Annales Univ. Sci. Budapest., Sect. Comp. 24 (2004) 103-118

MARCINKIEWICZ–Θ–SUMMABILITY OF DOUBLE FOURIER SERIES

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Dedicated to Professor Imre Kátai on his 65-th birthday

Abstract. A general summability method of two-dimensional Fourier series is given with the help of an integrable function θ. Under some conditions on θ we show that if the maximal Marcinkiewicz-Fejér operator is bounded from a Banach space $\mathbf{X}$ to $\mathbf{Y}$, then the maximal Marcinkiewicz-θ-operator is also bounded. As special cases the trigonometric and Walsh-Fourier series and the Fourier transforms are considered. It is proved that the maximal operator of the Marcinkiewicz-θ-means of these Fourier series is bounded from H_p to L_p $(p_0 < p \leq \infty)$ and is of weak type (1,1), where $p_0 < 1$ is depending only on the type of the Fourier series. As a consequence we obtain a generalization of a summability result due to Marcinkievicz and Zhizhiashvili, more exactly, the Marcinkiewicz-θ-means of a function $f \in L_1$ converge a.e. to the function in question. Some special cases of the θ-summation are considered, such as the Weierstrass, Picar, Bessel, Riesz, de La Vallée-Poussin, Rogosinski and Riemann summations.

The research was supported by the Hungarian Scientific Research Funds (OTKA) No. T043769.

AMS Subject classification: Primary 42B08, 42A24, 42C10; Secondary 42B30, 42A38.

1. Introduction

Fine [4] proved that the Fejér means $\sigma_n f$ of the Walsh-Fourier series of a one-dimensional function $f \in L_1$ converge a.e. to f as $n \to \infty$. It is known that the maximal operator of the Fejér means is of weak type $(1,1)$, i.e.

$$\sup_{\rho>0} \rho\lambda(\sigma_* f > \rho) \le C\|f\|_1 \qquad (f \in L_1)$$

(see Schipp [13]) and that σ_* is bounded from the dyadic H_1 Hardy space to L_1 (see Fujii [5]), where $\sigma_* := \sup_{n\in\mathbb{N}} |\sigma_n|$. The author [17] verified that σ_* is also bounded from H_p to L_p whenever $1/2 < p < \infty$. The same results are known for the trigonometric system (see Zygmund [26], Móricz [8, 9] and Weisz [18]).

One way to generalize these results to the two-dimensional case is the following. It is known that the diagonal partial sums $s_{n,n} f$ of the double trigonometric or Walsh-Fourier series of f converge a.e. to f as $n \to \infty$, whenever $f \in L_2$ (see Fefferman [3] and Schipp, Wade, Simon, Pál [12]). This result holds also for all functions from L_p $(1 < p < 2)$ for the trigonometric Fourier series, however, for Walsh series it is an unknown problem. Marcinkiewicz [7] has investigated the arithmetic means $\sigma_n f$ (in other words, the Marcinkiewicz-Fejér means) of the sequence $(s_{k,k} f)$. He [7] verified that the means $\sigma_n f$ of the two-dimensional trigonometric Fourier series of a function $f \in L \log L$ converge a.e. to f as $n \to \infty$. Later Zhizhiashvili [24, 25] generalized this result for all $f \in L_1$. Recently the author [19, 20, 21] has extended this result for the trigonometric and Walsh-Fourier series and for Fourier transforms by proving that the maximal Marcinkiewicz-Fejér operator σ_* is of weak type $(1,1)$. Moreover, we verified that σ_* is bounded from the Hardy space H_p to L_p whenever $p_0 < p \le \infty$. Note that $p_0 < 1$ is depending only on the type of the Fourier series. A usual density argument implies that the Marcinkiewicz-Fejér means $\sigma_n f$ converge a.e. to f in all three cases, as $n \to \infty$ and $f \in L_1$.

Butzer and Nessel [2] and recently Bokor, Schipp, Szili and Vértesi [1, 10, 11, 15, 16] considered a general method of summation of one-dimensional Fourier series, the so called θ-summability, where θ is an integrable function. Motivated by this idea the Marcinkiewicz-θ-summability for two-dimensional functions is considered in this paper. We investigate general orthogonal series and show that if the maximal Marcinkiewicz-Fejér operator σ_* is bounded from $\mathbf{X}$ to $\mathbf{Y}$, then the maximal Marcinkiewicz-θ-operator σ_*^θ is also bounded, where $\mathbf{X}$ and $\mathbf{Y}$ are two complete normed spaces. As special cases the trigonometric Fourier and Walsh-Fourier series and the Fourier transforms are examined. It is proved that σ_*^θ with respect to these Fourier series is bounded from H_p

to L_p ($p_0 < p \leq \infty$) and is of weak type (1,1), where $p_0 < 1$ is depending on the type of the Fourier series and different Hardy spaces are considered for different function systems. As a consequence we obtain that the Marcinkiewicz-θ-means of the above Fourier series of a function $f \in L_1$ converge a.e. to f. Some special cases of the Marcinkiewicz-θ-summation are considered, such as the Weierstrass, Picar, Bessel, Riesz, de La Vallée-Poussin, Rogosinski and Riemann summations.

2. θ-summability of Fourier series

We consider the unit square $[0,1)^2$ and the Lebesgue measure λ on it. We briefly write L_p instead of the real $L_p([0,1)^2, \lambda)$ space while the norm (or quasi-norm) of this space is defined by

$$\|f\|_p := \left(\int_{[0,1)^2} |f|^p \, d\lambda \right)^{1/p} \qquad (0 < p \leq \infty).$$

The space $L_{p,\infty} = L_{p,\infty}([0,1)^2, \lambda)$ $(0 < p < \infty)$ consists of all measurable functions f for which

$$\|f\|_{p,\infty} := \sup_{\rho > 0} \rho \lambda(|f| > \rho)^{1/p} < \infty,$$

while we set $L_{\infty,\infty} = L_\infty$. Note that $L_{p,\infty}$ is a quasi-normed space. It is known that

$$L_p \subset L_{p,\infty} \qquad \text{and} \qquad \|\cdot\|_{p,\infty} \leq \|\cdot\|_p$$

for each $0 < p \leq \infty$.

Let $\mathbb{M}$ denote either $\mathbb{Z}$ or $\mathbb{N}$. Suppose that ϕ_n ($n \in \mathbb{M}$) is a real or complex valued **uniformly bounded** orthonormal system over the unit interval. We consider the two-dimensional orthonormal system $\Phi = (\phi_n \times \phi_m; n, m \in \mathbb{M})$. For a function $f \in L_1$ the (n,m)-th Fourier coefficient with respect to Φ is defined by

$$\hat{f}(n,m) := \int_{[0,1)^2} f \overline{\phi_n \times \phi_m} \, d\lambda.$$

Denote by $s_{n,m}f := s_{n,m}^{\Phi}f$ $(n,m \in \mathbb{N})$ the (n,m)-th partial sum of the Fourier series of $f \in L_1$, namely,

$$s_{n,m}^{\Phi}f(x) := \sum_{k\in\mathbb{M},|k|\le n} \sum_{l\in\mathbb{M},|l|\le m} \hat{f}(k,l)\phi_k(x_1)\times\phi_l(x_2) = \\ = \int_0^1\int_0^1 f(t)D_n(t_1,x_1)D_m(t_2,x_2)\,dt,$$

where $x=(x_1,x_2)$, $t=(t_1,t_2)\in[0,1)^2$ and the *Dirichlet kernels* are defined by

$$D_n(t_i,x_i) := D_n^{\Phi}(t_i,x_i) := \sum_{k\in\mathbb{M},|k|\le n} \overline{\phi_k}(t_i)\phi_k(x_i) \qquad (n\in\mathbb{N}, i=1,2).$$

We suppose that $|D_n(t,x)| \le C(t,x)$ for all $n\in\mathbb{N}$ $(t,x\in[0,1), t\ne x)$, where $C(t,x)$ is independent of n. It is known that the trigonometric and Walsh system satisfy this condition.

The *Marcinkiewicz-Fejér means* $\sigma_n f := \sigma_n^{\Phi} f$ $(n\in\mathbb{N})$ of an integrable function f are given by

$$\sigma_n^{\Phi} f := \frac{1}{n+1}\sum_{k=0}^{n} s_{k,k}f = \int_0^1\int_0^1 f(t)K_n(t,x)\,dt,$$

where

$$K_n(t,x) := K_n^{\Phi}(t,x) := \frac{1}{n+1}\sum_{k=0}^{n} D_k(t_1,x_1)D_k(t_2,x_2)$$

denotes the *Marcinkiewicz-Fejér kernels.* The *maximal Fejér operator* is defined by

$$\sigma_* f := \sigma_*^{\Phi} f := \sup_{n\in\mathbb{N}} |\sigma_n f|.$$

We are going to introduce the Marcinkiewicz-θ-summability. In what follows the following conditions are always supposed.

$$(1) \quad \left\{ \begin{array}{l} \theta \in L_1(\mathbb{R}) \text{ is even and continuous, } \theta(0)=1, \\ \left(\theta\left(\frac{k}{n+1}\right)\right)_{k\in\mathbb{Z}} \in \ell_1, \ \lim_{x\to\infty}\theta(x)=0, \\ \theta \text{ is twice continuously differentiable on } \mathbb{R} \text{ except of} \\ \qquad \text{finitely many points,} \\ \theta'' \ne 0 \text{ except of finitely many points and finitely many intervals,} \\ \text{the left and right limits } \lim_{x\to y\pm 0} x\theta'(x)\in\mathbb{R} \text{ does exist at} \\ \qquad \text{each point } y\in\mathbb{R}, \\ \lim_{x\to\infty} x\theta'(x)=0. \end{array} \right.$$

Note that the second condition of (1) is satisfied if θ is non-increasing on (c,∞) for some $c \geq 0$ or if it has compact support.

Let

$$\Delta_1\theta\left(\frac{k}{n+1}\right) := \theta\left(\frac{k}{n+1}\right) - \theta\left(\frac{k+1}{n+1}\right),$$
$$\Delta_2\theta\left(\frac{k}{n+1}\right) := \Delta_1\theta\left(\frac{k}{n+1}\right) - \Delta_1\theta\left(\frac{k+1}{n+1}\right).$$

The *Marcinkiewicz-θ-means* of $f \in L_1$ are defined by

$$\sigma_n^\theta f(x) := \sigma_n^{\Phi,\theta} f(x) := \sum_{k=0}^{\infty} \Delta_1\theta\left(\frac{k}{n+1}\right) s_{k,k}f(x) =$$
$$= \int_0^1\int_0^1 f(t)K_n^\theta(t,x)\,dt,$$

where the K_n^θ kernels satisfy

$$K_n^\theta(t,x) := K_n^{\Phi,\theta}(t,x) := \sum_{k=0}^{\infty} \Delta_1\theta\left(\frac{k}{n+1}\right) D_k(t_1,x_1)D_k(t_2,x_2)$$

($n \in \mathbb{N}, t, x \in [0,1)^2$), which is well defined by (1). We define the *maximal Marcinkiewicz-θ-operator* by

$$\sigma_*^\theta f := \sigma_*^{\Phi,\theta} f := \sup_{n\in\mathbb{N}} |\sigma_n^\theta f| \qquad (f \in L_1).$$

If $\theta(x) := (1-|x|) \vee 0$, then we get the Fejér kernels and means.

We assume that

$$\int_0^1\int_0^1 |K_n(t,x)|\,dt \leq C \qquad (n \in \mathbb{N}, \quad x \in [0,1)^2) \tag{2}$$

which implies

$$\|\sigma_* f\|_\infty \leq C\|f\|_\infty \qquad (f \in L_\infty).$$

The constants C are absolute constants and the constants C_p are depending only on p and may denote different constants in different contexts.

Let $\mathbf{X}$ and $\mathbf{Y}$ be two complete quasi-normed spaces of measurable functions, L_∞ be continuously embedded into $\mathbf{X}$ and L_∞ be dense in $\mathbf{X}$. Suppose that if $0 \leq f \leq g$, $f, g \in \mathbf{Y}$ then $\|f\|_{\mathbf{Y}} \leq \|g\|_{\mathbf{Y}}$. If $f_n, f \in \mathbf{Y}$, $f_n \geq 0$ ($n \in \mathbb{N}$)

and $f_n \nearrow f$ a.e. as $n \to \infty$, then assume that $\|f - f_n\|_{\mathbf{Y}} \to 0$. Note that the spaces L_p and $L_{p,\infty}$ $(0 < p \le \infty)$ satisfy these properties.

Theorem 1. *Assume that (1) and (2) are satisfied. If $\sigma_* : \mathbf{X} \to \mathbf{Y}$ is bounded, i.e.*

$$\|\sigma_* f\|_{\mathbf{Y}} \le C\|f\|_{\mathbf{X}} \qquad (f \in \mathbf{X} \cap L_\infty), \tag{3}$$

then σ_*^θ is also bounded,

$$\|\sigma_*^\theta f\|_{\mathbf{Y}} \le C\|f\|_{\mathbf{X}} \qquad (f \in \mathbf{X}). \tag{4}$$

Proof. By Abel rearrangement,

$$\sum_{k=0}^{m} \Delta_1\theta\left(\frac{k}{n+1}\right) D_k(t_1,x_1) D_k(t_2,x_2) =$$
$$= \sum_{k=0}^{m-1} \Delta_2\theta\left(\frac{k}{n+1}\right) kK_k(t,x) + \Delta_1\theta\left(\frac{m}{n+1}\right) mK_m(t,x).$$

Observe that for a fixed t and x $K_m(t,x)$ is uniformly bounded in m. By Lagrange's mean value theorem there exists $m < \xi(m) < m+1$, such that

$$m\Delta_1\theta\left(\frac{m}{n+1}\right) = -\frac{m}{n+1}\theta'\left(\frac{\xi(m)}{n+1}\right)$$

and this converges to zero, if $m \to \infty$ (cf. (1)). Thus,

$$K_n^\theta(t,x) = \sum_{k=0}^{\infty} k\,\Delta_2\theta\left(\frac{k}{n+1}\right) K_k(t,x). \tag{5}$$

In [23] we have proved that

$$\sup_{n\in\mathbb{N}} \sum_{k=0}^{\infty} k \left|\Delta_2\theta\left(\frac{k}{n+1}\right)\right| \le C < \infty. \tag{6}$$

For the sake of the completeness we give the sketch of the proof of (6). If $\theta'' \geq 0$ on the interval $(i/(n+1), (j+2)/(n+1))$, then θ is convex on this interval and this yields that $\Delta_2\theta\left(\frac{k}{n+1}\right) \geq 0$ for $i \leq k \leq j$. Hence

$$\begin{aligned}\sum_{k=i}^{j} k \left|\Delta_2\theta\left(\frac{k}{n+1}\right)\right| =& \sum_{k=i}^{j} k\,\Delta_2\theta\left(\frac{k}{n+1}\right) = \\ =& \theta\left(\frac{i}{n+1}\right) + (i-1)\,\Delta_1\theta\left(\frac{i}{n+1}\right) - \\ &- j\,\Delta_1\theta\left(\frac{j+1}{n+1}\right) - \theta\left(\frac{j+1}{n+1}\right).\end{aligned}$$

Applying again Lagrange's mean value theorem we have

$$(i-1)\left|\Delta_1\theta\left(\frac{i}{n+1}\right)\right| = \frac{i-1}{n+1}\left|\theta'\left(\frac{\xi(i)}{n+1}\right)\right| = \frac{i-1}{\xi(i)}\left|\frac{\xi(i)}{n+1}\theta'\left(\frac{\xi(i)}{n+1}\right)\right| \leq C,$$

where $i < \xi(i) < i+1$. Here we used the fact that the function $x \mapsto |x\theta'(x)|$ is bounded, which follows from (1).

If $\theta'' = 0$ at an isolated point u or if θ'' is not twice continuously differentiable at u, $u \in (k/(n+1), (k+1)/(n+1))$, then the boundedness of $k\left|\Delta_2\theta\left(\frac{k}{n+1}\right)\right|$ can be seen in the same way. Since there are only finitely many intervals and isolated points satisfying the above properties, we have shown (6).

It follows from (2), (5) and (6) that

$$\sigma_n^\theta f(x) = \int_0^1 f(t)K_n^\theta(t,x)\,dt = \sum_{k=0}^{\infty}\int_0^1 k\,\Delta_2\theta\left(\frac{k}{n+1}\right) f(t)K_k(t,x)\,dt$$

for all $f \in L_\infty$. Thus $\sigma_*^\theta f \leq C\sigma_* f$ $(f \in L_\infty)$ and so

$$\|\sigma_*^\theta f\|_{\mathbf{Y}} \leq C\|f\|_{\mathbf{X}} \qquad (f \in \mathbf{X} \cap L_\infty).$$

By a usual density argument we finish the proof of the theorem.

3. Some summability methods

In this section we consider some summability methods as special cases of the Marcinkiewicz-θ-summation. Of course, there are a lot of other summability methods which could be considered as special cases. It is easy to see that (1) is satisfied all in the next examples. The elementary computations are left to the reader.

Example 1. Marcinkiewicz-Weierstrass summation. Let $\theta_1(x) = e^{-|x|^\gamma}$ for some $0 < \gamma < \infty$. Note that if $\gamma = 1$ then we obtain the Marcinkiewicz-Abel means

Example 2. Marcinkiewicz-Picar and Marcinkiewicz-Bessel summations. Let $\theta_2(x) = (1 + |x|^\gamma)^{-\alpha}$ for some $0 < \alpha, \gamma < \infty$ such that $\alpha\gamma > 1$.

Example 3. For some $1 < \alpha < \infty$ let

$$\theta_3(x) := \begin{cases} 1 & \text{if } |x| \leq 1, \\ |x|^{-\alpha} & \text{if } |x| > 1. \end{cases}$$

Example 4. For some $1 < \alpha < \infty$ let

$$\theta_4(x) := \begin{cases} 1 & \text{if } x = 0, \\ \dfrac{1 - e^{-|x|^\alpha}}{|x|^\alpha} & \text{if } |x| > 0. \end{cases}$$

Example 5. Marcinkiewicz-Riesz summation. Let

$$\theta_5(x) := \begin{cases} (1 - |x|^\gamma)^\alpha & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| > 1 \end{cases}$$

for some $1 \leq \alpha < \infty$ and $0 < \gamma < \infty$. The means are called Marcinkiewicz-Fejér means if $\alpha = \gamma = 1$.

Example 6. Marcinkiewicz-de La Vallée-Poussin summation. Let

$$\theta_6(x) = \begin{cases} 1 & \text{if } |x| \leq 1/2, \\ -2|x| + 2 & \text{if } 1/2 < |x| \leq 1, \\ 0 & \text{if } |x| > 1. \end{cases}$$

Example 7. Marcinkiewicz-Jackson-de La Vallée-Poussin summation. Let

$$\theta_7(x) = \begin{cases} 1 - 3x^2/2 + 3|x|^3/4 & \text{if } |x| \le 1, \\ (2-|x|)^3/4 & \text{if } 1 < |x| \le 2, \\ 0 & \text{if } |x| > 2. \end{cases}$$

Example 8. The summation method of cardinal B-splines. For $m \ge 2$ let

$$M_m(x) := \frac{1}{(m-1)!} \sum_{k=0}^{l} (-1)^k \binom{m}{k} (x-k)^{m-1}$$

$(x \in [l, l+1), l = 0, 1, \ldots, m-1)$ and

$$\theta_8(x) = \frac{M_m(m/2 + mx/2)}{M_m(m/2)}.$$

Note that θ_8 is even and $\theta_8(x) = 0$ for $|x| \ge 1$ (see also Schipp and Bokor [10]).

Example 9. This example generalizes Examples 6, 7, 8. Let $0 = \alpha_0 < < \alpha_1 < \ldots < \alpha_m$ and $\beta_0, \ldots, \beta_m$ $(m \in \mathbb{N})$ be real numbers, $\beta_0 = 1$, $\beta_m = 0$. Suppose that θ_9 is even, $\theta_9(\alpha_j) = \beta_j$ $(j = 0, 1, \ldots, m)$, $\theta_9(x) = 0$ for $x \ge \alpha_m$, θ_9 is a polynomial on the interval $[\alpha_{j-1}, \alpha_j]$ $(j = 1, \ldots, m)$.

Example 10. Marcinkiewicz-Rogosinski summation. Let

$$\theta_{10}(x) = \begin{cases} \cos \pi x/2 & \text{if } |x| \le 1 + 2j, \\ 0 & \text{if } |x| > 1 + 2j \end{cases} \qquad (j \in \mathbb{N}).$$

This summation was originally defined for $j = 0$.

Example 11. Marcinkiewicz-Riemann summation. For $\alpha > 1$ let

$$\theta_{11}(x) = \begin{cases} \left(\dfrac{\sin \pi x}{\pi x} \right)^\alpha & \text{if } |x| \le j, \\ 0 & \text{if } |x| > j \end{cases} \qquad (j \in \mathbb{N}, j \ne 0).$$

4. Orthonormal systems

In this section we consider the trigonometric and Walsh system and the Fourier transforms.

For the *trigonometric system*

$$\mathcal{T} := (\exp(2\pi \imath n \cdot), n \in \mathbb{Z}) \qquad (\imath := \sqrt{-1}),$$

the inequality (2) is proved in Zhizhiashvili [24, 25] or Weisz [20].

To define the *Walsh system* let

$$r(x) := \begin{cases} 1 & \text{if } x \in [0, \frac{1}{2}), \\ \\ -1 & \text{if } x \in [\frac{1}{2}, 1) \end{cases}$$

extended to $\mathbb{R}$ by periodicity of period 1. The *Rademacher system* $(r_n, n \in \mathbb{N})$ is defined by

$$r_n(x) := r(2^n x) \qquad (x \in [0,1), n \in \mathbb{N}).$$

The *Walsh functions* are given by

$$w_n(x) := \prod_{k=0}^{\infty} r_k(x)^{n_k} \qquad (x \in [0,1), n \in \mathbb{N}),$$

where $n = \sum_{k=0}^{\infty} n_k 2^k$ $(0 \leq n_k < 2)$. Let

$$\mathcal{W} := (w_n, n \in \mathbb{N}).$$

Condition (2) is proved in Weisz [21]. It is known that in these examples we can write $D_n(t,x) = D_n(x-t)$.

4.1. Fourier transforms

The *Fourier transform* of a function $f \in L_1(\mathbb{R})$ is defined by

$$\hat{f}(t,u) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} f(x,y) e^{-\imath t x - \imath u y} \, dx \, dy \qquad (t, u \in \mathbb{R}).$$

This definition can be extended to $f \in L_p(\mathbb{R})$ $(1 \le p \le 2)$ (see e.g. Butzer and Nessel [2]). It is known that if $f \in L_p(\mathbb{R})$ $(1 \le p \le 2)$ and $\hat{f} \in L_1(\mathbb{R})$ then

$$f(x,y) = \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{f}(t,u) e^{\imath xt} e^{\imath yu} \, dt \, du \qquad (x, y \in \mathbb{R}).$$

This motivates the definition of the *Dirichlet integral* $s_{t,u} f := s_{t,u}^{\mathcal{F}} f$ $(t, u > 0)$:

$$s_{t,u}^{\mathcal{F}} f(x,y) := \frac{1}{2\pi} \int_{-t}^{t} \int_{-u}^{u} \hat{f}(v,w) e^{\imath xv + \imath yw} \, dv \, dw =$$
$$= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} f(v,w) D_t(x-v) D_u(y-w) \, dv \, dw,$$

where

$$D_t(x) := D_t^{\mathcal{F}}(x) := \frac{1}{\sqrt{2\pi}} \int_{-t}^{t} e^{\imath xu} \, du = \frac{2}{\sqrt{2\pi}} \frac{\sin xt}{x}$$

is the *Dirichlet kernel.* Then $|D_t^{\mathcal{F}}(x)| \le C/x$ $(t > 0, x \ne 0)$.

The *Marcinkiewicz-θ-means* $\sigma_T^\theta f := \sigma_T^{\mathcal{F},\theta} f$ $(T > 0)$ of $f \in L_p(\mathbb{R})$ $(1 \le p \le$ $\le 2)$ are defined by

$$\sigma_T^{\mathcal{F},\theta} f(x,y) := \frac{-1}{T} \int_0^\infty \theta' \left(\frac{t}{T} \right) s_{t,t} f(x,y) \, dt =$$
$$= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} f(v,w) K_T^\theta (x-v, y-w) \, dv \, dw,$$

where

$$K_T^\theta(x,y) := K_T^{\mathcal{F},\theta}(x,y) := \frac{-1}{T} \int_0^\infty \theta' \left(\frac{t}{T} \right) D_t(x) D_t(y) \, dt.$$

The definition of the θ-means can be extended to tempered distributions as follows:

$$\sigma_T^{\mathcal{F},\theta} f := f * K_T^\theta \qquad (T > 0),$$

where $*$ denotes the convolution. One can show that $\sigma_T^{\mathcal{F},\theta} f$ is well defined for all tempered distributions $f \in H_p^{\mathcal{F}}$ $(0 < p \le \infty)$ and for all functions $f \in L_p$

$(1 \le p \le \infty)$ (cf. Stein [14]). Note that the Hardy spaces $H_p^{\mathcal{F}}$ are defined in the next section.

The *maximal Marcinkiewicz-θ-operator* is defined by

$$\sigma_*^\theta f := \sigma_*^{\mathcal{F},\theta} f := \sup_{T>0} |\sigma_T^{\mathcal{F},\theta} f|.$$

If $\theta(x) := (1-|x|) \vee 0$, then we get the Fejér means and operator and in this case we leave the θ in the notation. Note that (2) is shown in Weisz [19].

Theorem 2. *If (1) holds and if*

$$\|\sigma_* f\|_{\mathbf{Y}} \le C\|f\|_{\mathbf{X}} \qquad (f \in \mathbf{X} \cap L_\infty),$$

then

$$\|\sigma_*^\theta f\|_{\mathbf{Y}} \le C\|f\|_{\mathbf{X}} \qquad (f \in \mathbf{X}),$$

where $\mathbf{X}$ *and* $\mathbf{Y}$ *is defined in Theorem 1.*

Proof. Let $(a,b) \subset (0,\infty)$ be an interval such that $\theta''\left(\frac{t}{T}\right)$ exists for all $t \in (a,b)$. Integrating by parts we obtain

$$-\frac{1}{T}\int_a^b \theta'\left(\frac{t}{T}\right) D_t(x) D_t(y)dt =$$

$$= -\left[\theta'\left(\frac{t}{T}\right)\frac{t}{T}K_t(x,y)\right]_a^b + \frac{1}{T^2}\int_a^b \theta''\left(\frac{t}{T}\right) tK_t(x,y)dt =$$

$$= C_1 K_b(x,y) + C_2 K_a(x,y) + \frac{1}{T^2}\int_a^b \theta''\left(\frac{t}{T}\right) tK_t(x,y)dt,$$

which implies

$$\sigma_*^\theta f \le C\sigma_* f + \sigma_* f \frac{1}{T^2}\int_0^\infty t\left|\theta''\left(\frac{t}{T}\right)\right| dt,$$

because θ'' exists except of finitely many points. Since

$$\frac{1}{T^2}\int_a^b t\theta''\left(\frac{t}{T}\right) dt = \left[\frac{t}{T}\theta'\left(\frac{t}{T}\right)\right]_a^b - \frac{1}{T}\int_a^b \theta'\left(\frac{t}{T}\right) dt,$$

which is bounded by (1), we conclude

$$\frac{1}{T^2}\int_0^\infty t\left|\theta''\left(\frac{t}{T}\right)\right| dt \leq C.$$

This finishes the proof of the theorem.

5. Hardy spaces

First we define the *Poisson kernels* $P_t^{\mathcal{G}}$ for all function systems $\mathcal{G} \in \in \{\mathcal{T}, \mathcal{W}, \mathcal{F}\}$. Set

$$P_t^{\mathcal{T}}(x,y) := \sum_{j_1=-\infty}^{\infty}\sum_{j_2=-\infty}^{\infty} e^{-t(|j_1|+|j_2|)}e^{2\pi\imath(j_1x+j_2y)} \qquad (x,y \in \mathbb{R}, t>0),$$

$$P_t^{\mathcal{F}}(x,y) := \frac{ct}{(t^2+x^2+y^2)^{3/2}} \qquad (x,y \in \mathbb{R}, t>0),$$

$$P_t^{\mathcal{W}}(x,y) := 2^{2n}1_{[0,2^{-n})\times[0,2^{-n})}(x,y) \quad \text{if} \quad n \leq t < n+1 \qquad (x,y\in\mathbb{R}).$$

For a tempered distribution f the *non-tangential maximal function* is defined by

$$f_*^{\mathcal{G}}(x,y) := \sup_{t>0}|(f * P_t^{\mathcal{G}})(x,y)| \qquad (x,y\in\mathbb{R}),$$

where $\mathcal{G} \in \{\mathcal{T}, \mathcal{W}, \mathcal{F}\}$.

The *Hardy space* $H_p^{\mathcal{G}}(\mathbb{R}^2)$ $(0<p<\infty)$ consists of all tempered distributions f for which

$$\|f\|_{H_p^{\mathcal{G}}(\mathbb{R}^2)} := \|f_*^{\mathcal{G}}\|_p < \infty.$$

Now let $H_p^{\mathcal{F}} := H_p^{\mathcal{F}}(\mathbb{R}^2)$ and

$$H_p^{\mathcal{G}} := H_p^{\mathcal{G}}([0,1)^2) := \{f \in H_p^{\mathcal{G}}(\mathbb{R}^2) : \operatorname{supp} f \subset [0,1)^2\},$$

where $\mathcal{G} \in \{\mathcal{T}, \mathcal{W}\}$. Define $H_\infty^{\mathcal{G}} := L_\infty$.

Note that $H_p^{\mathcal{W}}$ is the dyadic Hardy space. It is known (see Stein [14], Weisz [22]) that the space H_p is equivalent to L_p if $1<p\leq\infty$.

Theorem 3. *If $\mathcal{G} \in \{\mathcal{T}, \mathcal{W}, \mathcal{F}\}$ and (1) is satisfied then*

$$\|\sigma_*^{\mathcal{G},\theta} f\|_p \leq C_p \|f\|_{H_p^{\mathcal{G}}} \qquad (f \in H_p^{\mathcal{G}})$$

for all $p_0 < p \leq \infty$, where $p_0 = 8/9$ for the trigonometric Fourier series and for the Fourier transforms and $p_0 = 2/3$ for the Walsh-Fourier series. Moreover,

$$\sup_{\rho>0} \rho\lambda(\sigma_*^{\mathcal{G},\theta} f > \rho) \leq C\|f\|_1 \qquad (f \in L_1).$$

This theorem for the Marcinkiewicz-Fejér operators was proved by the author [19, 20, 21]. Theorem 3 follows from these results and from Theorems 1 and 2. It is unknown whether the constant p_0 is sharp in the inequalities.

A usual density argument of Marcinkiewicz and Zygmund [6] implies

Corollary 1. *If (1) is satisfied and if $f \in L_1$ then*

$$\sigma_n^{\mathcal{G},\theta} f \to f \qquad a.e. \ \ as \ \ n \to \infty,$$

where $\mathcal{G} \in \{\mathcal{T}, \mathcal{W}\}$. Moreover

$$\sigma_T^{\mathcal{F},\theta} f \to f \qquad a.e. \ \ as \ \ T \to \infty.$$

This Corollary for the trigonometric Fourier series is due to Zhizhiashvili [24, 25].

Similar results can also be proved for the conjugate θ-means and for the Hardy-Lorentz spaces (cf. Weisz [22]).

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(Received September 15, 2003)

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