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DIFFERENCE EQUATIONS VIA SPECTRAL SYNTHESIS

L. Székelyhidi (Debrecen, Hungary)

Dedicated to Professor I. Kátai on his 65. birthday

Abstract. The paper presents a new method for the solution of partial difference equations. The method is based on discrete spectral synthesis on finitely generated free Abelian groups. It makes possible to reduce the solution problem of linear systems of partial difference equations to the determination of polynomial solutions of systems of linear partial differential equations.

1. Introduction

In this paper we shall consider linear difference equations in several variables. As the theory of linear difference equations on the integers is well-known and highly developed, our main emphasis is mainly on the higher dimensional case. However, the results we obtain generalize the known results in the one dimensional case as well.

Our main purpose is to give a general method for the solution of convolution type partial difference equations and of systems of such equations. The method is based on a classical result on spectral synthesis for finitely generated free Abelian groups, as it has been proved by M. Lefranc in [1].

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Let k be a fixed positive integer. If $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ is an element of $\mathbb{C}^k$ with nonzero components and $x = (x_1, x_2, \dots, x_k)$ is in $\mathbb{Z}^k$, then λ^x is defined as the product $\lambda_1^{x_1} \lambda_2^{x_2} \dots \lambda_k^{x_k}$.

For any $i = 1, 2, \dots, k$ we use the notation e_1 for the vector whose i-th component is 1 and all the others are 0.

The *partial translation operators* τ_i are defined for $i = 1, 2, \dots, k$ on functions $f : \mathbb{Z}^k \to \mathbb{C}$ by

$$\tau_i f(x) = f(x + e_i),$$

where x is in $\mathbb{Z}^k$. If we write symbolically $\tau = (\tau_1, \tau_2, \dots, \tau_k)$, then - according to the above notation - for any $y = (y_1, y_2, \dots, y_k)$ in $\mathbb{Z}^k$ we can write

$$\tau^y = \tau_1^{y_1} \tau_2^{y_2} \dots \tau_k^{y_k},$$

and for any $f : \mathbb{Z}^k \to \mathbb{C}$ we have

$$f(x + y) = \tau^y f(x),$$

whenever x, y are in $\mathbb{Z}^k$. We call τ^y the *translation operator with increment y.* In particular, τ^0 is the identity operator.

Similar notation will be used for partial differential operators, acting on complex polynomials in k variables. The *partial differential operators* ∂_i for $i = 1, 2, \dots, k$ are defined on polynomials $p : \mathbb{Z}^k \to \mathbb{C}$ in the usual way: $\partial_i p$ is the partial derivative of p with respect to the i-th variable. If we write symbolically $\partial = (\partial_1, \partial_2 \dots, \partial_k)$, then for any y in $\mathbb{Z}^k$ with nonnegative components we have

$$\partial^y = \partial_1^{y_1} \partial_2^{y_2} \dots \partial_k^{y_k},$$

and for any polynomial $p : \mathbb{Z}^k \to \mathbb{C}$ we have

$$\partial^y p(x) = \partial_1^{y_1} \partial_2^{y_2} \dots \partial_k^{y_k} p(x),$$

whenever x, y are in $\mathbb{Z}^k$. For any y in $\mathbb{Z}^k$ we shall use the notation $\langle y, \partial \rangle$ for the differential operator $y_1 \partial_1 + y_2 \partial_2 + \ldots + y_k \partial_k$. Hence, for any nonnegative integer j the operator $\langle y, \partial \rangle$ is the j-th *differential*, which acts on the polynomial $p : \mathbb{Z}^k \to \mathbb{C}$ in the obvious manner. By the Taylor-formula we have for any polynomial $p : \mathbb{Z}^k \to \mathbb{C}$ and for any x, y in $\mathbb{Z}^k$

$$p(x + y) = \sum_{j=0}^{\infty} \frac{1}{j!} \langle y, \partial \rangle^j p(x).$$

As p is a polynomial, this is a finite sum.

If λ is in $\mathbb{C}^k$ with nonzero components and $p : \mathbb{Z}^k \to \mathbb{C}$ is a polynomial, then the function $f : x \mapsto (x)\lambda^x$ on $\mathbb{Z}^k$ is called an *exponential monomial.* A linear combination of exponential monomials is called an *exponential polynomial.* In particular, $x \mapsto \lambda^x$ is called an *exponential.*

2. Spectral synthesis

Spectral synthesis deals with the description of translation invariant function spaces on Abelian groups. As translation invariant function spaces are closely related to solution spaces of systems of linear difference equations, it seems to be reasonable to apply spectral synthesis results in the theory of difference equations. Here we summarize the concepts and results we shall use in the sequel.

For a fixed positive integer k the set of all complex valued functions defined on $\mathbb{Z}^k$ will be denoted by $\mathcal{C}(\mathbb{Z}^k)$. This is clearly a linear space which we equip with the topology of pointwise convergence, hence topological concepts, like closed set, limit, etc., will refer to this topology. A subset of $\mathcal{C}(G)$ will be called *translation invariant*, if it contains all translates of its each element. More exactly, the set V of complex valued functions on $\mathbb{Z}^k$ is translation invariant if and only if for each f in V and for all $i = 1, 2, \ldots, k$ the function $\tau_i f$ belongs to V. If V is a closed translation invariant linear subspace of $\mathcal{C}(\mathbb{Z}^k)$, then V is called a *variety.* For instance, if f is any complex valued function on $\mathbb{Z}^k$, then the variety $\tau(f)$ generated by all translates of f is the smallest variety containing f. If f is an exponential, then obviously $\tau(f)$ is the one-dimensional subspace consisting of the scalar multiples of f.

An important special class of varieties arises in the following way. Let $c : \mathbb{Z}^k \to \mathbb{C}$ be a function with finite support, that is c vanishes outside a finite set. Then the set of all functions $f : \mathbb{Z}^k \to \mathbb{C}$ satisfying the equation

$$\sum_{y \in \mathbb{Z}^k} c(y) f(x + y) = 0, \tag{1}$$

for all x in $\mathbb{Z}^k$ is obviously a variety, which is proper if and only if c is nonidentically zero. Equation (1) is a linear difference equation, which can be written in the form

$$\left(\sum_{y \in \mathbb{Z}^k} c(y) \tau^y \right) f = 0,$$

hence the solution space of (1) is identical with the kernel of the linear operator $\sum_{y\in\mathbb{Z}^k} c(y)\tau^y$. More generally, if Γ is a nonempty set and $c_\gamma : \mathbb{Z}^k \to \mathbb{C}$ is a finitely supported function for each γ in Γ, then all solutions $f : \mathbb{Z}^k \to \mathbb{C}$ of the system of linear difference equations

$$\sum_{y\in\mathbb{Z}^k} c_\gamma(y) f(x+y) = 0, \tag{2}$$

where x is in $\mathbb{Z}^k$ and γ is in Γ, form a variety, which is proper if and only if at least one of the functions c_γ is nonindentically zero. This variety is equal to the intersection of the kernels of all operators $\sum_{y\in\mathbb{Z}^k} c(y)\tau^y$ with γ in Γ. Hence any system of linear difference equations of the form (2) generates a variety, which is the solution space of the system. Conversely, any variety arises in this way. For further details concerning varieties see [2].

The fundamental theorem on spectral synthesis for varieties in $\mathcal{C}(\mathbb{Z}^k)$ is the following (see [1]).

Theorem 1. *The linear hull of all exponential monomials in any variety in $\mathcal{C}(\mathbb{Z}^k)$ is dense in this variety.*

In the language of difference equations this means that the exponential monomial solutions of a system of linear difference equations generate a dense set in the solution space. In other words, any solution is the pointwise limit of a sequence of exponential polynomial solutions. Hence the exponential monomial solutions of a system of the form (2) characterize the whole solution space, and it seems to be reasonable to find methods for the determination of exponential monomial solutions.

3. The spectrum and the spectral set of difference equations

The set of all exponentials in a variety is called the *spectrum* of the variety. Similarly, the set of all exponential solutions of a system of the form (2) is called the spectrum of the system. We recall that the exponentials can be identified by elements of $\mathbb{C}^k$ with nonzero components. The following result is a reformulation of a well-known fact, and it is straightforward to prove.

Theorem 2. *The spectrum of (2) is the set of all solutions λ in $\mathbb{C}^k$ with nonzero components of the system of algebraic equations*

$$\sum_{y\in\mathbb{Z}^k} c_\gamma(y)\lambda^y = 0 \qquad (\gamma\in\Gamma). \tag{3}$$

The left hand side of (3) is a polynomial in the components of λ and their reciprocals, for any fixed γ in Γ. As the functions c_γ vanish off a finite set (depending on γ), each equation of the system (3) can be multiplied by an appropriate power of λ so that the new equations contain only nonnegative powers of the components of λ. As the new system, obtained in this way is obviously equivalent to the original one, we may always suppose that in (3) $c_\gamma(y) = 0$, if y has a negative component. Then the left hand sides of (3) are polynomials in y. The system (3) of algebraic equations is called the *system of characteristic equations* of (2), and the left hand sides are the *characteristic polynomials* of (3). If $k = 1$ and Γ is a singleton, that is if we have the case of a single linear homogeneous difference equation in one variable, then the system of characteristic equations reduces to a single equation, to the characteristic equation, corresponding to the characteristic polynomial. The nonzero roots of this polynomial form the spectrum of the equation. We know from the general theory that in order to have a complete description on the solutions of the difference equation in this case we also need the multiplicities of these roots. The general solution is a linear combination of exponential monomial solutions corresponding to the different roots together with their multiplicities. In the general case the common roots of the characteristic polynomials describe the spectrum, but the role of the multiplicities of a given spectrum element is played by all exponential monomial solutions corresponding to it. The set of all exponential monomials in a variety is called the *spectral set* of the variety. Similarly, the set of all exponential monomial solutions of a system of the form (2) is called the spectral set of the system. Hence the spectral set can be identified with a set of pairs (λ, p), where λ is in $\mathbb{C}^k$ with nonzero components, and p is a polynomial for which the exponential monomial $x \mapsto \lambda^x p(x)$ is a solution of (2). The following result shows that the elements (λ, p) of the spectral set are "built up" from the spectrum.

Theorem 3. *If (λ, p) is in the spectral set of (2) and p is a nonzero polynomial, then λ is in the spectrum of (2).*

Proof. Substituting the exponential monomial $x \mapsto \lambda^x p(x)$ into (2) we obtain

$$\sum_{y\in\mathbb{Z}^k} c_\gamma(y)\lambda^x\lambda^y p(x+y) = \sum_{y\in\mathbb{Z}^k} c_\gamma(y)\lambda^{x+y} p(x+y) = 0,$$

and hence

$$\sum_{y\in\mathbb{Z}^k} c_\gamma(y)\lambda^y p(x+y) = 0 \tag{4}$$

holds for any x in $\mathbb{Z}^k$ and γ in Γ. Here the left hand side is a polynomial in x and p is nonzero, and comparing the terms of the highest degree on both sides we get our statement.

This theorem shows that the determination of the spectral set of a system of the form (2) should start with the determination of the spectrum. This requires to find the common roots of a set of given polynomials in k variables. However, this is only a part of the work to be done: we have to find all exponential monomials corresponding to these roots, which are solutions. This leads to find polynomial solutions of the system (4). We show that this problem can be reduced to find polynomial solutions of systems of linear homogeneous partial differential equations. In other words we show that the annihilator ideal of the polynomial solution space of (4) in the algebra of all linear operators of the polynomial ring $\mathbb{C}[x] = \mathbb{C}[x_1, x_2, \ldots, x_k]$ is generated by a set of homogeneous linear partial differential operators. Moreover, we present a generating set of this annihilator ideal in terms of the given functions c_γ.

For a given λ in $\mathbb{C}^k$ with nonzero coefficients the set of all polynomials $p : \mathbb{Z}^k \to \mathbb{C}$ satisfying (4) for any x in $\mathbb{Z}^k$ and γ in Γ is a translation invariant linear space of polynomials. It is not necessarily closed. A polynomial $p : \mathbb{Z}^k \to \mathbb{C}$ is a solution of (4) if and only if the exponential monomial $x \mapsto \lambda^x p(x)$ belongs to the spectral set of (2). The set of all linear operators of $\mathbb{C}[x]$ which are zero on the polynomial solutions of (4) is obviously an ideal. We call this the *annihilator ideal* oh (4). We shall present a generating set of this annihilator ideal consisting of homogeneous linear partial differential operators. Then the polynomial solutions of the system of difference equations (4) can be found by solving systems of linear partial differential equations in the ring of polynomials.

4. The characteristic differential equation system

Our basic result is the following

Theorem 4. *The annihilator ideal of (4) is generated by the differential operators*

$$\sum_{y\in\mathbb{Z}^k} c_\gamma(y)\lambda^y \langle y, \partial\rangle^j \tag{5}$$

for all γ *in* Γ *and for* $j = 0, 1, \ldots$.

Proof. First of all we note that polynomial in k variables with complex coefficients is a solution of (4) on $\mathbb{Z}^k$ if and only if it is a solution of (4) on $\mathbb{R}^k$. Hence we have to show that a polynomial $p : \mathbb{Z}^k \to \mathbb{C}$ is a solution of (4) if and only if

$$\sum_{y \in \mathbb{Z}^k} c_\gamma(y) \lambda^y \langle y, \partial \rangle^j p(x) = 0 \tag{6}$$

holds for all x in $\mathbb{R}^k$, for all γ in Γ and for $j = 0, 1, \ldots$. By the Taylor-formula we have

$$p(x + y) = \sum_{j=0}^{\infty} \frac{1}{j!} \langle y, \partial \rangle^j p(x)$$

for all x, y in $\mathbb{R}^k$. Obviously, the sum is finite. Substituting into (4) we have

$$\sum_{j=0}^{\infty} \frac{1}{j!} \sum_{y \in \mathbb{Z}^k} c_\gamma(y) \lambda^y \langle y, \partial \rangle^j p(x) = 0 \tag{7}$$

for each x in $\mathbb{R}^k$ and γ in Γ. On the left hand side the j-th term is either zero or a homogeneous polynomial of degree exactly j. Hence (7) is equivalent to (6) for all x in $\mathbb{R}^k$, for all γ in Γ and for $j = 0, 1, \ldots$ and our theorem is proved.

If we use the convention $0^0 = 1$, then the system (6) includes the system of characteristic equations of (3) for $j = 0$. We may call the system of partial differential equations (6) the *characteristic differential equation system* of the system of difference equations (2). In this system λ and p are the unknowns and the pairs (λ, p) of solutions characterize the spectral set of (2). We note that λ has nonzero coefficients. If we have the spectral set of (2) then by Theorem 1 we know that any solution of (2) is the pointwise limit of linear combinations of spectral elements. So any property, enjoyed by all the elements of the spectral set, which is preserved under taking linear combinations and pointwise limits, is possessed by any solution. Depending on the particular form of the equations (2) this may lead to the complete description of all solutions. We shall exhibit some particular examples for the application of this method in the following section.

5. Applications

We have seen above that Theorem 1 on spectral synthesis and Theorem 4 above make it possible to reduce the problem of solving a system of linear difference equations to the solution of algebraic and partial differential equations. We note that we actually do not have to solve the system (6) of partial differential equations, because we need only its polynomial solutions. It seems to be quite reasonable to assume that there are effective computer algebraic methods to find polynomial solutions of systems of linear homogeneous partial differential equations with constant coefficients. We also note that in the original problem (2) methods depending on differentiation cannot be used even if we are looking only for solutions, which are restrictions of smooth functions to $\mathbb{Z}^k$. Differentiation of the equations in (2) with respect to the components of x results similar equations for the partial derivatives of the unknown function, but does not help to find these functions. Here we present some simple examples which may illustrate the application of this method. In some cases the general solutions of the given problem can be found by using different methods, too.

5.1. Example 1

In this example we show how this method relates to the well-known solution method of linear difference equations in one variable. We let $k = 1$ and consider the difference equation

$$\sum_{l=0}^{n} c_l f(x+l) = 0 \tag{8}$$

supposing that n is a nonnegative integer, c_l is a complex number for $l = 0, 1, \ldots, n$ and $c_0 \neq 0$, $c_n \neq 0$.

The characteristic differential equation system of (8) reduces now to the system of differential equations

$$\sum_{l=0}^{n} c_l \lambda^l l^j p^{(j)}(x) = 0 \tag{9}$$

for $j = 0, 1, \ldots$ and x in $\mathbb{R}$. If $l^{(j)} = \frac{l!}{j!}$, then the system (9) is equivalent to the system

$$P^{(j)}(\lambda) \cdot p^{(j)}(x) = \sum_{l=0}^{n} c_l \lambda^{l-j} l^{(j)} p^{(j)}(x) = 0 \tag{10}$$

for $j = 0, 1, \dots$ and x in $\mathbb{R}$, where

$$P(\lambda) = \sum_{l=0}^{n} c_l \lambda^l$$

is the characteristic polynomial of (8). By Theorem 4 we have that exponential monomial solution f of (8) has the form

$$f(x) = \sum_{i=1}^{s} \lambda_i^x p_i(x) \tag{11}$$

for each x in $\mathbb{R}$, where the complex numbers $\lambda_1, \lambda_2, \dots, \lambda_s$ are the different roots of the characteristic polynomial P with multiplicities $n_1, n_2, \dots, n_s$, and p_i is a polynomial of degree $n_i - 1$ ($i = 1, 2, \dots, s$). As the linear space of all functions of the form (11) is of finite dimension, hence it is a closed subspace in $\mathcal{C}(\mathbb{Z})$. Then Theorem 1 implies that any solution of (8) has the form (11). Thus from Theorem 4 one can derive the classical results for linear homogeneous difference equations with constant coefficients.

5.2. Example 2

We consider the difference equation

$$\sum_{l=0}^{n+1} \binom{n+1}{l} (-1)^{n+1-l} f(x + ly) = 0 \tag{12}$$

for all x, y in $\mathbb{Z}^k$, where n is a fixed nonnegative integer. Equation (12) can be rewritten in the form

$$(\tau^y - I)^{n+1} f(x) = 0$$

for all x, y in $\mathbb{Z}^k$. Here I stands for the identity operator. The characteristic differential equation system of (12) has the form

$$\sum_{l=0}^{n+1} \binom{n+1}{l} (-1)^{n+1-l} \lambda^{ly} l^j \langle y, \partial \rangle^j p(x) = 0 \tag{13}$$

for $j = 0, 1, \dots$ and x in $\mathbb{R}^k$. For$j = 0$ we have

$$\left(\sum_{l=0}^{n+1} \binom{n+1}{l} (-1)^{n+1-l} (\lambda^y)^l \right) p(x) = 0$$

for all x in $\mathbb{R}^k$ and y in $\mathbb{Z}^k$, hence there are nonzero solutions if and only if

$$(\lambda^y - 1)^{n+1} = \sum_{l=0}^{n+1} \binom{n+1}{l} (-1)^{n+1-l} (\lambda^y)^l = 0$$

holds for any y in $\mathbb{Z}^k$. This implies $\lambda_1 = \lambda_2 = \ldots = \lambda_k = 1$, hence the spectral set of (12) consists of polynomials. Then (13) takes the form

$$\left(\sum_{l=0}^{n+1} \binom{n+1}{l} (-1)^{n+1-l} l^j \right) \langle y, \partial \rangle^j p(x) = 0$$

for each x in $\mathbb{R}^k$ and y in $\mathbb{Z}^k$, and for $j = 0, 1, \ldots$. The expression in the bracket is zero for $j \leq n$, and is equal to $(n+1)!$ for $j = n+1$, hence the differential operators $\langle y, \partial \rangle^{n+1} p$ annihilate any polynomial solution of (12) for any y in $\mathbb{Z}^k$. In other words, the differential $\langle y, \partial \rangle^{n+1} p$ is zero for any y in $\mathbb{Z}^k$, hence the spectral set of (12) consists of polynomials of degree at most n. Thus, by Theorem 1 the solution space of (12) is the set of polynomials of degree at most n.

5.3. Example 3

We consider the difference equation

$$f(x+2, y) - 2f(x+1, y+1) + f(x, y+2) = 0 \tag{14}$$

on $\mathbb{Z}^2$. The characteristic differential equation of (14) has the form

$$\left[\, \lambda^2 (2 \cdot \partial_1 + 0 \cdot \partial_2)^j - 2\lambda\mu (1 \cdot \partial_1 + 1 \cdot \partial_2)^j + \mu^2 (0 \cdot \partial_1 + 2 \cdot \partial_2)^j \,\right] p(x, y) = 0$$

for all x, y in $\mathbb{R}$ and $j = 0, 1, \ldots$. For $j = 0$ we have $\lambda^2 - 2\lambda\mu + \mu^2 = 0$ which implies $\lambda = \mu$. Using this, it follows for $j = 1, 2, \ldots$

$$(2^j \partial_1^j - 2(\partial_1 + \partial_2)^j + 2^j \partial_2^j) p(x, y) = 0$$

whenever x, y are in $\mathbb{R}$. For $j = 1$ we get no restriction on p, but for $j = 2$ we have the partial differential equation

$$(\partial_1 - \partial_2)^2 p(x, y) = 0$$

for all x, y in $\mathbb{R}$. If $q = (\partial_1 - \partial_2)p$, then $(\partial_1 - \partial_2)q = 0$, and this means that $q(x, y) = a(x+y)$ for all x, y in $\mathbb{R}$, where $a : \mathbb{R} \to \mathbb{C}$ is an arbitrary polynomial. Then again we have that

$$(\partial_1 - \partial_2)\,\big(\,p(x, y) - xa(x + y)\,\big) = 0,$$

hence $p(x, y) = xa(x + y) + b(x + y)$ for all x, y in $\mathbb{R}$, where $b : \mathbb{R} \to \mathbb{C}$ is an arbitrary polynomial, too. Hence the spectral set of (14) consists of functions of the form

$$(x, y) \mapsto \big(\,xa(x + y) + b(x + y)\,\big)\,\lambda^{x+y},$$

where λ is any nonzero complex number, and $a, b : \mathbb{R} \to \mathbb{C}$ are arbitrary polynomials. In particular, every function φ in the spectral set has the form

$$\varphi(x, y) = xA(x + y) + B(x + y) \tag{15}$$

for all x, y in $\mathbb{R}$, with some functions $A, B : \mathbb{R} \to \mathbb{C}$. As any such function satisfies also

$$\varphi(x, y) = x[\varphi(1, x + y - 1) + \varphi(0, x + y)] + \varphi(0, x + y)$$

for all x, y in $\mathbb{Z}$, this latter equation is also satisfied by pointwise limits of linear combinations of such functions. This means, that any solution of (14) has the form (15). On the other hand, any function of the form (15) satisfies (14), hence the general solution of (14) is

$$f(x, y) = xA(x + y) + B(x + y)$$

with arbitrary functions $A, B : \mathbb{Z} \to \mathbb{C}$.

5.4. Example 4

Now we consider in three dimensions the difference equation

$$f(x + 2, y, z) - 2f(x + 1, y + 1, z + 1) = 0. \tag{16}$$

Here the characteristic differential equation system has the form

$$\Big(\lambda^2(2\partial_1)^j - 2\lambda\mu\nu(\partial_1 + \partial_2 + \partial_3)^j\Big)p(x, y, z) = 0 \tag{17}$$

for all x, y, z in $\mathbb{R}$. For $j = 0$ we have $\lambda = 2\mu\nu$, which means that the exponential solutions of (16) have the form

$$(x, y, z) \mapsto 2^x\mu^{x+y}\nu^{x+z}$$

with some nonzero complex numbers μ, ν. This form may suggest that the general form of the solutions is

$$(x, y, z) \mapsto 2^x a(x + y, x + z) \tag{18}$$

with an arbitrary function $a : \mathbb{Z}^2 \to \mathbb{C}$, which are solutions, indeed. This suggestion is reinforced by the solutions of the characteristic differential equation system, which has the form

$$\left(2^j \partial_1^j - (\partial_1 + \partial_2 + \partial_3)^j \right) p(x, y, z) = 0$$

for all x, y, z in $\mathbb{R}$. Namely, for $j = 1$ we have the linear homogeneous partial differential equation

$$(\partial_1 - \partial_2 - \partial_3) p(x, y, z) = 0,$$

which implies that $p(x, y, z) = q(x+y, x+z)$ for all x, y, z with some polynomial $q : \mathbb{R}^2 \to \mathbb{C}$. This means that the spectral set of (16) consists of functions of the form (18), and as this form is preserved under taking linear combinations and pointwise limits, functions of the form (18) represent all solutions of (16), where $a : \mathbb{Z}^2 \to \mathbb{C}$ is an arbitrary function.

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L. Székelyhidi
Institute of Mathematics
University of Debrecen
H-4010 Debrecen, P.O.B. 12
Hungary
szekely@math.klte.hu

Department of Mathematics
University of Louisville