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DISTRIBUTION OF MULTIPLICATIVE FUNCTIONS: THE SYMMETRIC CASE

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To Professor Imre Kátai on his 65th birthday

Abstract. We conclude our sequence of papers on proving by purely probabilistic arguments the existence of limiting distributions for arithmetical functions. In the present paper we cover multiplicative functions in which the emphasis is on the case of symmetric distributions. The fact that we can proceed without analytic methods and sieve arguments is made possible by our recent proof for a special case of Wirsing's theorem and, on the probability side, we again apply Simonelli's theorem on products of independent random variables. In our concluding remarks an extension is indicated to the behavior of arithmetical functions on subsequences of integers.

1. Introduction

Let Ω_N denote the first N positive integers, $\mathcal{F}_N$ the collection of all subsets of Ω_N, and P_N the probability measure that assigns a mass of $1/N$ to each element of Ω_N. An arithmetical function $g(m)$ can be viewed as a random variable on the probability space $(\Omega_N, \mathcal{F}_N, P_N)$, with distribution function $F_N(x)$ given by

$$F_N(x) = P_N\Big(\{m \leq N \mid \ g(m) \leq x\}\Big).$$

The arithmetical function $g(m)$ is said to have a limit distribution function if there is a distribution function $F(x)$ such that $\lim\limits_{N\to+\infty} F_N(x) = F(x)$, for all continuity points of $F(x)$. The distribution function $F(x)$ is said to be symmetric about zero (or simply symmetric) if $F(x) = 1 - F(-x)$ for all continuity points of $F(x)$.

An arithmetical function $g(m)$ is called multiplicative if for any coprime integers m and n,

$$g(mn) = g(n)g(m).$$

Let $s_p(m)$ denote the exponent of p in the unique prime factorization of m. Then a multiplicative function $g(m)$ can be represented as

$$g(m) = \prod_p g(p^{s_p(m)}),$$

and for any fixed N the distribution function of $g(m)$ is determined by the distribution of the random variables $s_{p_i}(m)$, as p_i ranges over all prime integers such that $s_{p_i}(m) \leq (\log N)/\log p_i$, which both in formula (1) below, and in all subsequent applications, can be changed to $s_{p_i}(m) \leq N$. By using the definition of P_N we obtain

$$P_N\Big(m \;\Big|\; s_{p_i}(m) \geq k_i,\ 1 \leq i \leq t\Big) = \frac{1}{N}\left[\frac{N}{p_1^{k_1}\cdots p_t^{k_t}}\right], \tag{1}$$

where $[y]$ denotes the integer part of the real number y, from which we immediately get the asymptotic relation

$$P_N\Big(m \;\Big|\; s_{p_i}(m) \geq k_i,\ 1 \leq i \leq t\Big) = \frac{1}{p_1^{k_1}\cdots p_t^{k_t}} + O\left(\frac{1}{N}\right) \cdot \tag{2}$$

In some abstract probability space let $e_{p_i}(\omega) = e_{p_i}$, $i = 1, 2, \cdots$, be independent random variables, $e_{p_i} = 0, 1, \cdots$, and

$$P(e_{p_i} \geq k) = \frac{1}{p_i^k} \cdot$$

Hence for arbitrary nonnegative integers k_i, $1 \leq i \leq t$, and arbitrary prime integers $p_{j_1}, \cdots, p_{j_t}$,

$$\lim_{N\to+\infty} P_N\Big(s_{p_{j_i}}(m) = k_i,\ i = 1, \cdots, t\Big) = P\Big(e_{p_{j_i}} = k_i,\ i = 1, \cdots, t\Big). \tag{3}$$

Our basic tools in the present paper are the following theorems. For arbitrary N we define

$$G_N(\omega) = \prod_{p \le N} g(p^{e_p(\omega)}).$$

Since $G_N(\omega)$ is a product of independent random variables, a theorem of Simonelli (2001) immediately gives the following result.

Theorem 1. *Let $g(m)$ be a multiplicative function, $g(m) \neq 0$ for all m, and assume*

$$\sum_{p\colon g(p)<0} \frac{1}{p} = +\infty.$$

Then

(i) $G_N(\omega)$ converges weakly to a symmetric random variable continuous at zero if, and only if, $|G_N(\omega)|$ converges weakly to a random variable continuous at zero;

(ii) $G_N(\omega)$ converges weakly to a random variable discontinuous at zero if, and only if, $G_N(\omega)$ converges to zero almost surely.

If $g(m)$ is a multiplicative function, then $sign(g(m))$ is also multiplicative. The next result is a special case of a theorem of Wirsing (1967), for which a purely probabilistic proof is given in Galambos and Simonelli (2003).

Theorem 2. *Let $g(m)$ be a multiplicative function, $g(m) \neq 0$ for all m. Then $sign(g(m))$ always has a limit distribution function, and this limit is symmetric if, and only if, either $sign(g(2^k)) = -$ for all $k \ge 1$, or*

$$\sum_{p\colon g(p)<0} \frac{1}{p} = +\infty.$$

2. Results

Our aim is to give a new proof for the following theorem of Galambos (1971), see also Timofeev, Tuliaganov and Levin (1973) and Elliott (1979), p. 280.

Theorem 3. *Let $g(m)$ be a multiplicative function, $g(m) \neq 0$ for all m, and assume*

$$\sum_{p: g(p)<0} \frac{1}{p} = +\infty.$$

Then $g(m)$ has a limit symmetric distribution function $F(x)$ continuous at zero, if, and only if, $|g(m)|$ has a limit distribution function continuous at zero. Equivalently, $g(m)$ has a limit symmetric distribution function continuous at zero, if, and only if, for arbitrary M, $0 < M < +\infty$, the three series

$$(i) \quad \sum_{p:|ln|g(p)||>M} \frac{1}{p}, \qquad (ii) \quad \sum_{p:|ln|g(p)||<M} \frac{ln|g(p)|}{p},$$

$$(iii) \quad \sum_{p:|ln|g(p)||<M} \frac{ln^2|g(p)|}{p}$$

converge. Moreover $g(m)$ has a limit distribution function discontinuous at zero if, and only if, $g(m)$ has a limit distribution function degenerate at zero.

In the case $\sum_{p:\, g(p)<0} 1/p$ converges to a finite value, a theorem of Bakstys' gives necessary and sufficient conditions for a multiplicative function $g(m)$ to have a limit distribution function. Recently Galambos and Simonelli (2002) gave a purely probabilistic proof for this theorem. With minor changes their proof can be used to show that $|g(m)|$ converges weakly to a random variable continuous at zero if, and only if, $|G_N(\omega)|$ does, and that this holds independently of the convergence or divergence of the sum $\sum_{p:\, g(p)<0} 1/p$. In fact from their proof one obtains the validity of the following result, which is itself new in this form.

Theorem 4. *Let $g(m) \neq 0$. Then $|g(m)|$ has a limit distribution function continuous at zero, if, and only if, $|G_N(\omega)|$ does. Equivalently, $|g(m)|$ has a limit distribution function continuous at zero, if, and only if, for arbitrary M, $0 < M < +\infty$, the three series*

$$(i) \quad \sum_{p:|ln|g(p)||>M} \frac{1}{p}, \qquad (ii) \quad \sum_{p:|ln|g(p)||<M} \frac{ln|g(p)|}{p},$$

$$(iii) \quad \sum_{p:|ln|g(p)||<M} \frac{ln^2|g(p)|}{p}$$

converge. Moreover $|g(m)|$ has a limit distribution function discontinuous at zero if, and only if, $|g(m)|$ has a limit distribution function degenerate at zero.

Proof. See Galambos and Simonelli (2002, pp. 180-185).

To simplify the proof of Theorem 3 we prove the following result.

Lemma 1. *Let $g(m)$ be a multiplicative function, $g(m) \neq 0$ for all m. Then for any finite collection of prime integers $q_1, q_2, \cdots, q_t$, and arbitrary nonnegative integers k_i, $i = 1, \cdots, t$*

$$\lim_{N\to+\infty} P_N\left(\prod_{\substack{p\le N\\ p\neq q_j}} g(p^{s_p(m)}) < 0,\ s_{q_j}(m) = k_j,\ j = 1,\cdots,t\right) =$$

(4)

$$= \lim_{N\to+\infty} P_N\left(\prod_{\substack{p\le N\\ p\neq q_j}} g(p^{s_p(m)}) < 0\right) \lim_{N\to+\infty} P_N\Big(s_{q_j}(m) = k_j,\ j = 1,\cdots,t\Big).$$

Proof. We start our proof by observing that the two limits in (4) exist. The first limit exists by Theorem 2 (see the paragraph after (6) for details), whereas the second exists because of (3). For any fixed N, $q_i \le N$ for $i = 1, \cdots, t$, let $p_1, \cdots, p_l$ denote all prime integers less than or equal to N different from the q_is. By applying the inclusion exclusion principle one can express

$$P_N\left(\prod_{\substack{p\le N\\ p\neq q_j}} g(p) < 0,\ s_{q_j}(m) = k_j,\ j = 1,\cdots,t\right) \tag{5}$$

as sums or differences of terms of the type

$$\frac{\left[\frac{N}{Q_i Pj}\right]}{N},$$

where $Q_i = q_1^{i_1}\cdots q_t^{i_t}$, $i_j = k_j$ or $k_j + 1$, and $P_j = p_1^{j_1}\cdots p_l^{j_l}$, $j_k \ge 0$. If one writes each of these terms as

$$\frac{\left[\frac{N}{Q_i Pj}\right]}{N} = \frac{\left[\frac{N}{Q_i}\right]}{N}\,\frac{\left[\frac{\left[\frac{N}{Q_i}\right]}{P_j}\right]}{\left[\frac{N}{Q_i}\right]},$$

and let $A_{i,N}$ be the collection of all terms in the inclusion exclusion expansion of (5) containing Q_i, then it is easy to see that

$$\lim_{N\to+\infty} A_{i,N} = \frac{1}{Q_i} \lim_{N\to+\infty} P_{\left[\frac{N}{Q_i}\right]} \left(\prod_{\substack{p\le\left[\frac{N}{Q_i}\right] \\ p\neq q_j}} g(p^{s_p(m)}) < 0 \right). \tag{6}$$

Let $\tilde{g}(m)$ be such that $\tilde{g}(p^{s_p(m)}) = g(p^{s_p(m)})$, if $p \neq q_i$, $i = 1, \cdots, t$, and $\tilde{g}(p^{s_p(m)}) = |g(p^{s_p(m)})|$, if $p = q_i$, for some i. Then $\tilde{g}(m)$ is a multiplicative function, $\tilde{g}(m) \neq 0$ for all m, and

$$P_{\left[\frac{N}{Q_i}\right]} \left(\prod_{\substack{p\le\left[\frac{N}{Q_i}\right] \\ p\neq q_j}} g(p^{s_p(m)}) < 0 \right) = P_{\left[\frac{N}{Q_i}\right]} \left(\prod_{p\le\left[\frac{N}{Q_i}\right]} \tilde{g}(p^{s_p(m)}) < 0 \right).$$

This and Theorem 2 imply that

$$\lim_{N\to+\infty} A_{i,N} = \frac{1}{Q_i} \lim_{N\to+\infty} P_N \left(\prod_{p\le N} \tilde{g}(p^{s_p(m)}) < 0 \right) =$$

$$= \frac{1}{Q_i} \lim_{N\to+\infty} P_N \left(\prod_{\substack{p\le N \\ p\neq q_j}} g(p^{s_p(m)}) < 0 \right).$$

If we repeat the above calculations for every Q_i, and then combine the obtained limits, we obtain that

$$\lim_{N\to+\infty} P_N \left(\prod_{\substack{p\le N \\ p\neq q_j}} g(p^{s_p(m)}) < 0,\ s_{q_j}(m) = k_j,\ j = 1, \cdots, t \right) =$$

$$= \lim_{N\to+\infty} P_N \left(\prod_{\substack{p\le N \\ p\neq q_j}} g(p^{s_p(m)}) < 0 \right) \lim_{N\to+\infty} P_N \Big(s_{q_j}(m) = k_j,\ j = 1, \cdots, t \Big),$$

thus proving the lemma.

It is interesting to note that Lemma 1 implies that for any finite collection of $q_1, q_2, \cdots, q_t$, and arbitrary nonnegative integers k_i, $i = 1, \cdots, t$,

$$\lim_{N\to+\infty} P_N\left(\prod_{\substack{p\le N\\ p\ne q_j}} g(p^{s_p(m)}) < 0 \;\middle|\; s_{q_j}(m) = k_j,\ j = 1, \cdots, t\right) =$$

$$= \lim_{N\to+\infty} P_N\left(\prod_{\substack{p\le N\\ p\ne q_j}} g(p^{s_p(m)}) < 0\right),$$

from which we obtain the following result, which will be used in the proof of Theorem 3.

Lemma 2. *Let $g(m)$ be a multiplicative function, $g(m) \ne 0$ for all m, and for an arbitrary collection of prime integers $q_1, q_2, \cdots, q_t$, and arbitrary nonnegative integers k_i, $i = 1, \cdots, t$, let $\mathcal{A} = \{s_{q_j}(m) = k_j,\ j = 1, \cdots, t\}$. Then $sign(g(m))|\mathcal{A}$ always has a limit distribution function, and this limit is symmetric if, and only if, either $sign(g(2^k)) = -$ for all $k \ge 1$ and $q_i \ne 2$ for $i = 1, \cdots, t$, or*

$$\sum_{p\colon g(p)<0} \frac{1}{p} = +\infty.$$

We now turn to the proof of Theorem 3.

Proof of Theorem 3. Let

$$g_N(m) = \prod_{p\le N} g(p^{s_p(m)}).$$

Clearly, the weak convergence of $g_N(m)$ and $G_N(\omega)$ imply the weak convergence of $|g_N(m)|$ and $|G_N(\omega)|$, respectively. This and Theorem 4 imply that in what follows we can assume that $|g_N(m)|$ and $|G_N(\omega)|$ both converge weakly, and that their limits are continuous at zero. Under this assumption, Theorem 1 further gives that $G_N(\omega)$ converges weakly to a random variable X. Let

$$\nu_{i,j} = sign\left(\prod_{i<p\le j} g(p^{s_p(m)})\right), \qquad \nu_j = sign\left(\prod_{p\le j} g(p^{s_p(m)})\right),$$

and for arbitrary continuity point a of X we consider

$$L = \lim_{T\to+\infty} \lim_{N\to+\infty} E_N\left[1_{\{|g_T(m)|\le a\}} 1_{\{\nu_N = -\}}\right].$$

We claim that the above quantity is well defined. That is, each of the above limits exists. Let

$$\mathcal{A}_\eta = \left\{ m : 1_{\{|g_T(m)|\le a\}} 1_{\{\nu_T=\eta\}} \prod_{p\le T} 1_{\{s_p(m)\le k\}} \right\},$$

$\eta = +, -$. Then

$$\begin{aligned} L = \lim_{k\to+\infty} \lim_{T\to+\infty} \lim_{N\to+\infty} E_N \left[1_{\{|g_T(m)|\le a\}} 1_{\{\nu_N=-\}} \prod_{p\le T} 1_{\{s_p(m)\le k\}} \right] = \\ = \lim_{k\to+\infty} \lim_{T\to+\infty} \lim_{N\to+\infty} \Big(E_N \Big[1_{\{\nu_{T,N}=-\}} \;\Big|\; \mathcal{A}_+ \Big] P_N \Big(\mathcal{A}_+\Big) + \\ + E_N \Big[1_{\{\nu_{T,N}=+\}} \Big| \mathcal{A}_- \Big] P_N \Big(\mathcal{A}_-\Big) \Big). \end{aligned}$$

Since

$$\lim_{N\to+\infty} P_N\Big(\mathcal{A}_\eta\Big) = E\left[1_{\{|G_T(\omega)|\le a\}} 1_{\{sign(G_T(\omega))=\eta\}} \prod_{p\le T} 1_{\{e_p(\omega)\le k\}} \right],$$

and by Theorem 1 the right hand side of the above equation has a limit as T and K go to infinity, then to prove our claim it suffices to show that

$$\lim_{N\to+\infty} E_N \Big[1_{\{\nu_{T,N}=\eta\}} \;\Big|\; \mathcal{A}_{-\eta} \Big] = \frac{1}{2}.$$

This and Theorem 1 will further imply that

$$\prod_{p\le T} g(p^{s_p}(m)) \quad \text{and} \quad \prod_{p\le T} |g(p^{s_p}(m))| \nu_N$$

have the same limiting distribution (as N, T, and K go to infinity, in this order).

By utilizing the prime factorization of the elements in $\mathcal{A}_\eta$ we are going to partition $\mathcal{A}_\eta$ into a finite number of subsets, say $\mathcal{A}_{\eta,i}$, $i = 1, 2, \cdots, l$. The sets $\mathcal{A}_{\eta,i}$ are defined as follows. Let $p_1 < p_2 < \cdots < p_t$ denote the collection of all prime integers less than or equal to T, and consider the collection of all t-element sequences of the form $\{(p_j, l_j)\}_{j=1}^t$, $0 \le l_j \le k$. For each sequence $I_i = \{(p_j, l_j)\}_{j=1}^t$, $i = 1, 2, \cdots, (k+1)^t$, we define $\mathcal{A}_{\eta,i}$ to be the collection of all $m \in \mathcal{A}_\eta$ such that $s_{p_j}(m) = l_j$, $1 \le j \le t$. Clearly the collection of the nonempty $\mathcal{A}_{\eta,i}$ constitutes a partition of A_η, and hence we only need to show that for any nonempty $\mathcal{A}_{\eta,i}$,

$$\lim_{T\to+\infty}\lim_{N\to+\infty} E_N\left[I_{\nu_{T,N}=-1} \mid \mathcal{A}_i\right] = \frac{1}{2}.$$

But this immediately follows from Lemma 2. Hence our claim holds.

The above discussion and

$$\lim_{T\to+\infty}\limsup_{N\to+\infty} E_N\left[\left||g_T(m)|1_{\{\nu_N=\eta\}} - |g_N(m)|1_{\{\nu_N\}=\eta}\right| \geq a\right] \leq$$
$$\leq \lim_{T\to+\infty}\limsup_{N\to+\infty} E_N\left[\left||g_T(m)| - |g_N(m)|\right| \geq a\right] = 0,$$

which follows from the proof Theorem 4 (see Galambos and Simonelli, 2002, p. 191), complete the proof of the theorem.

3. Concluding remarks

Our method both in the present paper and in the previous ones on the distribution of arithmetical functions is applicable in a more general situation than just considering the sequence of consecutive integers. In fact, if we choose Ω_N as a finite set of N integers with P_N the uniform distribution on all subsets of Ω_N, then the distribution $P_N(g(n) \leq x)$ now becomes an investigation of the behavior of $g(n)$ where n is running through the members of Ω_N, that is, on specified subsequences of the integers. If on such subsequences an asymptotic formula similar to (2) is valid, that is, if the major term in (2) has a multiplicative character, and thus an asymptotic independence is implicit in such a formula, then all of our estimates remain valid (we should add here that the Kubilius-Turán inequality is valid whenever (2) is replaced by another asymptotic formula which expresses almost independence; see a discussion in Indlekofer and Kátai (2001) and their references). However, the end result cannot be claimed to be a purely probabilistic proof, because the estimates leading to the formula replacing (2) involves (some time) deep number theoretic results. A good representative of such a direction of investigation is a work of Kátai (1968), when one chooses the special sequence $p+1$, where p runs through the prime numbers.

Another direction of using purely probabilistic methods for the investigation of arithmetical functions is a 'random truncation' method, well developed within probability theory. We start with a special case initiated by Kátai

(1969); see also Elliott (1970). Take the probability space as before on the consecutive integers. Let $\mathcal{P}$ be an infinite set of prime numbers $r_1 < r_2 < \cdots$, and let $N = N_t = r_1 r_2 \cdots r_t$. We use these special N's in the probability space for N. With preassigned values $f(r_j)$, $j \geq 1$, we introduce additive functions $f_t(n) = f(n) = \sum_{j=1}^{t} \epsilon_j(n) f(r_j)$, where $\epsilon_j(n) = 1$ if r_j divides n, and $= 0$ otherwise. Note that, on our probability space, the ϵ_j are independent (not just asymptotically), since in (1) the 'fractions' inside the integer part sign are always integers. Hence, classical results are readily available. Assume that the choice of the sequence $f(r_j)$ is such that, with proper normalization, $(f_t(n) - A_t)/B_t$ is asymptotically normally distributed. Then, one can replace t by a random variable Z_t such that Z_t/t converges to a positive random variable ν, in which case the limiting distribution of the randomly truncated additive function $f_{Z_t}(n)$, with normalization, is the mixture of the normal distribution and that of ν (see Rényi (1960) and Mogyoródi (1962)). The same idea can be carried to a modification of the probability model of the preceding example by considering an abstract probability space in which there are independent random variables η_j which take the values 0 and 1 with probability 1/2 each, and $f^*(r_j) = \log r_j$. Then the sum $\sum_{j=1}^{t} \eta_j f^*(r_j) = f_t^*$, say, takes the divisors d of N_t, defined as in the previous example, each with probability 2^{-t}, where $2^t = d(N_t)$, the number of divisors of N_t. By Liapunov's form of the central limit theorem (see Galambos (1995), p.119) one gets that $(f_t^* - (1/2)\log N_t)/B_t$, with $B_t = o(\log N_t)$ is asymptotically normal (this is the original Kátai model; see also Galambos (1995), p.125 for details of calculation). Once again, one can change t to a random normalization. The exact independence of the terms are essential in the random truncation method in the preceding two examples. Namely, if the terms are only almost independent, then in limit we get only finitely additive set functions, for which the results of Rényi and Mogyoródi are not directly applicable. On the extent to which their method can be extended to finitely additive measures such as the concept of density of a sequence of positive integers we plan to report in another paper.

We conclude by mentioning that, instead of additive functions in the last two models in which the terms are independent one can discuss multiplicative functions as well whose distributions reduce to the direct application of the results of Simonelli (2001). This possibility is expanded further in our forthcoming book Galambos and Simonelli (2004).

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