in a natural way. These equations, however, can be reduced to one of equations (1.4) - (1.7) (see [2]).

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