ON TRANSLATIVE AND QUASI–COMMUTATIVE OPERATIONS

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Dedicated to my friend, Imre Kátai, on the occasion of his 65th birthday

Abstract. We determine all the continuous operations $\circ : \mathbb{R}^2 \to \mathbb{R}$ that are translative $((x + z) \circ (y + z) = x \circ y + z)$ quasi-commutative $(x \circ (y \circ z) = y \circ (x \circ z))$.

1. Introduction

Let $(G, +)$ be an Abelian group. The operation $\circ : G^2 \to G$ is called translative if

\[(1) \quad (x + z) \circ (y + z) = x \circ y + z\]

holds for all $x, y, z \in G$. If $f(x) := -x \circ 0$ $(x \in G)$ then substituting $z := -y$ (1) implies

\[(2) \quad x \circ y = y - f(x - y)\]

for all $x, y \in G$, and conversely, if $f : G \to G$ is arbitrary then the operation $\circ$ given by formula (2) is translative. This means that defining a translative

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operation is equivalent to defining a function \( f : G \to G \). It is an interesting problem to examine what can be stated about translative operations that have further properties. The operation \( \circ : G^2 \to G \) is called quasi-commutative if

\[
x \circ (y \circ z) = y \circ (x \circ z)
\]

holds for all \( x, y, z \in G \). In this case we can ask which are the translative and quasi-commutative operations on the Abelian group \((G, +)\). From equation (2) we have

\[
x \circ (y \circ z) = y \circ z - f(x - y \circ z) = z - f(y - z) - f(x - z + f(y - z)]
\]

for the unknown function \( f : G \to G \), and (3) implies

\[
f(y - z) + f[x - z + f(y - z)] = f(x - z) + f[y - z + f(x - z)]
\]

for all \( x, y, z \in G \). This yields the functional equation

\[
f(x + f(y)) + f(y) = f[y + f(x)] + f(x) \quad (x, y \in G)
\]

for the unknown function \( f : G \to G \). Conversely, one can easily see that if \( f : G \to G \) solves (4) then the operation \( \circ \) given by (2) is translative and quasi-commutative on the Abelian group \((G, +)\). Let \( S(G) \) denote the set of all the solutions \( f : G \to G \) of equation (4). The following assertions can be easily checked:

(i) If \( f(x) := c \; (x \in G) \) for some fix \( c \in G \) then \( f \in S(G) \).
(ii) If \( f \in S(G) \) and \( a \in G \) then with the notation \( f_a(x) := f(x + a) \; (x \in G) \), \( f_a \in S(G) \).
(iii) If \( f \in S(G) \) then with the notation \( f^*(x) := -f(-x) \; (x \in G) \), \( f^* \in S(G) \).

In this paper our aim is to determine the continuous translative and quasi-commutative operations defined on the group \((G, +) = (\mathbb{R}, +)\), that is on the additive group of real numbers. According the remarks above, this is equivalent to giving the continuous functions: \( f : \mathbb{R} \to \mathbb{R} \) satisfying \( f \in S(\mathbb{R}) \). This problem was first considered (under further additional conditions) by Kampé de Feriet-Forte [7], whose investigations were motivated by information theory. Therefore we call the functional equation (4) Kampé de Feriet-Forte equation.

The following result was proved independently and using basically different methods by C. Baiocchi [2], [3] and Z. Daróczy [4].

**Theorem 1.** If \( f \in S(\mathbb{R}) \) is continuous and

\[
f(-x) = f(x) + x \quad (x \in \mathbb{R})
\]
holds, then $f$ is either of the following forms

$$f(x) = \pm \frac{1}{2} \{ |x| \mp x \} \quad (x \in \mathbb{R})$$

or

$$f(x) = -\frac{1}{A} \ln (1 + e^{Ax}) \quad (x \in \mathbb{R}),$$

where $A \neq 0$ is a constant.

On the basis of the previous results, our aim is to show the following: If $f \in S(\mathbb{R})$ is continuous and not constant then there exists $a \in \mathbb{R}$ such that $f_a(-x) = f_a(x) + x \quad (x \in \mathbb{R})$, and we can apply Theorem 1, since $f_a \in S(\mathbb{R})$ is continuous.

2. On the Kampé de Feriet-Forte equation on the additive group of real numbers

Let $S(\mathbb{R})$ denote the set of all the functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the Kampé de Feriet-Forte equation

$$(6) \quad f[x + f(y)] + f(y) = f[y + f(x)] + f(x) \quad (x, y \in \mathbb{R}).$$

In what follows we prove the existence of functions $f \in S(\mathbb{R})$ nowhere continuous. For this purpose let $A : \mathbb{R} \to \mathbb{R}$ be an additive function for which $A(1) = 0$, $A(x) \in \mathbb{Q}$, and $A$ is not constant. Such a function $A$ exists, since we can choose not zero rational numbers as values of $A$ on the Hamel basis (Hamel [5], Kuczma [6]). Then $f(x) := A(x) \quad (x \in \mathbb{R})$ solves (6), since

$$f[x + f(y)] + f(y) = A[x + A(y)] + A(y) = A(x) + A[A(y)] + A(y) = A(x) + A(y)A(1) = A(x) + A(y),$$

from which our assertion follows. This solution is obviously nowhere continuous (and not measurable) (Aczél [1], Kuczma [6]).

Therefore it is natural to assume that $f \in S(\mathbb{R})$ is continuous, since finding the general solutions seems hopeless.

Our aim is to prove the following theorem, from which, applying the previous results, we obtain all the continuous solutions of equation (6).
**Theorem 2.** If $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous and nonconstant solution of the Kampé de Feriet-Forte equation (6) then there exists $a \in \mathbb{R}$ for which

$$f(-x + a) = f(x + a) + x$$

holds for all $x \in \mathbb{R}$.

To prove the theorem we need the following results.

**Lemma 1.** If $f \in S(\mathbb{R})$ is continuous then $f$ is either nonnegative or nonpositive.

**Proof.** Supposing the contrary that there exists a continuous $f \in S(\mathbb{R})$ which does not satisfy Lemma 1. Then there are real numbers $x_1 \neq x_2$ such that $f(x_1) < 0$ and $f(x_2) > 0$. Continuity then implies the existence of a number $x_0$ between $x_1$ and $x_2$ for which $f(x_0) = 0$. Now substitute $x = x_0$ in (6), then

$$f(x_0 + f(y)) = 0$$

holds for all $y \in \mathbb{R}$. Since $f$ is continuous, it assumes every value in the interval $[f(x_1), f(x_2)]$, that is, by (8),

$$f(t) = 0 \quad \text{if} \quad t \in [x_0 + f(x_1), x_0 + f(x_2)].$$

We show by induction that

$$f(t) = 0 \quad \text{if} \quad t \in [x_0 + nf(x_1), x_0 + nf(x_2)]$$

for any natural number $n$. For $n = 1$ our assertion is true because of (9). Now suppose that (10) holds for some integer $n \geq 1$. Let $x = u \in [x_0 + nf(x_1), x_0 + nf(x_2)]$ be arbitrary in (6), then the assumptions imply

$$f[u + f(y)] = 0$$

for all $y \in \mathbb{R}$. Since $f$ takes every value from the interval $[f(x_1), f(x_2)]$, with the notation $t = u + f(y)$, from (11) we obtain

$$f(t) = 0 \quad \text{if} \quad t \in [x_0 + (n + 1)f(x_1), x_0 + (n + 1)f(x_2)],$$

that is (10) is true for $(n + 1)$. Since $x_0 + nf(x_1) \rightarrow -\infty$ and $x_0 + nf(x_2) \rightarrow \infty$, (10) implies $f(t) = 0$ for all $t \in \mathbb{R}$, which is a contradiction. This completes the proof of the Lemma.
In the following we determine all the nonconstant and continuous functions \( f \in S(\mathbb{R}) \) satisfying
\[
N_f := \{ x : x \in \mathbb{R}, f(x) = 0 \} \neq \emptyset.
\]

**Lemma 2.** If \( f \in S(\mathbb{R}) \) is nonnegative, nonconstant, continuous and \( N_f \neq \emptyset \), then for any \( \xi \in N_f \)
\[
f(t) = 0 \quad \text{if} \quad t \geq \xi.
\]

**Proof.** Since \( f \) is not constant, there exists \( \eta \in \mathbb{R} \) such that \( f(\eta) = b > 0 \).
Continuity then implies \([0, b] \subset F\), where \( F := \{ f(x) : x \in \mathbb{R} \} \). We prove by induction that for any natural number \( n \) we have
\[
(12) \quad f(\xi + nz) = 0 \quad \text{for all} \quad z \in [0, b].
\]
Let \( x = \xi \) in (6), then
\[
f[\xi + f(y)] = 0
\]
for all \( y \in \mathbb{R} \), that is with the notation \( z = f(y) \),
\[
(13) \quad f(\xi + z) = 0
\]
for all \( z \in F \). This implies (13) for all \( z \in [0, b] \). With this we have shown (12) for \( n = 1 \). Now suppose that (12) holds for some natural number \( n \geq 1 \).
Substituting \( \xi + nz \) (\( z \in [0, b] \) arbitrary) in (6) we have
\[
f[\xi + nz + f(y)] = 0
\]
for all \( y \in \mathbb{R} \), that is, with the notation \( z = f(y) \),
\[
f[\xi + (n+1)z] = 0
\]
holds for all \( z \in [0, b] \). With this we have proved (12) for all \( n \in \mathbb{N} \).

Now take any \( t > \xi \). Then there exists a natural number \( n \) for which
\[
\frac{t - \xi}{n} \in [0, b], \quad \text{that is} \ t \ \text{can be written as} \ t = \xi + nz \ \text{with} \ z \in [0, b].
\]
Thus by (12), \( f(t) = 0 \).

**Lemma 3.** If \( f \in S(\mathbb{R}) \) is nonnegative, nonconstant, continuous and \( N_f \neq \emptyset \), then
\[
a := \inf N_f > -\infty.
\]
**Proof.** If \( a = -\infty \) choose an arbitrary \( x \in \mathbb{R} \). Then there exists \( \xi \in N_f \) such that \( \xi < x \). By Lemma 2 \( f(x) = 0 \), so \( f \) is everywhere zero. This contradicts the assumption that \( f \) is nonconstant, which proves the Lemma.

**Lemma 4.** Let \( f \in S(\mathbb{R}) \) be nonnegative, nonconstant, continuous and \( N_f \neq \emptyset \). With the notation \( a := \inf N_f > -\infty \), let

\[(14)\quad g(x) := f(x + a) \quad (x \in \mathbb{R}).\]

Then \( g : \mathbb{R} \to \mathbb{R} \) is nonnegative, nonconstant, continuous, and \( g \in S(\mathbb{R}) \) having the following properties:

\[
\begin{align*}
(i) & \quad g(t) = 0 \quad \text{if} \quad t \geq 0; \\
(ii) & \quad g(t) > 0 \quad \text{if} \quad t < 0.
\end{align*}
\]

**Proof.** (14) obviously implies that \( g \) is nonnegative, nonconstant, continuous. On the other hand, \( g \in S(\mathbb{R}) \). Properties (i) and (ii) follow from Lemmas 2 and 3.

**Lemma 5.** If \( g \in S(\mathbb{R}) \) is continuous and properties (i) and (ii) of Lemma 4 are fulfilled then

\[(15)\quad g(t) = -t \quad \text{if} \quad t < 0.\]

**Proof.** Let \( x < 0 \) be fixed and take an arbitrary \( y \in [-g(x), 0] \). Then \( 0 \leq y + g(x) \), and since \( g \in S(\mathbb{R}) \), (6) and (i) imply

\[g[x + g(y)] + g(y) = g(x).\]

Hence, with the notation \( z := x + g(y) \), we have

\[g(z) = g(x) - g(y) = g(x) - (z - x) = -z + g(x) + x\]

for all \( z \in [x, x + g(-g(x))] \). This means that for any \( x < 0 \) there exists a number \( \varepsilon_x := g(-g(x)) > 0 \) such that

\[g(z) = -z + b_x\]

holds in the closed interval \([x, x + \varepsilon_x]\). On the other hand, from the continuity of \( g \) necessarily \( b_x = b \) (\( b \) is constant) follows for all \( x \). However, since \( g(0) = 0 \), this yields \( b = 0 \), that is \( g(x) = -x \) for all \( x \). Moreover, the function \( g \) defined as above satisfies (6).
On the basis of our previous results we can state the following

**Theorem 3.** If \( f : \mathbb{R} \to \mathbb{R} \) is a nonconstant continuous solution of the Kampé de Feriet-Forte equation (6) with

\[
N_f := \{ x \mid x \in \mathbb{R}, f(x) = 0 \} \neq \emptyset,
\]

then there exists a number \( a \in \mathbb{R} \) such that either

\[
f(x) = \frac{1}{2}(|x - a| - (x - a))
\]

or

\[
f(x) = -\frac{1}{2}(|x - a| + (x - a))
\]

holds for all \( x \in \mathbb{R} \). In both cases the existing \( a \in \mathbb{R} \) satisfies (7) for all \( x \in \mathbb{R} \).

**Proof.** By Lemma 1, \( f \) keeps the sign and we distinguish between two cases.

(i) If \( f(x) \geq 0 \) for all \( x \in \mathbb{R} \) then, with the notation \( a = \inf N_f > -\infty \) (Lemma 3), applying Lemmas 4 and 5 we have that the function \( g \) defined in (14) satisfies

\[
g(t) = \frac{1}{2}(|t| - t) \quad (t \in \mathbb{R}).
\]

This gives the solution (16).

(ii) If \( f(x) \leq 0 \) for all \( x \in \mathbb{R} \) define the following function

\[
f^*(x) := -f(-x) \quad (x \in \mathbb{R}).
\]

Then \( f^* \in S(\mathbb{R}) \), \( f^* \) is continuous, nonconstant, moreover \( N_f^* \neq \emptyset \). Thus, with the notation \( -a := \inf N_f^* > -\infty \), we have case (i), and we obtain the solution (17).

At last, an easy computation shows that in both cases there exists \( a \in \mathbb{R} \) for which (7), that is

\[
f(-x + a) = f(x + a) + x \quad (a \in \mathbb{R})
\]

holds.

This also means that if \( f \in S(\mathbb{R}) \) is nonconstant, continuous and \( N_f \neq \emptyset \), then Theorem 2 holds, but in addition, with the help of the existing \( a \in \mathbb{R} \), \( f \) can be completely given in either of the forms (16) or (17).
In what follows we have to examine the case when $f \in S(\mathbb{R})$ is nonconstant, continuous and $N_f \neq \emptyset$. By Lemma 3, then $f$ is either everywhere positive or everywhere negative.

**Lemma 6.** If $f \in S(\mathbb{R})$ is positive and continuous, then $f$ is monotone decreasing.

**Proof.** Let

\begin{equation}
  g_n(x) := nf(x) - x \quad (x \in \mathbb{R})
\end{equation}

for any natural number $n$. We show that $g_n : \mathbb{R} \to \mathbb{R}$ is injective. If $g_1(x) = g_1(y)$ then $f(x) + y = f(y) + x$. We have two possible cases, namely, $f(x) = f(y)$ and $f(x) \neq f(y)$. In the first case, $x = y$ and the second case contradicts (6). Thus $g_1 : \mathbb{R} \to \mathbb{R}$ is injective.

Now suppose that $g_n : \mathbb{R} \to \mathbb{R}$ is injective for some natural number $n \geq 1$ and let $g_{n+1}(x) = g_{n+1}(y)$. Then (6) implies

\[
g_n[x + f(y)] - g_n[y + f(x)] =
\]

\[
= nf[x + f(y)] - x - f(y) - nf[y + f(x)] + y + f(x) =
\]

\[
= n(f[x + f(y)] + f(y) - f[y + f(x)] - f(x)) +
\]

\[
+ (n + 1)f(x) - x - (n + 1)f(y) + y =
\]

\[
= g_{n+1}(x) - g_{n+1}(y) = 0.
\]

Since $g_n$ is injective, we have $x + f(y) = y + f(x)$, that is $g_1(x) = g_1(y)$, from which $x = y$ follows. Thus we have proved by induction that $g_n$ is injective for all $n \in \mathbb{N}$.

On the other hand,

\[
f(x) = \frac{g_n(x) + x}{n} > 0
\]

yields

\[
g_n(x) > -x \quad (x \in \mathbb{R}),
\]

from which we obtain

\[
\lim_{x \to \infty} g_n(x) = \infty.
\]

Since $g_n : \mathbb{R} \to \mathbb{R}$ is continuous and injective, the above assertion implies that $g_n$ is strictly monotone decreasing. If $x < y$ then

\[
f(x) - f(y) = \frac{1}{n} [g_n(x) - g_n(y)] + \frac{1}{n}(x - y) > \frac{1}{n}(x - y)
\]
for all \( n \in \mathbb{N} \), and taking the limit \( n \to \infty \), we have

\[
f(x) \geq f(y),
\]

that is \( f \) is monotone decreasing.

**Lemma 7.** If \( f \in S(\mathbb{R}) \) is positive, nonconstant and continuous, then

\[
\lim_{x \to \infty} f(x) = 0 \quad \lim_{x \to -\infty} f(x) = \infty,
\]

and \( f \) is strictly monotone decreasing.

**Proof.** By Lemma 6, the limits

\[
\lim_{x \to \infty} f(x) = \alpha \quad \lim_{x \to -\infty} f(x) = \beta
\]

exist in the extended set of real numbers, and \( 0 \leq \alpha \leq \beta \leq \infty \), because \( f \) is not constant. If \( \alpha > 0 \) or \( \beta < \infty \) then taking the limit \( y \to \infty \) (or \( y \to -\infty \)) in (6) we obtain \( f(x + \alpha) = f(x) \) (or \( f(x + \beta) = f(x) \)) for all \( x \in \mathbb{R} \), that is \( f \) periodic with positive period. Then, by Lemma 6, \( f \) is constant, which is a contradiction. Thus (19) is true.

Now suppose that there exist \( x < y \) with \( f(x) = f(y) \), i.e. \( f \) is not strictly monotone decreasing (but because of Lemma 6, monotone decreasing). Then by \( 0 < y - x \) and (19), there exists \((t \in \mathbb{R})\) for which

\[
y - x = f(t).
\]

We show that

\[
(20) \quad f[x + nf(t)] = f(y)
\]

for any natural number \( n \). For \( n = 1 \) (20) obviously holds. If (20) holds for some \( n \geq 1 \) then, by (6),

\[
\begin{align*}
f[x + (n + 1)f(t)] &= f[x + nf(t) + f(t)] = \\
&= f[t + f(x + nf(t)) + f(x + nf(t))] - f(t) = \\
&= f[t + f(y)] + f(y) - f(t) = f[t + f(x)] + f(x) - f(t) = \\
&= f[x + f(t)] = f(y).
\end{align*}
\]

Now taking the limit \( n \to \infty \), we have

\[
f(y) = 0,
\]
which is a contradiction.

**Lemma 8.** If \( f \in S(\mathbb{R}) \) is positive, nonconstant and continuous, then there exists the finite limit

\[
\lim_{x \to -\infty} [f(x) + x].
\]

**Proof.** By Lemma 7, there exists the inverse function \( f^{-1} : \mathbb{R}_+ \to \mathbb{R} \), which is continuous and strictly monotone decreasing. From (6) we have for all \( t > 0 \) and \( (x \in \mathbb{R}) \)

\[
f(x + t) + t - f(x) = f[f^{-1}(t) + f(x)],
\]

whence, by (19),

\[
\lim_{x \to -\infty} [f(x + t) + t - f(x)] = 0.
\]

The substitution \( t = 1 \) gives

\[
\lim_{x \to -\infty} [f(x + 1) + 1 - f(x)] = 0.
\]

(23) implies that there exists a real number \( K \) such that

\[2 > f(x + 1) - f(x) + 1 \quad \text{if} \quad x < K.\]

In equation (22) replace \( t \) by \( 2 - [f(x + 1) - f(x) + 1] \), which is positive if \( x < K \), and replace \( x \) by \( f(x + 1) + x \). Then, by (6),

\[
f\{f^{-1}[2 - (f(x + 1) - f(x) + 1)] + f[f(x + 1) + x]\} =
\]

\[
= f[2 - (f(x + 1) - f(x) + 1) + f(x + 1) + x] +
\]

\[
+ 2 - (f(x + 1) - f(x) + 1) - f[f(x + 1) + x] =
\]

\[
= f[1 + x + f(x)] + f(x) + 1 - f(x + 1) - f[x + f(x + 1)] =
\]

\[
= f[x + f(x + 1)] + f(x + 1) + 1 - f(x + 1) - f[x + f(x + 1)] = 1
\]

holds for all \( x < K \). From this last equation we have

\[
f(x + 1) + x + 1 = 1 + f^{-1}[f^{-1}(1) - f^{-1}(2 - (f(x + 1) - f(x) + 1))]\]

for all \( x < K \). Applying (23), equation (24) implies

\[
\lim_{x \to -\infty} [f(x) + x] = 1 + f^{-1}[f^{-1}(1) - f^{-1}(2)],
\]
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which proves the Lemma.

Now we can formulate the following result.

**Theorem 4.** If \( f \in S(\mathbb{R}) \) is positive, nonconstant and continuous, then there exists \( a \in \mathbb{R} \) such that

\[
f(-x + a) = f(x + a) + x \quad (x \in \mathbb{R})
\]

holds.

**Proof.** Let

\[
g(x) := f(x) + x \quad (x \in \mathbb{R}).
\]

Then equation (6) implies that

\[
f[x + f(y)] + f(y) = g[x + g(y) - y] - x - g(y) + g(y) - y = g[x - y + g(y)] - x
\]

is symmetric in \( x, y \), that is

\[
g[x - y + g(y)] - x = g[y - x + g(x)] - y.
\]

In this equation put \( t = x - y \), then

\[
(25) \quad g[t + g(y)] = t + g[-t + g(y + t)]
\]

holds for all \( t, y \in \mathbb{R} \). By Lemma 8, there exists the finite limit

\[
\lim_{y \to -\infty} g(y) = \lim_{y \to -\infty} [f(y) + y] =: a.
\]

Take \( y \to -\infty \) in (25), then we have

\[
g(t + a) = t + g(-t + a)
\]

for all \( t \in \mathbb{R} \), which implies the assertion of the Theorem.

Analogously we obtain the following

**Theorem 5.** If \( f \in S(\mathbb{R}) \) is negative, nonconstant and continuous, then there exists \( a^* \in \mathbb{R} \) such that (7), that is

\[
f(-x + a^*) = f(x + a^*) + x \quad (x \in \mathbb{R})
\]

holds.

**Proof.** In this case let

\[
f^*(x) := -f(-x) \quad (x \in \mathbb{R}),
\]
then $f^* \in S(\mathbb{R})$ is positive, nonconstant and continuous. Thus, by Theorem 4, there exists $a \in \mathbb{R}$ such that

$$f^*(-x + a) = f^*(x + a) + x \quad (x \in \mathbb{R}).$$

From this we have

$$f(-x - a) = f(x - a) + x \quad (x \in \mathbb{R}),$$

that is with the notation $a^* := -a$, the assertion of the Theorem follows.

With the help of the previous results we can easily prove Theorem 2.

**Proof (of Theorem 2).** If $f \in S(\mathbb{R})$ is nonconstant and continuous, then there are two possibilities: either $N_f \neq \emptyset$ or $N_f = \emptyset$. In the first case Theorem 3 implies the assertion. In the second case $f$ is everywhere positive or everywhere negative, and the assertion of Theorem 2 follows from Theorems 4 and 5.

3. Continuous, translative and quasi-commutative operations on the additive group of real numbers

On the basis of the above and previous results we can state the following theorem.

**Theorem 6.** If $\circ : \mathbb{R}^2 \to \mathbb{R}$ is a continuous, translative and quasi-commutative operation then it is one of the following operations:

(i) $x \circ y = y + c \quad (x, y \in \mathbb{R})$ and $c \in \mathbb{R}$ is constant;
(ii) $x \circ y = \min\{x - a; y\} \quad (x, y \in \mathbb{R})$ and $a \in \mathbb{R}$ is constant;
(iii) $x \circ y = \max\{x - a; y\} \quad (x, y \in \mathbb{R})$ and $a \in \mathbb{R}$ is constant;
(iv) $x \circ y = \frac{1}{A} \ln \left( e^{A(x-a)} + e^{Ay} \right) \quad (x, y \in \mathbb{R})$ and $a \in \mathbb{R}$, $A \neq 0$ are constant.

**Proof.** Under these conditions $f(x) := -x \circ 0 \quad (x \in \mathbb{R}$ is continuous and $f \in S(\mathbb{R})$, moreover

$$x \circ y = y - f(x - y) \quad (x, y \in \mathbb{R}).$$

There are the following possible cases:
(1) $f(x) = -c$ ($x \in \mathbb{R}$) for some constant $c \in \mathbb{R}$. Then (26) implies the solution (i);

(2) If $f$ is not constant, then according to Theorem 3, suppose that $N_f \neq \emptyset$. Then the solutions (16) and (17) give the solutions (ii) and (iii) for some constant $a \in \mathbb{R}$.

(3) If $f$ is not constant and $N_f \neq \emptyset$, then, by Theorem 2, there exists $a \in \mathbb{R}$ such that (7) holds, that is the function $f_a(x) := f(x+a)$ ($x \in \mathbb{R}$) satisfies

$$f_a(-x) = f_a(x) + x \quad (x \in \mathbb{R}).$$

On the other hand, $f_a \in S(\mathbb{R})$ and $f_a$ is continuous, thus by Theorem 1, there exists $A \neq 0$ for which

$$f_a(x) = -\frac{1}{A} \ln(1 + e^{Ax}) \quad (x \in \mathbb{R}).$$

From this the solutions (iv) follow.

References


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