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NOTE ON APPROXIMATE RING HOMOMORPHISMS IN ALGEBRAS OVER FIELDS WITH VALUATIONS

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Dedicated to Professor Imre Kátai
on the occasion of his 65th birthday

Abstract. We prove a stability theorem for ring homomorphisms that map a normed algebra over a field with an arbitrary valuation into a Banach algebra over a (possibly different) field with a valuation. We allow estimations involving the norms of the arguments.

1. Introduction

In connection with a problem posed by Ulam (see [7]), D.H. Hyers [4] proved the stability of the linear equation for mappings defined on a Banach space and mapping into another one. In fact, his argument also proves the stability of the homomorphisms of a commutative semigroup into the additive group of a Banach space. Motivated by Hyers' result, D.G. Bourgin proved the stability of ring homomorphisms of Banach algebras with identity elements.

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Bourgin's result is usually recalled as the superstability of ring homomorphisms, since it states that a surjective function with bounded additive and multiplicative differences is a ring homomorphism. Recently, R. Badora proved a generalization of Bourgin's theorem. He considered the problem without assuming the existence of the identity elements and the surjectivity of the function. Applying appropriate results by Th.M. Rassias [6] and Z. Gajda [3] for approximately additive mappings, Badora also established the asymptotic stability of ring homomorphisms (or algebra homomorphisms) for functions mapping a normed algebra into a Banach algebra. The second author [5] proved a generalization of Rassias' and Gajda's aforementioned results for functions that map a normed space over a field with an arbitrary valuation into a Banach space over a (possibly different) field with a valuation. Applying this result, in the present paper we prove a generalization of Badora's theorem for functions that map a normed algebra over a field with a valuation into a Banach algebra over a field with a valuation.

2. Basic concepts

Given a field F, a mapping $|\ |_F : F \to \mathbb{R}$ is called a valuation, if $|\ |_F$ is positive definite (i.e. $|0|_F = 0$ and $|t|_F > 0$ for every $t \in F \setminus \{0\}$), multiplicative and subadditive. If $|t|_F = 1$ for every $t \in F \setminus \{0\}$, $|\ |_F$ is said to be trivial, otherwise it is called non-trivial. Let $\mathcal{A}$ be an algebra over a field F with a valuation $|\ |_F$ and let $\|\ \| : \mathcal{A} \to \mathbb{R}$. We call the pair $(\mathcal{A}, \|\ \|)$ a normed algebra over F, if $\|\ \|$ is positive definite, subadditive, submultiplicative, and it fulfils $\|\lambda x\| = |\lambda|_F \|x\|$ for every $x \in \mathcal{A}$ and $\lambda \in F$. A Banach algebra is a normed algebra which is complete with respect the metric generated by the norm.

In our calculations we suppose that the function $\lambda \mapsto \lambda^\alpha$ maps the set of non-negative real numbers into itself and it is multiplicative for every real exponent α. Moreover, it is convenient to assume that $\lambda^0 = 1$ for non-negative real number λ. For this purpose, we define $0^\alpha = 0$ for every $\alpha \in \mathbb{R} \setminus \{0\}$, while $0^0 = 1$.

3. Results

Theorem 1. *Let $(\mathcal{A}, \|\ \|_1)$ be a normed algebra over a field F of characteristic zero with a valuation $|\ |_F$, $(\mathcal{B}, \|\ \|_2)$ be a Banach algebra over* *a field K of characteristic zero with a valuation $|\ |_K$, and α, β be real numbers. Let us suppose that there exists an $s \in \mathbb{Q}\backslash\{0\}$ such that $|s|_F^\alpha > |s|_K$, $|s|_F^\beta > |s|_K$ or $|s|_F^\alpha < |s|_K$, $|s|_F^\beta < |s|_K$. Then there exists $c \in \mathbb{R}$ such that, for arbitrary non-negative real numbers ε, δ, the following implication holds: if a function $f : \mathcal{A} \to \mathcal{B}$ satisfies*

$$\|f(x+y) - f(x) - f(y)\|_2 \le \varepsilon(\|x\|_1^\alpha + \|y\|_1^\alpha) \tag{1}$$

and

$$\|f(xy) - f(x)f(y)\|_2 \le \delta\|x\|_1^\beta\|y\|_1^\beta \tag{2}$$

for every $x, y \in \mathcal{A}$, then there exists a unique ring homomorphism $h : \mathcal{A} \to \mathcal{B}$ for which

$$\|f(x) - h(x)\|_2 \le c\varepsilon\|x\|_1^\alpha \tag{3}$$

for every $x \in \mathcal{A}$. Furthermore, we have

$$h(x)(f(y) - h(y)) = (f(x) - h(x))h(y) = 0 \tag{4}$$

for every $x, y \in \mathcal{A}$.

Proof. We note that it is enough to consider the case $|s|_F^\alpha < |s|_K$ and $|s|_F^\beta < |s|_K$. Namely, if the reversed inequalities are satisfied, we may replace s with $1/s$.

Using the stability theorem from [5] we get that there exists a unique additive mapping $h : \mathcal{A} \to \mathcal{B}$ and a real constant c such that

$$\|f(x) - h(x)\|_2 \le c\varepsilon\|x\|_1^\alpha \qquad (x \in \mathcal{A}).$$

Therefore

$$\|f(s^n x) - h(s^n x)\|_2 \le c\varepsilon\|s^n x\|_1^\alpha \qquad (x \in \mathcal{A}, n \in \mathbb{N}),$$

consequently

$$\left\|\frac{1}{s^n}f(s^n x) - h(x)\right\|_2 \le c\varepsilon\left(\frac{|s|_F^\alpha}{|s|_K}\right)^n\|x\|_1^\alpha \qquad (x \in \mathcal{A}, n \in \mathbb{N}),$$

which means that

$$h(x) = \lim_{n\to\infty}\frac{1}{s^n}f(s^n x) \qquad (x \in \mathcal{A}).$$

Let $r(x,y) = f(xy) - f(x)f(y)$. Using (2) we get that

$$\lim_{n\to\infty} \left\| \frac{1}{s^n} r(s^n x, y) \right\|_2 = \lim_{n\to\infty} \left\| \frac{1}{s^n} \left(f(s^n xy) - f(s^n x) f(y) \right) \right\|_2 =$$
$$= \lim_{n\to\infty} \frac{1}{|s|_K^n} \| f(s^n xy) - f(s^n x) f(y) \|_2 \leq$$
$$\leq \lim_{n\to\infty} \left(\frac{|s|_F^\beta}{|s|_K} \right)^n \delta \|x\|_1^\beta \|y\|_1^\beta = 0,$$

therefore $\lim\limits_{n\to\infty} \frac{1}{s^n} r(s^n x, y) = 0$. Now we have

$$h(xy) = \lim_{n\to\infty} \frac{1}{s^n} f(s^n xy) = \lim_{n\to\infty} \frac{1}{s^n} \Big(f(s^n x) f(y) + f(s^n xy) - f(s^n x) f(y) \Big) =$$

$$= \lim_{n\to\infty} \frac{1}{s^n} \Big(f(s^n x) f(y) + r(s^n x, y) \Big) = \lim_{n\to\infty} \frac{1}{s^n} f(s^n x) f(y) = h(x) f(y).$$

Using the additivity of h we get the following:

$$h(x) f(s^n y) = h(x(s^n y)) = h((s^n x) y) = h(s^n x) f(y) = s^n h(x) f(y).$$

Therefore

$$h(x) \frac{1}{s^n} f(s^n y) = h(x) f(y).$$

Consequently, sending n to infinity, we have

$$\text{(5)} \qquad h(x)h(y) = h(x)f(y) = h(xy) \qquad (x, y \in \mathcal{A}),$$

so we get that h is multiplicative function.

Moreover, an analogous argument yields

$$\text{(6)} \qquad h(x)h(y) = f(x)h(y) = h(xy) \qquad (x, y \in \mathcal{A}).$$

From equations (5) and (6) we obtain (4).

Remark. Note that a possible value of the coefficient c occuring in Theorem 1 is explicitly given in [5]. Namely, the assumption that there exists a non-zero rational number s satisfying $|s|_F^\alpha \neq |s|_K$ immediately implies the existence of an integer $p > 1$ fulfilling $|p|_F^\alpha \neq |p|_K$. Then

$$c = \frac{2}{\big| |p|_K - |p|_F^\alpha \big|} \left(p - 1 + \sum_{k=1}^{p-1} |k|_F^\alpha \right).$$

A sufficient condition for the linearity of the approximating additive mapping is also given in [5, Theorem 3]. Combining it with Theorem 1 we obtain the following result.

Theorem 2. *Let F be a field of characteristic zero with some non-trivial valuation $|\ |_F$ such that $\mathbb{Q}$ is dense in F with respect to this valuation. Furthermore, let $(\mathcal{A}, \|\ \|_1)$ be a normed algebra over F, $(\mathcal{B}, \|\ \|_2)$ be a Banach algebra over F, and α, β be real numbers. Let us suppose that there exists an $s \in \mathbb{Q} \setminus \{0\}$ such that $|s|_F^\alpha > |s|_F, |s|_F^\beta > |s|_F$ or $|s|_F^\alpha < |s|_F, |s|_F^\beta < |s|_F$. Then there exists $c \in \mathbb{R}$ such that, for arbitrary non-negative real numbers ε, δ, the following implication holds: if a function $f : \mathcal{A} \to \mathcal{B}$ satisfies (1) and (2) for every $x, y \in \mathcal{A}$ and, for every $x \in \mathcal{A}$, the mapping $f_x : t \mapsto f(tx) \quad (t \in F)$ is bounded on an open ball $B_{\delta_x}(t_x)$ of non-zero center $t_x \in F$ and radius $\delta_x > 0$, then there exists a unique algebra homomorphism $h : \mathcal{A} \to \mathcal{B}$ for which (3) and(4) hold every $x, y \in \mathcal{A}$.*

Finally, we note that in the case

$$|s|_F^\alpha = |s|_K \qquad (s \in \mathbb{Q})$$

the conclusion of Theorem 1 may fail to hold. A counterexample with $F = K = \mathbb{R}$, $\alpha = 1$ is presented in [1].

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