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ON A PROBLEM OF KÁTAI AND SUBBARAO

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Dedicated to Professor I. Kátai on his 65. birthday

Abstract. Two conjectures of Kátai and Subbarao concerning unimodular completely multiplicative functions f for which the sequence $f(n+1)/f(n)$ has finitely many limit points are related and partly proved in the more general setting of (restrictedly) multiplicative functions mapping into a locally compact Abelian group. En passant a theorem on integer valued additive functions with bounded differences is given.

1. Introduction

For arithmetic functions f let the quotient operator $\mathcal{Q}$ be defined by

$$\mathcal{Q}f(n) := \frac{f(n+1)}{f(n)}.$$

Let us further use the following notations:

$M(\mathcal{G}), CM(\mathcal{G})$:	the sets of multiplicative/completely multiplicative functions $\mathrm{F} : \mathbb{N} \to \mathcal{G}$, where $\mathcal{G}$ is any multiplicative Abelian group,
μ_k :	the group of $\mathrm{k-th}$ roots of unity in $\mathbb{C}$,
$\mathbb{T}$:	the group $\{\mathrm{z} \in \mathbb{C} : \lvert\mathrm{z}\rvert = 1\}$,
$f(\mathbb{N})'$:	the set of limit points of the sequence $\big(\mathrm{f(n)}\big)_{\mathrm{n}\in\mathbb{N}}$,
$\langle \mathcal{S} \rangle$:	the group generated by a set $\mathcal{S}$.

In [7], see also [8], it was proved that a function $f \in M(\mathbb{T})$ with the property
$$\mathcal{Q}f(n) \to 1$$
is necessarily of the form $f(n) = n^{i\tau}$ with some $\tau \in \mathbb{R}$.

It is well-known that the functions $\chi_\tau(n) = n^{i\tau}$ are the characters of $\mathbb{R}_+^*$, i.e. the continuous homomorphisms of $\mathbb{R}_+^*$ into $\mathbb{T}$. The role that these functions play in our context is made clearer by a generalization of the above result that was given first by Daróczy and Kátai [1] for certain locally compact Abelian groups and then by Mauclaire [4] for all of them. They show:

Theorem DKM. *If G is a locally compact Abelian group, if $f \in M(\mathcal{G})$ and $\mathcal{Q}f(n) \to 1$ as $n \to \infty$, then f can be extended to a continuous homomorphism $\chi : \mathbb{R}_+^* \to \mathcal{G}$.*

Kátai and Subbarao in their paper [2] consider more generally functions f for which the set $\mathcal{Q}f(\mathbb{N})'$ of limit points of $\mathcal{Q}f(n)$ is finite, being more resrictive on the other hand by asking the functions to be completely multiplicative. They formulate three conjectures, of which we are concerned with numbers 1 and 3 here.

Conjecture KS1. *If $f \in CM(\mathbb{T})$ and $\mathcal{Q}f(\mathbb{N})'$ is finite with exactly k elements, then $f(n) = n^{i\tau}F(n)$ where $\tau \in \mathbb{R}$ and $F^k(n) = 1$ for all n.*

One might add, because it easily follows, that $\mathcal{Q}f(\mathbb{N})' = F(\mathbb{N}) = \mu_k$: The finite homomorphic image $F(\mathbb{N})$ is a group, so it is μ_ℓ with some $\ell \mid k$, therefore $\mathcal{Q}f(\mathbb{N})' = \mathcal{Q}F(\mathbb{N})' \subset \mu_\ell$ so that $k \leq \ell$. It follows $k = \ell$.

In [2] the authors prove this conjecture for $k \leq 3$ and in a further paper [3] partly for $k = 4$.

Conjecture KS3. *If $F \in CM(\mu_k)$ and k is minimal then $\mathcal{Q}F(\mathbb{N})' = \mu_k$, in other words: $\mathcal{Q}f(n)$ attains every $\zeta \in \mu_k$ infinitely often.*

This is actually the special case of Conjecture KS1, where the range of f is finite: Let an $f \in CM(\mathbb{T})$ be given that has finite image, $f(\mathbb{N}) = \mu_k$, say, and assume that $\mathcal{Q}f(\mathbb{N})'$ has ℓ elements. Then Conjecture KS1 would give $f(n) = n^{i\tau}F(n)$ with $F(\mathbb{N}) = \mu_\ell$. Obviously $\tau = 0$, thus $f = F$, $\ell = k$, and $\mathcal{Q}f(\mathbb{N})' = \mu_k$.

2. Results

We shall present two results which are related to these conjectures plus one that is auxiliary here but looks interesting apart from that.

Theorem 1. *If $F \in M(\mathcal{G})$ where $\mathcal{G}$ is any Abelian group, and the set $\mathcal{Q}F(\mathbb{N})$ is finite, then $F(\mathbb{N})$ too is finite.*

The theorem obviously implies that if F is *completely* multiplicative then $F(\mathbb{N})$ is a finite subgroup of $\mathcal{G}$.

Theorem 2. *Let $f : \mathbb{N} \to \mathcal{G}$ be multiplicative with values in a locally compact Abelian group $\mathcal{G}$. Assume that the sequence $\big(\mathcal{Q}f(n)\big)$ has only finitely many limit points and that the set $\mathcal{Q}f(\mathbb{N})$ is relatively compact. Then $f = \chi F$, where χ is the restriction to $\mathbb{N}$ of a continuous homomorphism from $\mathbb{R}_+^*$ to $\mathcal{G}$, the function F is multiplicative and has a finite image $F(\mathbb{N}) \subset \langle \mathcal{Q}f(\mathbb{N})' \rangle$.*

Moreover, if f is completely multiplicative then F too is completely multiplicative and $F(\mathbb{N}) = \langle \mathcal{Q}f(\mathbb{N})' \rangle$ is a finite subgroup of $\mathcal{G}$.

Note that the second part is an easy consequence of the body of the theorem: If $f \in CM(\mathcal{G})$ then so is $F = \chi^{-1} f$, and $F(\mathbb{N})$ is a group. $\mathcal{Q}\chi(n) = \chi(1 + 1/n) \to \chi(1) = 1$, hence $\mathcal{Q}f(\mathbb{N})' = \mathcal{Q}F(\mathbb{N})' \subset \mathcal{Q}F(\mathbb{N}) \subset F(\mathbb{N})$, thus $\langle \mathcal{Q}f(\mathbb{N})' \rangle \subset F(\mathbb{N})$.

In contrast to the completely multiplicative case, the group generated by $\mathcal{Q}f(\mathbb{N})'$ need not be finite, not even for $\mathcal{G} = \mathbb{T}$, if $f$is only multiplicative: Just look at the function defined by $f(n) = 1$ for odd and $f(n) = \omega$ for even n, where ω is any non-torsion element of $\mathbb{T}$.

As far as $\mathcal{G} = \mathbb{T}$ is concerned Theorem 2 is a weakened form of Conjecture KS1, which for $f \in CM(\mathcal{G})$ claims just this with $F(\mathbb{N}) = \mathcal{Q}f(\mathbb{N})'$.

One might well generalize Conjecture KS3 and formulate

Conj $(\mathcal{G})$ *If $F : \mathbb{N} \to \mathcal{G}$ is a surjective homomorphism then $\mathcal{Q}F(\mathbb{N})' = \mathcal{G}$.*

With this notation Conjecture KS3 amounts to Conj(μ_k) for all k, and if we assume Conj$(\mathcal{G})$ for all finite Abelian groups we could in the completely multiplicative case of Theorem 2 replace $\langle \mathcal{Q}f(\mathbb{N})' \rangle$ with $\mathcal{Q}f(\mathbb{N})'$: Simply apply Conj $\big(F(\mathbb{N})\big)$ and remember $\mathcal{Q}f(\mathbb{N})' = \mathcal{Q}F(\mathbb{N})'$.

In particular we mention

Theorem 3. *Conjecture KS3 implies Conjecture KS1.*

So the two are really equivalent.

The proof of Theorem 2 proceeds in two steps of which the first is formulated in the lemma below. For this lemma ideas are used that are present already in the proofs of the theorems from [7,8].

The second step in the proof of Theorem 2 consists in Theorem 1 and this in turn depends on

Theorem 4. *Let $g : \mathbb{N} \to \mathbb{Z}$ be additive and assume that*

$$\Delta g(n) := g(n+1) - g(n) \ll \log^\alpha n, \qquad \textit{where} \quad 0 \le \alpha < 1.$$

Then $g(n) \ll \log^\alpha n$.

If $g : \mathbb{N} \to \mathbb{Z}$ is completely additive and $\Delta(n) = o(\log n)$ then g is the null function. In either case a one-sided condition suffices.

This is a corollary, apparently not mentioned so far, of the theorems on real valued additive functions in [6] (1979). Actually we need only the case $\alpha = 0$, i.e. with the assumption $\Delta g(n) \ll 1$ here, but state the theorem in full because we think it interesting in itself and the proof is hardly more work for the full than for the weak version. The difference is that we have to quote the more elaborated [6] while otherwise [5] from 1968 would suffice.

3. Proofs

3.1. Proof of Theorem 4

In [6] it is shown that a completely additive function $g : \mathbb{N} \to \mathbb{R}$ with $\Delta g(n) = o(\log n)$ (one-side suffices) is of the form $g(n) = \tau \log n$ with some $\tau \in \mathbb{R}$. Clearly if g is integer valued then $\tau = 0$.

For an additive function $g : \mathbb{N} \to \mathbb{R}$ with the (one-sided) condition $\Delta g(n) \ll \log^\alpha n$ the paper gives $g(n) = \tau \log n + \rho(n)$ with $\rho(n) \ll \log^\alpha n$ (two-sided). If we assume $\tau \neq 0$ then without loss of generality $\tau = 1$.

With the real parameter x tending to infinity consider the intervals $I_x :=$ $:= (x, y]$ where $y = xe^{\log^\alpha x}$. Note that $\log y \sim \log x$. The function $\log n + \rho(n)$ varies on I_x only by $O(\log^\alpha x)$ and, since its values are integers, can take only $O(\log^\alpha x)$ different values. By the Prime Number Theorem and Dirichlet's principle there are $2k$ primes $p_j \in I_x$ for which g takes the same value $g(p_r) = a$, say, and where $k \gg y/(\log y \cdot \log^\alpha x) \gg y/\log^2 y$. Let the p_j be numbered in ascending order. At $n_1 := \prod_{j=1}^{k} p_j$ as well as at $n_2 := \prod_{j=k+1}^{2k} p_j$ the function g takes the value ka. Thus

$$
\begin{aligned}
|\rho(n_2) - \rho(n_1)| &= \log n_2 - \log n_1 = \\
&= \sum_{j=1}^{k} \log \frac{p_{j+k}}{p_j} \geq \\
&\geq \sum_{j=1}^{k} \log\left(1 + \frac{k}{y}\right) \gg \qquad (1)\\
&\gg \frac{k^2}{y} \gg \\
&\gg \frac{y}{\log^4 y}.
\end{aligned}
$$

On the other hand $\log n_1 < \log n_2 \leq k \log y \ll y$, so by assumption $r(n_i) \ll \ll \log^\alpha n_2 \ll y^\alpha$, $|r(n_2) - r(n_1)| \ll y^\alpha$. This, because of $\alpha < 1$, contradicts (1) and thereby proves $\tau = 0$.

3.2. Proof of Theorem 1

The subgroup $\langle F(\mathbb{N})\rangle$ of $\mathcal{G}$ generated by $F(\mathbb{N})$ is also generated by $\mathcal{Q}F(\mathbb{N})$ and thus, by assumption, is a finitely generated Abelian group, which by the well-known main theorem on such groups is a direct product

$$\langle F(\mathbb{N})\rangle = T \times Z_1 \times \ldots \times Z_s, \quad \ell \in \mathbb{N}, \quad s \in \mathbb{N}_0,$$

of a finite group T and infinite cyclic groups $Z_j = \{z_j^n : n \in \mathbb{Z}\}$ with $z_j \in \mathcal{G}$ that are multiplicatively independent. The direct decomposition of $\langle F(\mathbb{N})\rangle$ implies a corresponding decomposition of the multiplcative function F mapping into it

$$F(n) = F_0(n) z_1^{\gamma_1(n)} \ldots z_s^{\gamma_s(n)}, \qquad (2)$$

where $F_0 : \mathbb{N} \to T$ is multiplicative and the $\gamma_j : \mathbb{N} \to \mathbb{Z}$, $1 \leq j \leq s$, are additive. Furthermore for each $n \in \mathbb{N}$

$$\mathcal{Q}F(n) = \mathcal{Q}F_0(n) \prod_j z_j^{\Delta\gamma_j(n)}$$

is the unique decomposition of one of the finitely many elements of $\mathcal{Q}F(\mathbb{N})$, which shows that the $\Delta\gamma_j(n)$ can take only finitely many different values. By the case $\alpha = 0$ of Theorem 4 the γ_j are bounded, and now (2) shows that $F(\mathbb{N})$ is finite.

3.3. Proof of Theorem 2

Some further notations that will be used:

$$\begin{aligned}
\mathcal{G}_0 :=& \langle \mathcal{Q}f(\mathbb{N})' \rangle, \\
\mathcal{A} :=& \{q, q^{-1} : q \in \mathcal{Q}f(\mathbb{N})'\}, \\
\mathcal{M}^k :=& k - \text{fold elementwise product with itself,} \\
& \text{for a subset } \mathcal{M} \text{ of a group,} \\
\mathbb{N}_a :=& \{n \in \mathbb{N} : (n, a) = 1\}, \\
(n, a^\infty) :=& \prod_{p|a} p^{\nu_p(n)} \quad \text{if} \quad n = \prod_p p^{\nu_p(n)}, \\
n|a^\infty :\Leftrightarrow& \text{ if } \ p|n \ \text{ then } \ p|a.
\end{aligned}$$

To prepare the application of Theorem 1 a function $F \in CM(\mathcal{G})$ is constructed for which $\mathcal{Q}F(n)$ attains rather than approaches the elements of $\mathcal{Q}f(\mathbb{N})'$.

Lemma. *Let $f \in CM(\mathcal{G})$ where $\mathcal{G}$ is a topological Abelian group. Assume that the sequence $\big(\mathcal{Q}f(n)\big)$ has only finitely many limit points and that the set $\mathcal{Q}f(\mathbb{N})$ is relatively compact. Then there is a factorization $f = hF$ of f into two functions $h \in CM(\mathcal{G})$ and $F \in M(\mathcal{G}_0)$ such that $\mathcal{Q}h(n) \to 1$ as $n \to \infty$, and $\mathcal{Q}F(\mathbb{N})$ is finite.*

Proof. Because of the relative compactness of $\mathcal{Q}f(\mathbb{N})$ and the finiteness of the number of limit points one can split $\mathbb{N}$ into finitely many classes on which $\mathcal{Q}f(n)$ converges to one of the elements of $\mathcal{Q}f(\mathbb{N})'$. Therefore there are functions $q : \mathbb{N} \to \mathcal{Q}f(\mathbb{N})'$ such that

$$\mathcal{Q}f(n) \sim q(n) \quad \text{as} \quad n \to \infty.$$

We fix any of them and form

$$g(n) := \prod_{m<n} q(m).$$

Due to the initial ambiguity in defining $q(n)$ the function $g(n)$ is given uniquely for large n and up to a constant factor only. Therefore we study $g(an)/g(n)$

as n tends to infinity. Let $a \in \mathbb{N}$ be fixed and $|n' - n|$ be bounded as $n \to \infty$, $n < n' \le n + a$, say. Then

$$\frac{g(an')}{g(n')}\frac{g(n)}{g(an)} = \prod_{n \le m < n'} q(m)^{-1} \prod_{an \le m < an'} q(m) \sim \tag{3}$$

$$\sim \prod_{n \le m < n'} \big(\mathcal{Q}f(m)\big)^{-1} \prod_{an \le m < an'} \mathcal{Q}f(m) = \tag{4}$$

$$= \frac{f(n)}{f(n)'}\frac{f(an')}{f(an)}. \tag{5}$$

For a first application consider consecutive elements n, n' of $\mathbb{N}_a$. Then the right hand side of (5) is 1. But, as (3) shows, the left hand side is an element of the finite set $\mathcal{B} := \mathcal{A}^{a^2+a} \subset \mathcal{G}_0$. So the asymptotic equation turns into an actual equation for large n and the quotients stabilize:

$$\frac{g(an)}{g(n)} =: F(a) \in \mathcal{G}_0 \quad \text{for} \quad n \ge n_a, \quad (n,a) = 1. \tag{6}$$

For $n \ne \mathbb{N}_a$ the behavior of $g(an)/g(n)$ can be linked to that with a coprime n'. Let $n' > n$ be minimal in $\mathbb{N}_a$ (which is $\le n + a$) and inspect equations (3) to (5). If $n \ge n_a$ then the left hand side becomes

$$F(a)\frac{g(n)}{g(an)} \in \mathcal{B}.$$

If, for the moment, we fix any $d|a^\infty$ and let n tend to ∞ through the numbers $n = dn_1,\ (n_1, a) = 1$, then the right hand side of (5) is constant

$$= f(a)\frac{f(n)}{f(an)} = f(a)\frac{f(d)}{f(ad)},$$

and we obtain (whatever d)

$$f(a)\frac{f(n)}{f(an)} \in \mathcal{B}$$

for all $n \in \mathbb{N}$.

There are neighborhoods $\mathcal{U}$ and $\mathcal{V}$ of 1 such that

$$(x, y \in \mathcal{B},\ x \ne y) \Rightarrow xy^{-1} \notin \mathcal{U},$$

$$\mathcal{V}^{a(a+1)} \subset \mathcal{U}.$$

The bound n_a (it is crucial for our proof that it does not depend on d!) can be taken so large that

$$\frac{\mathcal{Q}F(m)}{q(m)} \in \mathcal{V} \quad \text{for} \quad m \geq n_a.$$

Since this applies to all such quotients involved in passing from (3) to (4) and since their number does not exceed $a(a+1)$ we find

$$F(a)\frac{g(n)}{g(an)}\left(f(a)\frac{f(n)}{f(an)}\right)^{-1} \in \mathcal{V}^{a(a+1)} \subset \mathcal{U} \quad \text{for} \quad n \geq n_a.$$

This, by the choice of $\mathcal{U}$, implies equality

$$\frac{g(an)}{g(n)} = \frac{F(a)f(an)}{f(a)f(n)} \quad \text{for} \quad n \geq n_a. \tag{7}$$

Now we let $n = bk$ with arbitrary b and $(k, ab) = 1$ tend to infinity, apply (6), and find

$$\frac{F(ab)}{F(b)} = \frac{F(a)f(ab)}{f(a)f(b)},$$

which says that $h := f/F$ is completely multiplicative. With this knowledge (7) simplifies and gives

$$\frac{g(an)}{g(n)} = \frac{F(an)}{f(n)} \quad \text{for} \quad n \geq n_a$$

or, if we write $F/g := v$,

$$v(b) = v(ab) \quad \text{for} \quad b \geq n_a. \tag{8}$$

In particular this implies that if $b \geq n_2$ then $v(b) = v(2b)$. Now also $2b \geq n_2$ etc., hence $v(2b) = v(4b)$ etc., $v(b) = v(2^k b)$ for all k. Furthermore for all large k (as soon as $2^k \geq n_b$) another application of (8) gives $v(b) = v(2^k)$. Since this is independent of b we have:

The function $v = F/g$ is constant from some point $(= n_2)$ on.

From this point on $\mathcal{Q}F(n) = \mathcal{Q}g(n) = q(n) \in \mathcal{Q}f(\mathbb{N})'$, thus $\mathcal{Q}F(\mathbb{N})$ is finite. The multiplicativity of F follows from $F = fh^{-1}$.

Finally

$$\mathcal{Q}h(n) = \frac{\mathcal{Q}f(n)}{\mathcal{Q}F(n)} = \frac{\mathcal{Q}f(n)}{q(n)} \quad \text{for} \quad n \geq n_2,$$

$$\to 1 \quad \text{as} \quad n \to \infty.$$

For Theorem 2 it is assumed that $\mathcal{G}$ is locally compact. So by Theorem DKM the function h of the lemma can be extended to a continuous homomorphism χ of $\mathbb{R}_+^*$ into $\mathcal{G}$. By Theorem 1 the function F of the lemma maps to a finite set. This ends the proof of Theorem 2.

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