

DIOPHANTINE EQUATIONS WITH QUADRATIC FORMS

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Dedicated to Professor Imre Kátai on his 65th birthday

1. Introduction

Let $\varphi_1(u, v)$ and $\varphi_2(u, v)$ be two quadratic forms. We denote $I_h(\varphi_1, \varphi_2, N)$ the number representations of a natural k in the form

$$k = \varphi_1(u_1, v_1) - \varphi_2(u_2, v_2)$$

under the condition $|\varphi_1(u_1, v_1)| \leq N$, $u_1, v_1 \in \mathbb{Z}$.

In the works [1]-[5] the diophantine equation

$$(1) \quad u_1 v_1 - u_2 v_2 = 1, \quad u_1, v_1 \in \mathbb{N}, \quad u_1 v_1 \leq N$$

is investigated. This problem is equivalent to the problem on the estimation of the sum

$$\sum_{n \leq N} \tau(n) \tau(n+1),$$

where $\tau(n)$ is the divisor function.

Y.Motohashi [6] proved the asymptotic formula for $1 \leq h \leq x^{20/27}$

$$\sum_{n \leq N} \tau(n) \tau(n+1) = N \sum_{i=0}^2 (\log N)^i \sum_{j=0}^2 c_{ij} \sum_{d|k} \frac{(\log d)^j}{d} + O_\varepsilon \left(N^{2/3+\varepsilon} \right).$$

In 1981 I.Chalk [7] studied the distribution of solutions of the diophantine equation

$$(2) \quad a(u_1^2 + u_2^2) - b(u_3^2 + u_4^2) = k, \quad u_1^2 + u_2^2 \leq N, \quad a, b \in \mathbb{N}.$$

The number $A(N, a, b, k)$ of solutions of the equation (2) is defined by the sum

$$A(N, a, b, k) = \sum_{\substack{k/n < n \leq N \\ an \equiv k \pmod{b}}} r(n) r\left(\frac{an-k}{b}\right), \quad r(n) = \sum_{n=u^2+v^2} 1.$$

In the special case $a = b = k = 1$ G.Bantle [8] obtained an asymptotic formula for $A(N, a, b, k)$ with the remainder term $R(N) = O(N^{2/3+\varepsilon})$.

Unfortunately, we cannot apply the methods of works [5], [6], [8] in the general case.

In our paper we study the mean square of the remainder term in the asymptotic formula for the number of solutions of the diophantine equation $A_\varphi(N, a, b, k)$

$$a\varphi(u_1, v_1) - b(u_2^2 + v_2^2) = k, \quad 0 < \varphi(u_1, v_1) \leq N,$$

where $\varphi(u_1, v_1)$ is a positively definite quadratic form

$$a_1u_1^2 + 2b_1u_1v_1 + c_1v_1^2, \quad a_1c_1 - b_1^2 = D > 0, \quad (a_1, 2) = 1.$$

2. Preliminary results

Let $\rho(\varphi; l, q)$ denote the number of solutions of the congruence

$$\varphi(u, v) \equiv l \pmod{q}.$$

Lemma 1. *Let*

$$\rho(\varphi; l, p^\alpha) = \begin{cases} p^\alpha - p^{\alpha-1} \left(\frac{-D}{p}\right)^{\gamma+1} + p^\alpha \left(1 - \frac{1}{p}\right) \sum_{\beta=\alpha-\gamma}^{\alpha-1} \left(\frac{-D}{p}\right)^{\alpha-\beta} & \text{if } (D, p) = 1 \\ & \text{and } \gamma < \alpha, \\ p^\alpha + p^\alpha \left(1 - \frac{1}{p}\right) \sum_{\beta=0}^{\alpha-1} \left(\frac{-D}{p}\right)^{\alpha-\beta} & \text{if } (D, p) = 1 \\ & \text{and } \gamma = \alpha, \\ p^\alpha + \left(\frac{-a_1l_1}{p}\right) p^{\alpha-\gamma-1} \left(\frac{D_1}{p}\right)^\gamma & \text{if } D = D_1, \\ & l = l_1p^\gamma, \\ & \gamma < \alpha, \\ p^\alpha & \text{if } \gamma = \alpha, \end{cases}$$

where (n/p) denotes the Legendre symbol.

This lemma can be proved if we use the equality

$$\rho(\varphi; l, q) = \frac{1}{q} \sum_{h, l_1, l_2=0}^{q-1} e^{2\pi i \frac{h(\varphi(l_1, l_2) - 1)}{q}}$$

and the properties of the Gauss' sums.

Lemma 2. *Let $q = 2^\alpha$, $(l, q) = 2^\gamma$, $l = l_0 2^\gamma$. Then for $(D, 2) = 1$*

$$\begin{aligned} & \rho(\varphi; l, 2^\alpha) = \\ & = \begin{cases} 2^{\alpha+1} & \text{if } 2l_1 - a_1(1 + D) \equiv 0 \pmod{8}, \quad \gamma \leq \alpha - 2, \\ 2^\alpha & \text{if } 2l_1 - a_1(1 + D) \equiv 0 \pmod{8}, \quad \gamma \geq \alpha - 1, \\ 0 & \text{in other cases.} \end{cases} \end{aligned}$$

For $D = 2D_1$, $(D_1, 2) = 1$

$$\begin{aligned} & \rho(\varphi; l, 2^\alpha) = \\ & = \begin{cases} 2^{\gamma+3} & \text{if } l_1 + 2a_1 + a_1D \equiv 0 \pmod{8} \text{ or } l_1 + a_1D_1 = 0 \pmod{8} \\ & \text{or } l_1 + a_1 + 2a_1D_1 \equiv 0 \pmod{8} \text{ or } l_1 + a_1 \equiv 0 \pmod{8}, \\ 2^{\gamma+2} & \text{if } l_1 + 2a_1 + a_1D \equiv 4 \pmod{8} \text{ or } l_1 + a_1D_1 = 4 \pmod{8} \\ & \text{or } l_1 + a_1 + 2a_1D_1 \equiv 4 \pmod{8} \text{ or } l_1 + a_1 \equiv 0 \pmod{8}, \\ 0 & \text{in other cases.} \end{cases} \end{aligned}$$

Let $r_\varphi(n)$ denote the number of representations of n in form $n = \varphi(u, v)$, $u, v \in \mathbb{Z}$.

Lemma 3. *Uniformly for $q \ll N^{1-\varepsilon}$*

$$\begin{aligned} B(\varphi; l, q, N) &= \sum_{\substack{n \equiv l(q) \\ n \leq N}} r_\varphi(n) = \frac{\pi N}{\sqrt{D}q^2} \rho(\varphi; l, q) + \\ &+ \frac{\sqrt{DN}}{q} \sum_{n \leq M} \frac{\Phi(n, l, q)}{\sqrt{n}} I_1 \left(\frac{2\pi}{q} \sqrt{DnN} \right) + O(N^\varepsilon) + O \left(\left(\frac{D}{M} \right)^{1/2} N^{1/2+\varepsilon} \right), \end{aligned}$$

where

$$\Phi(n, l, q) = \sum_{\substack{n_1, n_2 \in \mathbb{Z} \\ D\psi(n_1, n_2) = n}} \sum_{\substack{l_1, l_2 \pmod{q} \\ \phi(l_1, l_2) \equiv l(q)}} e^{2\pi i \frac{n_1 l_1 + n_2 l_2}{q}},$$

ψ is the inverse form to φ , $I_1(z)$ is the Bessel function.

This assertion can be obtained by applying Perron's formula, the functional equation for Epstein zeta-function and the estimation of trigonometric sum

$$\sum_{\substack{l_1, l_2 \pmod{q} \\ \varphi(l_1, l_2) \equiv l \pmod{q}}} e^{2\pi i \frac{n_1 l_1 + n_2 l_2}{q}} \ll q^{1/2} (n_1, n_2, q)^{1/2} \tau(q).$$

3. Main results

We assume that $(a, b) = (a, k) = (b, k) = 1$ and $(a_1, 2) = 1$. We shall build an asymptotic formula for $A_\varphi(A, a, b, k)$. From the definitions of $r(n)$ and $r_\varphi(n)$ we have

$$\begin{aligned} (3) \quad & A_\varphi(N, a, b, k) = \\ & = \sum_{\substack{k/a \leq n \leq N \\ an \equiv k \pmod{b}}} r_\varphi(n) r \left(\frac{an - k}{b} \right) = 4 \sum_{k/a \leq n \leq N} r_\varphi(n) \sum_{d | \frac{an - k}{b}} \chi_4(d) = \\ & = 4 \sum_{\substack{k/a \leq n \leq N \\ an \equiv k \pmod{b}}} r_\varphi(n) \sum_{\substack{d | \frac{an - k}{b} \\ d \leq \sqrt{N_1}}} \chi_4(d) + 4 \sum_{\substack{k/a \leq n \leq N \\ an \equiv k \pmod{b}}} r_\varphi(n) \sum_{\substack{d | \frac{an - k}{b} \\ d > \sqrt{N_1}}} \chi_4(d) = \\ & = 4\Sigma_1 + 4\Sigma_2, \end{aligned}$$

where $N_1 = \frac{aN - k}{b}$, χ_4 is nonprincipal character mod 4.

At first we study Σ_1 . By Lemma 3 we obtain

$$\begin{aligned} \Sigma_1 & = \sum_{d \leq \sqrt{N_1}} \chi_4(d) \sum_{\substack{an \equiv k \pmod{bd} \\ n \leq N}} r_\varphi(n) - \sum_{d \leq \sqrt{N_1}} \chi_4(d) \sum_{\substack{an \equiv k \pmod{bd} \\ n \leq k/a}} r_\varphi(n) = \\ & = \sum_{\substack{d \leq \sqrt{N_1} \\ (a, d) = 1}} \chi_4(d) \left\{ \frac{\pi N}{\sqrt{D}(bd)^2} \rho(a\varphi; k, bd) + \right. \\ & \quad \left. + \frac{\sqrt{DN}}{bd} \sum_{n \leq M} \frac{\Phi(n, k, bd)}{\sqrt{n}} I_1 \left(\frac{2\pi}{bd} \sqrt{DnN} \right) \right\} - \\ & - \sum_{\substack{d \leq \sqrt{N_1} \\ (a, d) = 1}} \chi_4(d) \left\{ \frac{\pi k}{a\sqrt{D}(bd)^2} \rho(a\varphi; k, bd) + \right. \end{aligned}$$

$$\begin{aligned}
 & + \frac{\sqrt{D\frac{k}{a}}}{bd} \sum_{n \leq M} \frac{\Phi(n, k, bd)}{\sqrt{n}} I_1 \left(\frac{2\pi \sqrt{Dnk}}{bd} \sqrt{\frac{Dnk}{a}} \right) \Big\} + \\
 & + O \left(\left(\frac{D}{M} \right)^{1/2} N^{1/2} \left(\frac{aN - k}{b} \right)^{1/2} \right) + O \left(D^{1/2} N^{1/2+\varepsilon} \right) = \\
 & = \Sigma_{11} - \Sigma_{12} + O \left(\left(\frac{aD}{bM} \right)^{1/2} N \right) + O \left(N^{1/2+\varepsilon} D^{1/2} \right).
 \end{aligned}$$

Investigate the sum Σ_{11} . We have

$$\begin{aligned}
 (4) \quad \Sigma_{11} & = \frac{\pi N}{b\sqrt{D}} \sum_{\substack{d \leq \sqrt{N_1} \\ (a,d)=1}} \frac{\chi_4(d) \rho(a\varphi; k, bd)}{bd^2} + \\
 & + \frac{\sqrt{DN}}{b^2} \sum_{\substack{d \leq \sqrt{N_1} \\ (a,d)=1}} \frac{\chi_4(d)}{d} \sum_{n \leq M} \frac{\Phi(n, k, bd)}{\sqrt{n}} I_1 \left(\frac{2\pi \sqrt{DnN}}{bd} \right).
 \end{aligned}$$

Let us consider the function

$$F(s) = \sum_{d=1}^{\infty} \frac{\chi_4(d) \rho(a\varphi; k, bd)}{bd^{s+2}}$$

for $\text{Re } s > 1$. From Lemmas 1, 2 we deduce

$$F(s) = \frac{L(s+1, \chi_4)}{L(s+2, \chi_4^0)} \cdot g(s),$$

where χ_4^0 is the principal character *mod* 4 and the function $g(s)$ is defined by

$$\begin{aligned}
 g(s) & = \prod_{p|abk} \left(1 - \frac{\chi_4(p)}{p^{s+1}} \right) \left(1 - \frac{\chi_4(p)}{p^{s+2}} \right)^{-1} \times \\
 & \times \prod_{p|k} \left(1 + \frac{\chi_4(p) \rho(a\varphi; k, p)}{p^{s+2}} \right) \prod_l \left(\frac{\chi_4(l) \rho(a\varphi; k, bl)}{bl^{s+2}} \right),
 \end{aligned}$$

where $l = p_1^{\beta_1} \dots p_m^{\beta_m}$, $\beta_j = 0, 1, \dots$; $j = 1, \dots, m$, if $b = p_1^{\alpha_1} \dots p_m^{\alpha_m}$ is the canonical decomposition of b .

Notice that

$$(5) \quad (\log \log(abk))^{-3} \ll g(0) \ll (\log \log(abk))^2.$$

Hence, by Perron's formula we obtain

$$(6) \quad \Sigma_{11} = \frac{\pi}{b\sqrt{D}} \frac{L(1, \chi_4)}{L(2, \chi_4)} g(0) N + \\ + \frac{\sqrt{DN}}{b} \sum_{\substack{d \leq \sqrt{N_1} \\ (a, d)=1}} \frac{\chi_4(d)}{d} \sum_{n \leq M} \frac{\Phi(n, k, bd)}{\sqrt{n}} I_1 \left(\frac{2\pi\sqrt{DnN}}{bd} \right) + O(N^{1/2} b^{-1} D^{-1/2}).$$

Similarly one can see that

$$(7) \quad \Sigma_{12} = \frac{\pi}{b\sqrt{D}} \frac{L(1, \chi_4)}{L(2, \chi_4^0)} g(0) \frac{k}{a} + \\ + \sqrt{\frac{Dk}{ab^2}} \sum_{\substack{d \leq \sqrt{N_1} \\ (a, d)=1}} \frac{\chi_4(d)}{d} \sum_{n \leq M} \frac{\Phi(n, k, bd)}{\sqrt{n}} I_1 \left(\frac{2\pi\sqrt{Dnk}}{bd\sqrt{a}} \right) + O(N^{1/2} b^{-1} D^{-1/2}).$$

Now we investigate the sum Σ_2 . Observe that

$$\Sigma_2 = \sum_{\substack{k/a \leq n \leq N \\ an \equiv k \pmod{b}}} r_\varphi(n) \sum_{\substack{dt = \frac{an-k}{b} \\ d > \sqrt{N_1}}} \chi_4(d) = \\ = \sum_{t \leq \sqrt{N}} \left\{ \sum_{\substack{N_2 t + \frac{k}{a} < n \leq N \\ an \equiv k \pmod{bt} \\ \frac{an-k}{bt} \equiv 1 \pmod{4}}} r_\varphi(n) - \sum_{\substack{N_2 t + \frac{k}{a} < n \leq N \\ an \equiv k \pmod{bt} \\ \frac{an-k}{bt} \equiv -1 \pmod{4}}} r_\varphi(n) \right\} = \\ = \sum_{\substack{t \leq \sqrt{N} \\ (t, a)=1}} \left\{ \sum_{\substack{N_2 t + \frac{k}{a} < n \leq N \\ an \equiv k+bt \pmod{4bt}}} r_\varphi(n) - \sum_{\substack{N_2 t + \frac{k}{a} < n \leq N \\ an \equiv k-bt \pmod{4bt}}} r_\varphi(n) \right\} = \\ = \sum_{\substack{t \leq \sqrt{N} \\ (t, a)=1}} \left\{ \sum_{n \leq N}^+ - \sum_{n \leq N}^- \right\} - \sum_{\substack{t \leq \sqrt{N} \\ (t, a)=1}} \left\{ \sum_{n \leq N_2 t + k/a}^+ - \sum_{n \leq N_2 t + k/a}^- \right\} = \\ = \Sigma_{21} - \Sigma_{22}.$$

Here $N_2 = N_1^{1/2} \frac{b}{a}$ and the signs '+' and '-' correspond to the conditions of summing $an \equiv k + bt$ or $an \equiv k - bt$.

We study Σ_{21} . By Lemma 3 we infer

$$\begin{aligned} \Sigma_{21} &= \sum_{\substack{t \leq \sqrt{N_1} \\ (t, a) = 1}} \frac{\pi N}{D(4bt)^2} (\rho(a\varphi; k + bt, 4bt) - \rho(a\varphi; k - bt, 4bt)) + \\ &+ \sum_{\substack{t \leq \sqrt{N_1} \\ (t, a) = 1}} \sqrt{DN \cdot 4bt} \times \\ &\times \sum_{n \leq M} \frac{\Phi(n, k + bt, 4bt) - \Phi(n, k - 4bt, 4bt)}{\sqrt{n}} I_1 \left(\frac{2\pi\sqrt{DN\bar{N}}}{4bt} \right) + \\ &+ O \left(\left(\frac{D}{M} \right)^{1/2} \left(\frac{a}{b} \right)^{1/2} N \right) + O(D^{1/2} N^{1/2+\varepsilon}). \end{aligned}$$

Let for $\text{Re } s > 1$

$$F_+(s) = \sum_{\substack{t=1 \\ (t, a)=1}}^{\infty} \frac{\rho(a\varphi; k + bt, 4bt)}{t^{s+2}}, \quad F_-(s) = \sum_{\substack{t=1 \\ (t, a)=1}}^{\infty} \frac{\rho(a\varphi; k - bt, 4bt)}{t^{s+2}}.$$

Let p^{α_b} and p^{α_k} denote that $p^{\alpha_b} \parallel b$, $p^{\alpha_k} \parallel k$. Then

$$\begin{aligned} F_{\pm}(s) &= \prod_{(p, 2abk)=1} \left(1 + \frac{\rho(a\varphi; 1, p)}{p^{s+2}} + \frac{\rho(a\varphi; 1, p^2)}{p^{2(s+2)}} + \dots \right) \times \\ &\times \prod_{p|b} \left(\rho(a\varphi; 1, p^{\alpha_b}) + \frac{\rho(a\varphi; 1, p^{\alpha_b+1})}{p^{s+2}} + \dots \right) \times \\ &\times \prod_{p|k} \left(1 + \frac{\rho(a\varphi; 0, p)}{p^{s+2}} + \frac{\rho(a\varphi; 0, p^2)}{p^{2(s+2)}} + \dots + \right. \\ &\quad \left. + \frac{\rho(a\varphi; 0, p^{\alpha_k})}{p^{\alpha_k(s+2)}} + \frac{\rho(a\varphi; k, p^{\alpha_k+1})}{p^{(\alpha_k+1)(s+2)}} + \dots \right) \times \\ &\times \left(\rho(a\varphi; k \pm b, 2^2) + \frac{\rho(a\varphi; k + 2b, 2^3)}{2^{s+2}} + \frac{\rho(a\varphi; k + 4b, 2^4)}{2^{2(s+2)}} + \dots \right) = \\ &= \frac{\zeta(s+1)}{L(s+2, \chi_4)} G_{\pm}(s), \end{aligned}$$

where $G_{\pm}(s)$ is a regular function for $\text{Re } s > -1$.

Observe that

$$G_+(s) = G_0(s)g_+(s), \quad G_-(s) = G_0(s)g_-(s),$$

where

$$g_{\pm}(s) = \rho(a\varphi; k \pm b, 2^2) + \frac{\rho(a\varphi; k \pm 2b, 2^3)}{2^{s+2}} + \frac{\rho(a\varphi; k \pm 4b, 2^4)}{2^{2(s+2)}} + \dots$$

Hence,

$$F_+(s) - F_-(s) = \frac{\zeta(s+1)}{L(s+2, \chi_4)} G_0(s) (g_+(s) - g_-(s)).$$

It is obvious that for $a \equiv 0 \pmod{2}$ or $a \equiv 2 \pmod{4}$

$$\rho(a\varphi; k + 2^m b, 2^{m+2}) = \rho(a\varphi; k - 2^m b, 2^{m+2}).$$

For $a \equiv 0 \pmod{4}$ we have

$$(8) \quad \rho(a\varphi; 0, 4) = 16, \quad \rho(a\varphi; 2, 4) = 0, \quad \rho(a\varphi; k \pm 2^m b, 2^{m+2}) = 0 \quad \text{if } m \geq 1.$$

So we have

$$F_+(s) = F_-(s) \quad \text{if } a \equiv 0 \pmod{4}.$$

For $a \equiv 0 \pmod{4}$ and $k \equiv b \pmod{4}$

$$\begin{aligned} S &= \sum_{\substack{t \leq \sqrt{N_1} \\ (a,t)=1}} \frac{\rho(a\varphi; k+bt, 4bt)}{t^2} - \sum_{\substack{t \leq \sqrt{N_1} \\ (a,t)=1}} \frac{\rho(a\varphi; k-bt, 4bt)}{t^2} = \\ &= \sum_{\substack{t \leq \sqrt{N_1} \\ (a,t)=1}} \left[\frac{\rho(a\varphi; k+bt, bt)\rho(a\varphi; k+bt, 4)}{t^2} - \frac{\rho(a\varphi; k-bt, bt)\rho(a\varphi; k-bt, 4)}{t^2} \right] = \\ &= \left\{ \sum_{\substack{t \leq \sqrt{N_1} \\ (t,a)=1 \\ t \equiv 1 \pmod{4}}} + \sum_{\substack{t \leq \sqrt{N_1} \\ (t,a)=1 \\ t \equiv -1 \pmod{4}}} \right\} - \left\{ \sum_{\substack{t \leq \sqrt{N_1} \\ (t,a)=1 \\ t \equiv 1 \pmod{4}}} + \sum_{\substack{t \leq \sqrt{N_1} \\ (t,a)=1 \\ t \equiv -1 \pmod{4}}} \right\}. \end{aligned}$$

By (8) we obtain for $a \equiv 0 \pmod{4}$, $k \equiv b \pmod{4}$

$$(9) \quad S = -16 \sum_{\substack{t \leq \sqrt{N_1} \\ (a,t)=1}} \frac{\chi_4(t)\rho(a\varphi; k, bt)}{t^2}.$$

Similarly, if $a \equiv 0 \pmod{4}$, $k \equiv -b \pmod{4}$

$$(10) \quad S = 16 \sum_{\substack{t \leq \sqrt{N_1} \\ (a,t)=1}} \frac{\chi_4(t) \rho(a\varphi; k, bt)}{t^2}.$$

Hence,

$$(11) \quad \Sigma_{21} = \begin{cases} R_1(M, N) & \text{if } a \not\equiv 0 \pmod{4}, \\ \Sigma_{11} & \text{if } a \equiv 0 \pmod{4} \text{ and } k \equiv -b \pmod{4}, \\ -\Sigma_{11} & \text{if } a \equiv 0 \pmod{4} \text{ and } k \equiv b \pmod{4}, \end{cases}$$

where

$$\begin{aligned} R_1(M, N) = & \sum_{\substack{t \leq \sqrt{N_1} \\ (a,y)=1}} \frac{\sqrt{DN}}{4bt} \sum_{n \leq M} \frac{\Phi(n, k + bt, 4bt) - \Phi(n, k - bt, 4bt)}{\sqrt{n}} I_1 \left(\frac{2\pi}{4bt} \sqrt{DnN} \right) + \\ & + O \left(\left(\frac{aD}{bM} \right)^{1/2} N \right) + O(D^{1/2} N^{1/2+\varepsilon}). \end{aligned}$$

Now consider Σ_{22} . Similarly as sum Σ_{21} we have

$$\begin{aligned} \Sigma_{22} = & \frac{\pi \left(N_2 + \frac{k}{a} \right)}{4\sqrt{D}b^2} \left\{ \sum_{\substack{t \leq \sqrt{N_1} \\ (a,t)=1}} \frac{\rho(a\varphi; k + bt, 4bt) - \rho(a\varphi; k - bt, 4bt)}{t^2} \right\} + \\ & + \frac{\sqrt{D}}{4b} \sum_{t \leq \sqrt{N_1}} \frac{\sqrt{N_2 t + \frac{k}{a}}}{t} \sum_{n \leq M} \frac{\Phi(n, k + bt, 4bt) - \Phi(n, k - bt, 4bt)}{\sqrt{n}} \times \\ & \times I_1 \left(\frac{2\pi}{4bt} \sqrt{Dn \left(N_2 t + \frac{k}{a} \right)} \right) + O \left(\left(\frac{aD}{bM} \right)^{1/2} N \right) + O(N^{1/2+\varepsilon} D^{1/2}). \end{aligned}$$

Thus after analogous calculations we obtain

$$(12) \quad \Sigma_{22} = \begin{cases} R_2(M, N) & \text{if } a \not\equiv 0 \pmod{4}, \\ \Sigma_{11} & \text{if } a \equiv 0 \pmod{4} \text{ and } k \equiv -b \pmod{4}, \\ -\Sigma_{11} & \text{if } a \equiv 0 \pmod{4} \text{ and } k \equiv b \pmod{4}, \end{cases}$$

where

$$R_2(M, N) = \frac{\sqrt{D}}{4b} \sum_{\substack{t \leq \sqrt{N_1} \\ (a, t) = 1}} \frac{\sqrt{N_2 t + \frac{k}{a}}}{t} \sum_{n \leq M} \frac{\Phi(n, k + bt, 4bt) - \Phi(n, k - bt, 4bt)}{\sqrt{n}} \times \\ \times I_1 \left(\frac{2\pi}{4bt} \sqrt{Dn \left(N_2 t + \frac{k}{a} \right)} \right) + O \left(\left(\frac{aD}{M} \right)^{1/2} N \right) + O(N^{1/2+\varepsilon}).$$

Hence, we proved

Theorem 1. *Let $\varphi(u, v) = a_1 u^2 + 2b_1 uv + c_1 v^2$ be primitive positive definite quadratic form, $D = a_1 c_1 - b_1^2 > 0$, D is square-free number, $(a_1, 2) = 1$. Let $A_\varphi(N, a, b, k)$ be the number of the solutions of diophantine equation*

$$a\varphi(u_1, v_1) - b(u_1^2 + v_1^2) = k, \quad a, b, k \in \mathbb{N}, \quad (a, b) = (a, k) = (b, k) = 1$$

under the condition $\varphi(u_1, v_1) \leq N$. Then the asymptotic formula

$$A_\varphi(N, a, b, k) = E(a, b, k) \cdot \left(N - \frac{k}{a} \right) + S_1(N) - S_2(N) + S_3(N) + \\ + O \left(N^{1/2} \left(\frac{k}{a} \right)^\varepsilon \right) + O \left(\frac{kD}{ab} \log N \right) + \left(\left(\frac{aD}{bM} \right)^{1/2} N \right) + O(N^{1/2+\varepsilon})$$

holds, where

$$E(a, b, k) = \begin{cases} E_2(a, b, k) & \text{if } a \not\equiv 0 \pmod{4}, \\ 2E_2(a, b, k) & \text{if } a \equiv 0 \pmod{4}, k \equiv -b \pmod{4}, \\ 0 & \text{if } a \equiv 0 \pmod{4}, k \equiv b \pmod{4}, \end{cases}$$

$$E_2(a, b, k) = \frac{4\pi}{\sqrt{Db}} \cdot \frac{L(1, \chi_4)}{L(2, \chi_4^0)} g(0),$$

$$S_1(N) = \frac{\sqrt{DN}}{b} \sum_{n \leq M} \frac{1}{\sqrt{n}} \sum_{\substack{d \leq \sqrt{N_1} \\ (a, d) = 1}} \frac{\Phi(n, k, bt)}{d} \chi_4(d) I_1 \left(\frac{2\pi \sqrt{DnN}}{bd} \right),$$

$$S_2(N) =$$

$$\begin{aligned}
 &= \frac{\sqrt{DN}}{4b} \sum_{n \leq M} \frac{1}{\sqrt{n}} \sum_{\substack{t \leq \sqrt{N_1} \\ (a,t)=1}} \frac{\Phi(n, k + bt, 4bt) - \Phi(n, k - bt, 4bt)}{t} I_1 \left(\frac{2\pi\sqrt{DnN}}{4bt} \right), \\
 &S_3(N) = \\
 &\frac{\sqrt[4]{D^2N}}{4(ab)^{1/2}} \sum_{n \leq M} \frac{1}{\sqrt{n}} \sum_{\substack{t \leq \sqrt{N_1} \\ (a,t)=1}} \frac{\Phi(n, k + bt, 4bt) - \Phi(n, k - bt, 4bt)}{\sqrt{t}} I_1 \left(\frac{2\pi\sqrt{DnN_2}}{b\sqrt{t}} \right), \\
 &N_1 = \frac{aN - k}{b}, \quad N_2 = \frac{b}{a}N_1^{1/2}.
 \end{aligned}$$

Corollary. For every $\varepsilon > 0$, $(ab)^{3+\varepsilon} \ll N$, $k = o(aN)$ and $N \rightarrow \infty$ the asymptotic formula

$$A_\varphi(N, a, b, k) = E(a, b, k) \cdot \left(N - \frac{k}{a} \right) + O \left(\left(\frac{Da}{b} \right)^{1/2} N^{5/6+\varepsilon} \right)$$

holds.

Now we denote

$$\Delta(x) = A_\varphi(x, a, b, k) - E(a, b, k) \cdot \left(x - \frac{k}{a} \right)$$

and consider

$$\frac{1}{x} \int_{k/a}^x \Delta^2(x) dx.$$

Theorem 2. Let $b \equiv k \equiv 1 \pmod{2}$ and $a > b$. Then

$$\frac{1}{x} \int_1^x \Delta^2(x) dx \ll \frac{aD}{b^{5/3}} X^{4/3+\varepsilon} + \left(\frac{Dk}{ab} \right)^2 \log^2 X.$$

Proof. By Theorem 1 and the Cauchy inequality we obtain

$$\int_{k/a}^x \Delta^2(x) dx \ll$$

$$\ll \sum_{i=1}^3 \int_{k/a}^X |S_i(x)|^2 dx + O\left(\left(\frac{Dk}{ab}\right)^2 X \log^2 X\right) + O\left(\frac{aD}{bM} X^3\right) + O(X^{2+\varepsilon} D).$$

The integrals $\int_{k/a}^X |S_i(x)|^2 dx$ can be considered on the identical scheme. Thus we obtain the estimate only of the first integral

$$\begin{aligned} & \int_{k/a}^X |S_1(x)|^2 dx = \\ &= \frac{D}{b^2} \sum_{n_1, n_2 \leq M} \frac{1}{\sqrt{n_1 n_2}} \int_{k/a}^X \sum_{d_1, d_2 \leq \sqrt{M} \frac{ax-k}{b}} \frac{\Phi(n_1, k, bd_1) \overline{\Phi(n_2, k, bd_2)}}{d_1 d_2} \chi_4(d_1 d_2) \times \\ & \quad \times I_1\left(\frac{2\pi\sqrt{Dn_2 x}}{bd_2}\right) I_1\left(\frac{2\pi\sqrt{Dn_1 x}}{bd_1}\right) dx. \end{aligned}$$

We account that for $z \rightarrow +\infty$

$$I_1(z) = \sqrt{\frac{2}{\pi}} \frac{\cos\left(z - \frac{3\pi}{4}\right)}{\sqrt{z}} + O\left(\frac{1}{z^{3/2}}\right).$$

Apply a refinement of Hilbert's inequality given by Montgomery and Vaughan [9].

Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers, $b_i \neq b_j$ for $i \neq j$. Then

$$\left| \sum_{\substack{r, s=1 \\ r \neq s}}^M \frac{a_r a_s}{(rs)^{1/4} (\sqrt{r} - \sqrt{s})} e^{i(b_2 - b_3)} \right| \ll \sum_{r=1}^M a_r^2.$$

Thus the main contribution in the estimation of $\int_{k/a}^X |S_i(x)|^2 dx$ is given by the integral

$$I = \frac{\sqrt{D}}{b} \sum_{n_1, n_2 \leq M} \frac{1}{(n_1 n_2)^{3/4}} \int_{k/a}^X \sqrt{x} \sum_{\substack{d_1, d_2 \leq \sqrt{\frac{ax-k}{b}} \\ n_1 d_2^2 = n_2 d_1^2 \\ (d_1 d_2, a) = 1}} \frac{|\Phi(n_1, k, d_1) \Phi(n_2, k, d_2)|}{(d_1 d_2)^{1/2}} dx.$$

By the estimate $|\Phi(n, k, d)| = O(d^{1/2}(n, d)^{1/2}\tau(d))$ we obtain

$$I \ll \frac{\sqrt{D}}{b} \sum_{n_1, n_2 \leq M} \frac{1}{(n_1 n_2)^{3/4}} \int_{k/a}^X \sqrt{x} \sum_{\substack{d_1, d_2 \leq \sqrt{\frac{ax-b}{b}} \\ n_1 d_2^2 = n_2 d_1^2}} \sqrt{(n_1, d_1)(n_2, d_2)} \tau(d_1) \tau(d_2) dx.$$

Observe, that the collection of n_1, n_2, d_1 define at most one value of d_2 such that $n_1 d_2^2 = n_2 d_1^2$.

Now we have

$$\begin{aligned} & \sum_{n_1, n_2} \frac{1}{(n_1 n_2)^{3/4}} \sum_{\substack{d_1, d_2 \\ n_1 d_2^2 = n_2 d_1^2}} \sqrt{(n_1, d_1)(n_2, d_2)} \tau(d_1) \tau(d_2) \ll \\ & \ll \sum_{n_1, n_2} \sum_{\substack{t_1 | n_1 \\ t_2 | n_2}} \frac{1}{(n_1 n_2)^{3/4}} \sum_{\substack{1 \leq d_i \leq 1/t_i \sqrt{\frac{ax}{b}} \\ i=1,2}} \tau(d_1 t_1) \tau(d_2 t_2) (t_1, t_2)^{1/2} \ll \\ & \ll \left(\frac{ax}{b}\right)^\varepsilon \sum_{n_1, n_2 \leq M} \frac{\tau(n_1) \tau(n_2)}{(n_1 n_2)^{3/4}} \cdot \left(\frac{ax}{b}\right)^{1/2} \ll M^{1/2} \left(\frac{ax}{b}\right)^{1/2+2\varepsilon} \log^2 M. \end{aligned}$$

Hence,

$$\begin{aligned} I & \ll \frac{\sqrt{D}}{b} \left(\frac{a}{b}\right)^{1+2\varepsilon} M^{1/2} X^2 \log^2 M, \\ \frac{1}{X} \int_{k/a}^X |S_1(x)|^2 dx & \ll \frac{\sqrt{D} a^{1+2\varepsilon}}{b^2} M^{1/2} X \log^2 M. \end{aligned}$$

Similar estimates we obtain for $\frac{1}{X} \int_{k/a}^X |S_i(x)|^2 dx$ ($i = 2, 3$). And therefore

$$\begin{aligned} & \int_{k/a}^X \left(A_\varphi(x, a, b, k) - E(a, b, k) \cdot \left(x - \frac{k}{a}\right) \right)^2 dx \ll \\ & \ll \frac{aD}{bM} X^3 + \left(\frac{kD}{ab}\right)^2 X \log^2 M + X^{2+\varepsilon} D + \frac{\sqrt{D} n^{1+2\varepsilon}}{b^2} M^{1/2} X \log^2 M. \end{aligned}$$

We take $M = (bX)^{2/3}$ and then we obtain our theorem.

Corollary. *Let $k \ll N^{2/3}a^{2/3}b^{1/6}D^{-1/2}$. Then for almost all $N \leq X$ and any $\varepsilon > 0$ the asymptotic formula*

$$A_\varphi(N, a, b, k) - E(a, b, k) \cdot \left(N - \frac{k}{a}\right) + O_\varepsilon\left(N^{2/3+\varepsilon} \frac{a^{1/2}D^{1/2}}{b^{5/6}}\right)$$

holds.

References

- [1] **Inham A.E.**, Some asymptotic formulae in the theory of numbers, *J. London Math. Soc.*, **2** (1927), 202-208.
- [2] **Estermann T.**, Über die Darstellungen einer Zahl als Differenz von zwei Produkten, *J. reine Angew. Math.*, **164** (1931), 173-182.
- [3] **Исmoilов Д.**, Асимптотическая формула в теории чисел, *ДАН Тадж. ССР*, **25** (1982), 320-324. (Ismoilov D., Asymptotic formula in number theory, *DAN Tadj. SSR*)
- [4] **Heath-Brown D.R.**, The fourth power moment of the Riemann zeta-function, *Proc. London Math. Soc.*, **38** (3) (1979), 385-422.
- [5] **Deshouillers M. and Iwaniec H.**, An additive divisor problem, *Proc. London Math. Soc.*, **26** (2) (1982), 1-14.
- [6] **Motohashi Y.**, The binary additive divisor problem, *Ann. Sci. École Normale Supérieure*, **27** (4) (1994), 529-572.
- [7] **Chalk I.H.**, An asymptotic formula for the number of solutions of a quadratic diophantine equation, *C.R. Math. Rep. Acad. Sci. Canada*, **8** (4) (1981), 215-220.
- [8] **Bantle G.**, An asymptotic formula for B -twins, *Acta Arithm.*, **42** (4) (1986), 297-312.
- [9] **Montgomery H. and Vaughan R.**, Hilbert's inequality, *J. London Math. Soc.*, **8** (2) (1974), 48-113.

(Received May 20, 2003)

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