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ON TRANSLATIVE AND QUASI–COMMUTATIVE OPERATIONS

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Dedicated to my friend, Imre Kátai,
on the occasion of his 65th birthday

Abstract. We determine all the continuous operations $\circ : \mathbb{R}^2 \to \mathbb{R}$ that are translative ($(x+z)\circ(y+z) = x\circ y + z$) quasi-commutative ($x\circ(y\circ z) = y\circ(x\circ z)$).

1. Introduction

Let $(G,+)$ be an Abelian group. The operation $\circ : G^2 \to G$ is called *translative* if

$$(x+z)\circ(y+z) = x\circ y + z \tag{1}$$

holds for all $x,y,z \in G$. If $f(x) := -x\circ 0 \ \ (x \in G)$ then substituting $z := -y$ (1) implies

$$x\circ y = y - f(x-y) \tag{2}$$

for all $x,y \in G$, and conversely, if $f : G \to G$ is arbitrary then the operation $\circ$ given by formula (2) is translative. This means that defining a translative

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operation is equivalent to defining a function $f : G \to G$. It is an interesting problem to examine what can be stated about translative operations that have further properties. The operation $\circ : G^2 \to G$ is called *quasi-commutative* if

$$x \circ (y \circ z) = y \circ (x \circ z) \tag{3}$$

holds for all $x, y, z \in G$. In this case we can ask which are the translative and quasi-commutative operations on the Abelian group $(G, +)$. From equation (2) we have

$$x \circ (y \circ z) = y \circ z - f(x - y \circ z) = z - f(y - z) - f[x - z + f(y - z)]$$

for the unknown function $f : G \to G$, and (3) implies

$$f(y - z) + f[x - z + f(y - z)] = f(x - z) + f[y - z + f(x - z)]$$

for all $x, y, z \in G$. This yields the functional equation

$$f[x + f(y)] + f(y) = f[y + f(x)] + f(x) \qquad (x, y \in G) \tag{4}$$

for the unknown function $f : G \to G$. Conversely, one can easily see that if $f : G \to G$ solves (4) then the operation $\circ$ given by (2) is translative and quasi-commutative on the Abelian group $(G, +)$. Let $S(G)$ denote the set of all the solutions $f : G \to G$ of equation (4). The following assertions can be easily checked:

(*i*) If $f(x) := c \;\; (x \in G)$ for some fix $c \in G$ then $f \in S(G)$.
(*ii*) If $f \in S(G)$ and $a \in G$ then with the notation $f_a(x) := f(x + a) \;\; (x \in G)$, $f_a \in S(G)$.
(*iii*) If $f \in S(G)$ then with the notation $f^*(x) := -f(-x) \;\; (x \in G)$, $f^* \in S(G)$.

In this paper our aim is to determine the *continuous* translative and quasi-commutative operations defined on the group $(G, +) = (\mathbb{R}, +)$, that is on the additive group of real numbers. According the remarks above, this is equivalent to giving the *continuous* functions: $f : \mathbb{R} \to \mathbb{R}$ satisfying $f \in S(\mathbb{R})$. This problem was first considered (under further additional conditions) by Kampé de Feriet-Forte [7], whose investigations were motivated by information theory. Therefore we call the functional equation (4) Kampé de Feriet-Forte equation.

The following result was proved independently and using basically different methods by C. Baiocchi [2], [3] and Z. Daróczy [4].

Theorem 1. *If $f \in S(\mathbb{R})$ is continuous and*

$$f(-x) = f(x) + x \qquad (x \in \mathbb{R}) \tag{5}$$

holds, then f is either of the following forms

$$f(x) = \pm\frac{1}{2}\{|x| \mp x\} \qquad (x \in \mathbb{R})$$

or

$$f(x) = -\frac{1}{A}\ln\left(1 + e^{Ax}\right) \qquad (x \in \mathbb{R}),$$

where $A \neq 0$ is a constant.

On the basis of the previous results, our aim is to show the following: If $f \in S(\mathbb{R})$ is continuous and not constant then there exists $a \in \mathbb{R}$ such that $f_a(-x) = f_a(x) + x \ \ (x \in \mathbb{R})$, and we can apply Theorem 1, since $f_a \in S(\mathbb{R})$ is continuous.

2. On the Kampé de Feriet-Forte equation on the additive group of real numbers

Let $S(\mathbb{R})$ denote the set of all the functions $f : \mathbb{R} \to \mathbb{R}$ satisfying the Kampé de Feriet-Forte equation

$$f[x + f(y)] + f(y) = f[y + f(x)] + f(x) \qquad (x, y \in \mathbb{R}). \tag{6}$$

In what follows we prove the existence of functions $f \in S(\mathbb{R})$ nowhere continuous. For this purpose let $A : \mathbb{R} \to \mathbb{R}$ be an *additive* function for which $A(1) = 0$, $A(x) \in \mathbb{Q}$, and A is not constant. Such a function A exists, since we can choose not zero rational numbers as values of A on the Hamel basis (Hamel [5], Kuczma [6]). Then $f(x) := A(x) \ \ (x \in \mathbb{R})$ solves (6), since

$$f[x + f(y)] + f(y) = A[x + A(y)] + A(y) = A(x) + A[A(y)] + A(y) =$$

$$= A(x) + A(y) + A(y)A(1) = A(x) + A(y),$$

from which our assertion follows. This solution is obviously nowhere continuous (and not measurable) (Aczél [1], Kuczma [6]).

Therefore it is natural to assume that $f \in S(\mathbb{R})$ is *continuous*, since finding the general solutions seems hopeless.

Our aim is to prove the following theorem, from which, applying the previous results, we obtain all the continuous solutions of equation (6).

Theorem 2. *If $f : \mathbb{R} \to \mathbb{R}$ is a continuous and nonconstant solution of the Kampé de Feriet-Forte equation (6) then there exists $a \in \mathbb{R}$ for which*

$$f(-x+a) = f(x+a) + x \tag{7}$$

holds for all $x \in \mathbb{R}$.

To prove the theorem we need the following results.

Lemma 1. *If $f \in S(\mathbb{R})$ is continuous then f is either nonnegative or nonpositive.*

Proof. Supposing the contrary that there exists a continuous $f \in S(\mathbb{R})$ which does not satisfy Lemma 1. Then there are real numbers $x_1 \neq x_2$ such that $f(x_1) < 0$ and $f(x_2) > 0$. Continuity then implies the existence of a number x_0 between x_1 and x_2 for which $f(x_0) = 0$. Now substitute $x = x_0$ in (6), then

$$f[x_0 + f(y)] = 0 \tag{8}$$

holds for all $y \in \mathbb{R}$. Since f is continuous, it assumes every value in the interval $[f(x_1), f(x_2)]$, that is, by (8),

$$f(t) = 0 \qquad \text{if} \quad t \in [x_0 + f(x_1), x_0 + f(x_2)]. \tag{9}$$

We show by induction that

$$f(t) = 0 \qquad \text{if} \quad t \in [x_0 + nf(x_1),\ x_0 + nf(x_2)] \tag{10}$$

for any natural number n. For $n = 1$ our assertion is true because of (9). Now suppose that (10) holds for some integer $n \geq 1$. Let $x = u \in [x_0 + nf(x_1), x_0 + nf(x_2)]$ be arbitrary in (6), then the assumptions imply

$$f[u + f(y)] = 0 \tag{11}$$

for all $y \in \mathbb{R}$. Since f takes every value from the interval $[f(x_1), f(x_2)]$, with the notation $t = u + f(y)$, from (11) we obtain

$$f(t) = 0 \qquad \text{if} \quad t \in [x_0 + (n+1)f(x_1),\ x_0 + (n+1)f(x_2)],$$

that is (10) is true for $(n+1)$. Since $x_0 + nf(x_1) \to -\infty$ and $x_0 + nf(x_2) \to \infty$, (10) implies $f(t) = 0$ for all $t \in \mathbb{R}$, which is a contradiction. This completes the proof of the Lemma.

In the following we determine all the nonconstant and continuous functions $f \in S(\mathbb{R})$ satisfying

$$N_f := \{x \mid x \in \mathbb{R},\ f(x) = 0\} \neq \emptyset.$$

Lemma 2. *If $f \in S(\mathbb{R})$ is nonnegative, nonconstant, continuous and $N_f \neq \emptyset$, then for any $\xi \in N_f$*

$$f(t) = 0 \qquad \text{if} \quad t \geq \xi.$$

Proof. Since f is not constant, there exists $\eta \in \mathbb{R}$ such that $f(\eta) = b > 0$. Continuity then implies $[0, b] \subset F$, where $F := \{f(x) \mid x \in \mathbb{R}\}$. We prove by induction that for any natural number n we have

$$f(\xi + nz) = 0 \qquad \text{for all} \quad z \in [0, b]. \tag{12}$$

Let $x = \xi$ in (6), then

$$f[\xi + f(y)] = 0$$

for all $y \in \mathbb{R}$, that is with the notation $z = f(y)$,

$$f(\xi + z) = 0 \tag{13}$$

for all $z \in F$. This implies (13) for all $z \in [0, b]$. With this we have shown (12) for $n = 1$. Now suppose that (12) holds for some natural number $n \geq 1$. Substituting $\xi + nz$ ($z \in [0, b]$ arbitrary) in (6) we have

$$f[\xi + nz + f(y)] = 0$$

for all $y \in \mathbb{R}$, that is, with the notation $z = f(y)$,

$$f[\xi + (n + 1)z] = 0$$

holds for all $z \in [0, b]$. With this we have proved (12) for all $n \in \mathbb{N}$.

Now take any $t > \xi$. Then there exists a natural number n for which $\dfrac{t - \xi}{n} \in [0, b]$, that is t can be written as $t = \xi + nz$ with $z \in [0, b]$. Thus by (12), $f(t) = 0$.

Lemma 3. *If $f \in S(\mathbb{R})$ is nonnegative, nonconstant, continuous and $N_f \neq \emptyset$, then*

$$a := \inf N_f > -\infty.$$

Proof. If $a = -\infty$ choose an arbitrary $x \in \mathbb{R}$. Then there exists $\xi \in N_f$ such that $\xi < x$. By Lemma 2 $f(x) = 0$, so f is everywhere zero. This contradicts the assumption that f is nonconstant, which proves the Lemma.

Lemma 4. *Let $f \in S(\mathbb{R})$ be nonnegative, nonconstant, continuous and $N_f \neq \emptyset$. With the notation $a := \inf N_f > -\infty$, let*

$$g(x) := f(x + a) \qquad (x \in \mathbb{R}). \tag{14}$$

Then $g : \mathbb{R} \to \mathbb{R}$ is nonnegative, nonconstant, continuous, and $g \in S(\mathbb{R})$ having the following properties:

$$(i) \qquad g(t) = 0 \quad if \quad t \geq 0;$$

$$(ii) \qquad g(t) > 0 \quad if \quad t < 0.$$

Proof. (14) obviously implies that g is nonnegative, nonconstant, continuous. On the other hand, $g \in S(\mathbb{R})$. Properties (i) and (ii) follow from Lemmas 2 and 3.

Lemma 5. *If $g \in S(\mathbb{R})$ is continuous and properties (i) and (ii) of Lemma 4 are fulfilled then*

$$g(t) = -t \qquad if \quad t < 0. \tag{15}$$

Proof. Let $x < 0$ be fixed and take an arbitrary $y \in [-g(x), 0]$. Then $0 \leq y + g(x)$, and since $g \in S(\mathbb{R})$, (6) and (i) imply

$$g[x + g(y)] + g(y) = g(x).$$

Hence, with the notation $z := x + g(y)$, we have

$$g(z) = g(x) - g(y) = g(x) - (z - x) = -z + g(x) + x$$

for all $z \in [x, x + g(-g(x))]$. This means that for any $x < 0$ there exists a number $\varepsilon_x := g(-g(x)) > 0$ such that

$$g(z) = -z + b_x$$

holds in the closed interval $[x, x + \varepsilon_x]$. On the other hand, from the continuity of g necessarily $b_x = b$ (b is constant) follows for all x. However, since $g(0) = 0$, this yields $b = 0$, that is $g(x) = -x$ for all x. Moreover, the function g defined as above satisfies (6).

On the basis of our previous results we can state the following

Theorem 3. *If $f : \mathbb{R} \to \mathbb{R}$ is a nonconstant continuous solution of the Kampé de Feriet-Forte equation (6) with*

$$N_f := \{x \mid x \in \mathbb{R}, f(x) = 0\} \neq \emptyset,$$

then there exists a number $a \in \mathbb{R}$ such that either

$$f(x) = \frac{1}{2}\{|x-a| - (x-a)\} \tag{16}$$

or

$$f(x) = -\frac{1}{2}\{|x-a| + (x-a)\} \tag{17}$$

holds for all $x \in \mathbb{R}$. In both cases the existing $a \in \mathbb{R}$ satisfies (7) for all $x \in \mathbb{R}$.

Proof. By Lemma 1, f keeps the sign and we distinguish between two cases.

(i) If $f(x) \geq 0$ for all $x \in \mathbb{R}$ then, with the notation $a = \inf N_f > -\infty$ (Lemma 3), applying Lemmas 4 and 5 we have that the function g defined in (14) satisfies

$$g(t) = \frac{1}{2}\{|t| - t\} \qquad (t \in \mathbb{R}).$$

This gives the solution (16).

(ii) If $f(x) \leq 0$ for all $x \in \mathbb{R}$ define the following function

$$f^*(x) := -f(-x) \qquad (x \in \mathbb{R}).$$

Then $f^* \in S(\mathbb{R})$, f^* is continuous, nonconstant, moreover $N_{f^*} \neq \emptyset$. Thus, with the notation $-a := \inf N_{f^*} > -\infty$, we have case ($i$), and we obtain the solution (17).

At last, an easy computation shows that in both cases there exists $a \in \mathbb{R}$ for which (7), that is

$$f(-x+a) = f(x+a) + x \qquad (a \in \mathbb{R})$$

holds.

This also means that if $f \in S(\mathbb{R})$ is nonconstant, continuous and $N_f \neq \emptyset$, then Theorem 2 holds, but in addition, with the help of the existing $a \in \mathbb{R}$, f can be completely given in either of the forms (16) or (17).

In what follows we have to examine the case when $f \in S(\mathbb{R})$ is nonconstant, continuous and $N_f \neq \emptyset$. By Lemma 3, then f is either everywhere positive or everywhere negative.

Lemma 6. *If $f \in S(\mathbb{R})$ is positive and continuous, then f is monotone decreasing.*

Proof. Let

$$g_n(x) := nf(x) - x \qquad (x \in \mathbb{R}) \tag{18}$$

for any natural number n. We show that $g_n : \mathbb{R} \to \mathbb{R}$ is *injective*. If $g_1(x) = g_1(y)$ then $f(x) + y = f(y) + x$. We have two possible cases, namely, $f(x) = f(y)$ and $f(x) \neq f(y)$. In the first case, $x = y$ and the second case contradicts (6). Thus $g_1 : \mathbb{R} \to \mathbb{R}$ is injective.

Now suppose that $g_n : \mathbb{R} \to \mathbb{R}$ is injective for some natural number $n \geq 1$ and let $g_{n+1}(x) = g_{n+1}(y)$. Then (6) implies

$$\begin{aligned} &g_n[x + f(y)] - g_n[y + f(x)] = \\ &\quad = nf[x + f(y)] - x - f(y) - nf[y + f(x)] + y + f(x) = \\ &\quad = n\{f[x + f(y)] + f(y) - f[y + f(x)] - f(x)\} + \\ &\qquad\quad + (n + 1)f(x) - x - (n + 1)f(y) + y = \\ &\quad = g_{n+1}(x) - g_{n+1}(y) = 0. \end{aligned}$$

Since g_n is injective, we have $x + f(y) = y + f(x)$, that is $g_1(x) = g_1(y)$, from which $x = y$ follows. Thus we have proved by induction that g_n is injective for all $n \in \mathbb{N}$.

On the other hand,

$$f(x) = \frac{g_n(x) + x}{n} > 0$$

yields

$$g_n(x) > -x \qquad (x \in \mathbb{R}),$$

from which we obtain

$$\lim_{x \to \infty} g_n(x) = \infty.$$

Since $g_n : \mathbb{R} \to \mathbb{R}$ is *continuous* and *injective*, the above assertion implies that g_n is *strictly monotone decreasing.* If $x < y$ then

$$f(x) - f(y) = \frac{1}{n}[g_n(x) - g_n(y)] + \frac{1}{n}(x - y) > \frac{1}{n}(x - y)$$

for all $n \in \mathbb{N}$, and taking the limit $n \to \infty$, we have

$$f(x) \geq f(y),$$

that is f is monotone decreasing.

Lemma 7. *If $f \in S(\mathbb{R})$ is positive, nonconstant and continuous, then*

$$\lim_{x\to\infty} f(x) = 0 \qquad \lim_{x\to-\infty} f(x) = \infty, \tag{19}$$

and f is strictly monotone decreasing.

Proof. By Lemma 6, the limits

$$\lim_{x\to\infty} f(x) = \alpha \qquad \lim_{x\to-\infty} f(x) = \beta$$

exist in the extended set of real numbers, and $0 \leq \alpha \leq \beta \leq \infty$, because f is not constant. If $\alpha > 0$ or $\beta < \infty$ then taking the limit $y \to \infty$ (or $y \to -\infty$) in (6) we obtain $f(x+\alpha) = f(x)$ (or $f(x+\beta) = f(x)$) for all $(x \in \mathbb{R})$, that is f periodic with positive period. Then, by Lemma 6, f is constant, which is a contradiction. Thus (19) is true.

Now suppose that there exist $x < y$ with $f(x) = f(y)$, i.e. f is not strictly monotone decreasing (but because of Lemma 6, monotone decreasing). Then by $0 < y - x$ and (19), there exists $(t \in \mathbb{R})$ for which

$$y - x = f(t).$$

We show that

$$f[x + nf(t)] = f(y) \tag{20}$$

for any natural number n. For $n = 1$ (20) obviously holds. If (20) holds for some $n \geq 1$ then, by (6),

$$\begin{aligned} f[x + (n+1)f(t)] &= f[x + nf(t) + f(t)] = \\ &= f[t + f(x + nf(t))] + f(x + nf(t)) - f(t) = \\ &= f[t + f(y)] + f(y) - f(t) = f[t + f(x)] + f(x) - f(t) = \\ &= f[x + f(t)] = f(y). \end{aligned}$$

Now taking the limit $n \to \infty$, we have

$$f(y) = 0,$$

which is a contradiction.

Lemma 8. *If $f \in S(\mathbb{R})$ is positive, nonconstant and continuous, then there exists the finite limit*

$$\lim_{x\to-\infty}[f(x)+x]. \tag{21}$$

Proof. By Lemma 7, there exists the inverse function $f^{-1}:\mathbb{R}_+\to\mathbb{R}$, which is continuous and strictly monotone decreasing. From (6) we have for all $t>0$ and $(x\in\mathbb{R})$

$$f(x+t)+t-f(x)=f[f^{-1}(t)+f(x)], \tag{22}$$

whence, by (19),

$$\lim_{x\to-\infty}[f(x+t)+t-f(x)]=0.$$

The substitution $t=1$ gives

$$\lim_{x\to-\infty}[f(x+1)+1-f(x)]=0. \tag{23}$$

(23) implies that there exists a real number K such that

$$2>f(x+1)-f(x)+1 \qquad \text{if} \quad x<K.$$

In equation (22) replace t by $\{2-[f(x+1)-f(x)+1]\}$, which is positive if $x<K$, and replace x by $\{f(x+1)+x\}$. Then, by (6),

$$\begin{aligned} f\{f^{-1}&[2-(f(x+1)-f(x)+1)]+f[f(x+1)+x]\}=\\ &=f[2-(f(x+1)-f(x)+1)+f(x+1)+x]+\\ &\quad+2-(f(x+1)-f(x)+1)-f[f(x+1)+x]=\\ &=f[1+x+f(x)]+f(x)+1-f(x+1)-f[x+f(x+1)]=\\ &=f[x+f(x+1)]+f(x+1)+1-f(x+1)-f[x+f(x+1)]=1\end{aligned}$$

holds for all $x<K$. From this last equation we have

$$f(x+1)+x+1=1+f^{-1}[f^{-1}(1)-f^{-1}(2-(f(x+1)-f(x)+1))] \tag{24}$$

for all $x<K$. Applying (23), equation (24) implies

$$\lim_{x\to-\infty}[f(x)+x]=1+f^{-1}[f^{-1}(1)-f^{-1}(2)],$$

which proves the Lemma.

Now we can formulate the following result.

Theorem 4. *If $f \in S(\mathbb{R})$ is positive, nonconstant and continuous, then there exists $a \in \mathbb{R}$ such that*

$$f(-x+a) = f(x+a) + x \qquad (x \in \mathbb{R})$$

holds.

Proof. Let

$$g(x) := f(x) + x \qquad (x \in \mathbb{R}).$$

Then equation (6) implies that

$$f[x+f(y)] + f(y) = g[x+g(y)-y] - x - g(y) + g(y) - y = g[x-y+g(y)] - x$$

is symmetric in x, y, that is

$$g[x-y+g(y)] - x = g[y-x+g(x)] - y.$$

In this equation put $t = x - y$, then

$$g[t+g(y)] = t + g[-t+g(y+t)] \tag{25}$$

holds for all $t, y \in \mathbb{R}$. By Lemma 8, there exists the finite limit

$$\lim_{y\to-\infty} g(y) = \lim_{y\to-\infty} [f(y)+y] =: a.$$

Take $y \to -\infty$ in (25), then we have

$$g(t+a) = t + g(-t+a)$$

for all $t \in \mathbb{R}$, which implies the assertion of the Theorem.

Analogously we obtain the following

Theorem 5. *If $f \in S(\mathbb{R})$ is negative, nonconstant and continuous, then there exists $a^* \in \mathbb{R}$ such that (7), that is*

$$f(-x+a^*) = f(x+a^*) + x \qquad (x \in \mathbb{R})$$

holds.

Proof. In this case let

$$f^*(x) := -f(-x) \qquad (x \in \mathbb{R}),$$

then $f^* \in S(\mathbb{R})$ is positive, nonconstant and continuous. Thus, by Theorem 4, there exists $a \in \mathbb{R}$ such that

$$f^*(-x+a) = f^*(x+a) + x \qquad (x \in \mathbb{R}).$$

From this we have

$$f(-x-a) = f(x-a) + x \qquad (x \in \mathbb{R}),$$

that is with the notation $a^* := -a$, the assertion of the Theorem follows.

With the help of the previous results we can easily prove Theorem 2.

Proof (of Theorem 2). If $f \in S(\mathbb{R})$ is nonconstant and continuous, then there are two possibilities: either $N_f \neq \emptyset$ or $N_f = \emptyset$. In the first case Theorem 3 implies the assertion. In the second case f is everywhere positive or everywhere negative, and the assertion of Theorem 2 follows from Theorems 4 and 5.

3. Continuous, translative and quasi-commutative operations on the additive group of real numbers

On the basis of the above and previous results we can state the following theorem.

Theorem 6. *If $\circ : \mathbb{R}^2 \to \mathbb{R}$ is a continuous, translative and quasi-commutative operation then it is one of the following operations:*

(*i*) $\quad x \circ y = y + c \quad (x, y \in \mathbb{R})$ *and* $a \in \mathbb{R}$ *is constant*;

(*ii*) $\quad x \circ y = \min\{x - a, y\} \quad (x, y \in \mathbb{R})$ *and* $a \in \mathbb{R}$ *is constant*;

(*iii*) $\quad x \circ y = \max\{x - a, y\} \quad (x, y \in \mathbb{R})$ *and* $a \in \mathbb{R}$ *is constant*;

(*iv*) $\quad x \circ y = \frac{1}{A} \ln\left(e^{A(x-a)} + e^{Ay}\right) \quad (x, y \in \mathbb{R})$ *and* $a \in \mathbb{R}$, $A \neq 0$ *are constant.*

Proof. Under these conditions $f(x) := -x \circ 0 \ \ (x \in \mathbb{R}$ is continuous and $f \in S(\mathbb{R})$, moreover

$$x \circ y = y - f(x-y) \qquad (x, y \in \mathbb{R}). \tag{26}$$

There are the following possible cases:

(1) $f(x) = -c \quad (x \in \mathbb{R})$ for some constant $c \in \mathbb{R}$. Then (26) implies the solution (i);

(2) If f is not constant, then according to Theorem 3, suppose that $N_f \neq \emptyset$. Then the solutions (16) and (17) give the solutions (ii) and (iii) for some constant $a \in \mathbb{R}$.

(3) If f is not constant and $N_f \neq \emptyset$, then, by Theorem 2, there exists $a \in \mathbb{R}$ such that (7) holds, that is the function $f_a(x) := f(x+a) \quad (x \in \mathbb{R})$ satisfies

$$f_a(-x) = f_a(x) + x \qquad (x \in \mathbb{R}).$$

On the other hand, $f_a \in S(\mathbb{R})$ and f_a is continuous, thus by Theorem 1, there exists $A \neq 0$ for which

$$f_a(x) = -\frac{1}{A} \ln(1 + e^{Ax}) \qquad (x \in \mathbb{R}).$$

From this the solutions (iv) follow.

References

[1] **Aczél J.,** *Lectures on functional equations and their applications,* Mathematics in Science and Engineering **19**, Academic Press, New York-London, 1966.

[2] **Baiocchi C.,** Su un sistema di equazioni funzionali conesso alla teoria dell'informazione, *Boll. Un. Mat. Ital.,* **22** (3) (1967), 236-246.

[3] **Baiocchi C.,** Sur une équation fonctionelle liée à la théorie de l'information, *Boll. Un. Mat. Ital.,* **3** (4) (1970), 827-846.

[4] **Daróczy Z.,** Über ein Funktionalgleichungssystem der Informationstheorie, *Aequationes Math.,* **2** (1969), 144-149.

[5] **Hamel G.,** Eine Basis aller Zahlen und die unstätigen Lösungen der Funktionalgleichung $f(x+y) = f(x) + f(y)$, *Math. Ann.,* **60** (1905), 459-462.

[6] **Kuczma M.,** *An introduction to the theory of functional equations and inequalities,* Państwowe Wydawnictwo Naukowe - Uniwersytet Śląski, Warszawa-Kraków-Katowice, 1985.

[7] **Kampé de Feriet J. et Forte B.,** Information et probabilité, *C. R. Acad. Sci. Paris Sér. A,* **265** (1967), 110-114, 142-146, 350-353.

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