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A MEAN VALUE RESULT INVOLVING THE FOURTH MOMENT OF $|\zeta(\frac{1}{2}+it)|$

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To Prof. Imre Kátai on the occasion of his 65th birthday

Abstract. If (k,ℓ) is an exponent pair such that $k+\ell<1$, then we have

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 |\zeta(\sigma+it)|^2 t \ll_\varepsilon T^{1+\varepsilon}$$

$$\left(\sigma > \min\left(\frac{5}{6}, \max\left(\ell-k,\, \frac{5k+\ell}{4k+1}\right)\right)\right),$$

while if (k,ℓ) is an exponent pair such that $3k+\ell<1$, then we have

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 |\zeta(\sigma+it)|^4 t \ll_\varepsilon T^{1+\varepsilon} \qquad \left(\sigma > \frac{11k+\ell+1}{8k+2}\right).$$

1. Introduction

Let as usual $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ $(\sigma > 1)$ denote the Riemann zeta-function, where $s = \sigma + it$ is a complex variable. Mean values of $\zeta(s)$ in the so-called "critical strip" $\frac{1}{2} \le \sigma \le 1$ represent a central topic in the theory of the zeta-function (see [7] and [8] for an extensive account). No bound of the form

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2m} t \ll_{\varepsilon,m} T^{1+\varepsilon} \qquad (m \in \mathbb{N}) \tag{1.1}$$

is known to hold when $m \ge 3$, while in the cases $m = 1, 2$ precise asymptotic formulas for the integrals in question are known (see op. cit.). Here we shall prove two hybrid bounds involving the mean value of $|\zeta\left(\frac{1}{2}+it\right)|^4$ multiplied by $|\zeta(\sigma+it)|^{2j}$ $(j = 1, 2;\ \frac{1}{2} < \sigma < 1)$. The results are

Theorem 1. *If (k, ℓ) is an exponent pair such that $k + \ell < 1$, then we have*

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 |\zeta(\sigma+it)|^2 t \ll_\varepsilon T^{1+\varepsilon} \qquad \left(\sigma > \min\left(\frac{5}{6}, \max\left(\ell - k,\ \frac{5k+\ell}{4k+1}\right)\right)\right), \tag{1.2}$$

and in particular (1.2) *holds for* $\sigma \ge 5/6 = 0.8333\ldots$.

Theorem 2. *If (k, ℓ) is an exponent pair such that $3k + \ell < 1$, then we have*

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 |\zeta(\sigma+it)|^4 t \ll_\varepsilon T^{1+\varepsilon} \qquad \left(\sigma > \max\left(\frac{\ell - k + 1}{2},\ \frac{11k+\ell+1}{8k+2}\right)\right), \tag{1.3}$$

and in particular (1.3) *holds for* $\sigma \ge 1953/1984 = 0.984375$.

The merit of these results is that (1.2) and (1.3) hold for values of σ less than one; of course one expects the bounds to hold for $\sigma \ge \frac{1}{2}$, in which case we would obtain the (yet unproved) sixth and eighth moment of $|\zeta\left(\frac{1}{2}+it\right)|$ (namely (1.1) with $m = 3$ and $m = 4$, respectively).

2. Proof of Theorem 1

In the proof of both (1.2) and (1.3) it is sufficient to consider the integral over $[T, 2T]$, then to replace T by $T2^{-j}$ $(j = 1, 2, \ldots)$ and sum all the resulting estimates. Also, it is sufficient to suppose that $\sigma \le 1$, since one has (see e.g. [7])

$$\zeta(\sigma + it) \ll \log|t| \qquad (\sigma \ge 1).$$

To prove the bound on σ in (1.2) involving k, ℓ, we shall use the simple approximate functional equation for $\zeta(s)$ (see [7, Theorem 1.8]), which gives

$$\zeta(s) = \sum_{n \le T} n^{-s} + O(1) \qquad (s = \sigma + it,\ T \le t \le 2T). \tag{2.1}$$

The essential tool in our considerations is the following theorem for the fourth moment of $|\zeta\left(\frac{1}{2}+it\right)|$, weighted by a Dirichlet polynomial, due to N. Watt [9]. This is built on the works of J.-M. Deshouillers and H. Iwaniec [1], [2], involving the use of Kloosterman sums, but it contains the following sharper result: Let $a_1, a_2, \ldots$ be complex numbers. Then, for $\varepsilon > 0$, $M \ge 1$ and $T \ge 1$,

$$\int_0^T \left|\sum_{m \le M} a_m m^{it}\right|^2 \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 t \ll_\varepsilon T^{1+\varepsilon} M(1 + M^2 T^{-1/2}) \max_{m \le M} |a_m|^2. \tag{2.2}$$

Here and later ε denotes arbitrarily small, positive constants, not necessarily the same ones at each occurrence. We write (2.1) as

$$\begin{aligned} \zeta(s) &= \sum_{m \le Y} m^{-s} + \sum_{Y < n \le T} n^{-s} + O(1) = \\ &= \sum\nolimits_1 + \sum\nolimits_2 + O(1), \end{aligned} \tag{2.3}$$

say, where $1 \ll Y \le T$. The sum $\sum_1$ is split into $O(\log T)$ subsums with $N < m \le N' \le 2N \le Y$. In (2.1) we take $a_m = m^{-\sigma}$ for $N < m \le N'$, $a_m = 0$ otherwise. Then in view of $N \ll Y$, $\frac{1}{2} \le \sigma \le 1$ it follows that

$$\begin{aligned} \int_0^T \left|\sum\nolimits_1\right|^2 \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 t &\ll_\varepsilon T^{1+\varepsilon} \max_{N \ll Y} N^{1-2\sigma}(1 + N^2 T^{-1/2}) \ll_\varepsilon \\ &\ll_\varepsilon T^{1+\varepsilon}(1 + Y^{3-2\sigma} T^{-1/2}) \ll_\varepsilon T^{1+\varepsilon} \end{aligned} \tag{2.4}$$

for

$$Y = T^{\frac{1}{6-4\sigma}}. \tag{2.5}$$

To estimate $\sum_2$ in (2.3) we use the theory of (one-dimensional) exponent pairs (see e.g. [3], [5] and [7]). We split $\sum_2$ into $O(\log T)$ subsums with $N < m \le \le N' \le 2N$, $Y \le N \le T$ and $\sigma \ge \frac{1}{2}$. Removing the (monotonically decreasing) factor $n^{-\sigma}$ by partial summation from each subsum, it remains to estimate

$$S(N,t) := \sum_{N<n\le N'\le 2N} n^{it} \qquad (Y \le N \le T,\, T \le t \le 2T).$$

If (k,ℓ) is an exponent pair, then since $n^{it} = e^{iF(n,t)}$ with $\dfrac{\partial^r F(n,t)}{\partial n^r} \asymp_r TN^{-r}$, it follows that

$$S(N,t) \ll \left(\frac{T}{N}\right)^k N^\ell = T^k N^{\ell-k},$$

and consequently

$$\sum\nolimits_2 \ll T^k N^{\ell-k-\sigma}\log T \ll T^{k+\frac{\ell-k-\sigma}{6-4\sigma}}\log T$$

if $\sigma \ge \ell - k$, which is our assumption. Hence $\sum_2 \ll \log T$ for $k + \dfrac{\ell-k-\sigma}{6-4\sigma} \le 0$, i.e.

$$\sigma \ge \frac{5k+\ell}{4k+1},$$

giving

$$\int_0^T \Big|\sum\nolimits_2\Big|^2 \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 t \ll_\varepsilon T^{1+\varepsilon} \qquad \left(\sigma \ge \max\Big(\ell-k,\, \frac{5k+\ell}{4k+1}\Big)\right). \tag{2.6}$$

Combining (2.4) and (2.6) we obtain the second bound in (1.2); for $k+\ell<1$ we have $\dfrac{5k+\ell}{4k+1} < 1$. To obtain a specific result we choose M.N. Huxley's exponent pair (see [6]) $(\kappa,\lambda) = \left(\dfrac{32}{205}+\varepsilon,\, \dfrac{1}{2}+\dfrac{32}{205}+\varepsilon\right)$, which supersedes his exponent pair (see [4], [5]) $(\kappa,\lambda) = \left(\dfrac{89}{570}+\varepsilon,\, \dfrac{374}{570}+\varepsilon\right)$. This exponent pair is one of the many obtained by the Bombieri–Iwaniec method. With this pair we find that

$$\frac{5k+\ell}{4k+1} = \frac{589}{666} = 0.884384384\ldots. \tag{2.7}$$

As is often the case when one applies the theory of exponent pairs, the above exponent pair is not optimal, and small improvements may be obtained by more laborious calculations. Note that the algorithm of Graham-Kolesnik [3, Chapter 5] cannot be used when the exponent pairs are formed by the use (of variants) of the Bombieri–Iwaniec method, and not only by the classical A-, B-process and convexity, so the optimal value is hard to compute. However, in the above case the exponent pair is in a certain sense optimal. Namely if $\ell = k+\frac{1}{2}$, then one has (see [3, Theorem 4.1])

$$\mu\left(\frac{1}{2}\right) \le k, \quad \mu(\sigma) := \limsup_{t\to\infty} \frac{\log|\zeta(\sigma+it)|}{\log t} \quad (\sigma\in\mathbb{R}).$$

But if $\ell = k+\frac{1}{2}$, then

$$\frac{5k+\ell}{4k+1} = \frac{\frac{1}{2}+6k}{1+4k},$$

which is an upper bound for

$$\frac{\frac{1}{2}+6\mu(\frac{1}{2})}{1+4\mu(\frac{1}{2})}. \tag{2.8}$$

If we use the bound ([7, eq. (8.14)])

$$\sum_{N<n\le 2N} n^{-\sigma-it} \ll_\varepsilon N^{\sigma_0-\sigma}T^{\varepsilon-\sigma_0} + N^{\sigma_0-\sigma}\int_0^T |\zeta(\sigma_0+it+iv)|\,\frac{v}{v+1},$$

then we obtain

$$\sum_{N<n\le 2N} n^{-\sigma-it} \ll_\varepsilon 1 + N^{\frac{1}{2}-\sigma}T^{\mu(\frac{1}{2})+\varepsilon} \quad \left(T\le t\le 2T,\, \sigma>\frac{1}{2}\right).$$

But if $N \ge Y = T^{1/(6-4\sigma)}$, then the above bound gives

$$\sum_{N<n\le 2N} n^{-\sigma-it} \ll_\varepsilon T^\varepsilon \quad \left(T\le t\le 2T,\, N\ge Y,\, \sigma>\frac{1}{2}\right)$$

for

$$\sigma \ge \frac{\frac{1}{2} + 6\mu(\frac{1}{2})}{1 + 4\mu(\frac{1}{2})}, \tag{2.9}$$

which is (2.8). Huxley's work [6] brings forth precisely the new bound (hitherto the sharpest one of its kind) $\mu(\frac{1}{2}) \le 32/205$, corresponding to the exponent pair (k, ℓ) with $k = 32/205 + \varepsilon$, $\ell = k + \frac{1}{2}$, so that in this context the value given by (2.7) is the optimal one that can be obtained at present from exponent pairs satisfying the condition $\ell = k + \frac{1}{2}$.

To complete the proof of (1.2) we use the well-known Mellin inversion integral

$$e^{-x} = \frac{1}{2\pi i} \int\limits_{(c)} x^{-w}\Gamma(w)w \quad (c > 0,\, x > 0), \tag{2.10}$$

where $\int\limits_{(c)}$ denotes integration over the line $\Re e\ w = c$. In (2.10) we set $x = n/Y$ $(1 \ll Y \ll T^C)$, multiply by n^{-s} $(\frac{1}{2} < \sigma < 1)$ and sum over n. This gives

$$\sum_{n=1}^{\infty} e^{-n/Y} n^{-s} = \frac{1}{2\pi i} \int\limits_{(2)} Y^w \zeta(s+w)\Gamma(w)dw \quad (s = \sigma + it,\, T \le t \le 2T). \tag{2.11}$$

In (2.11) we shift the line of integration to $\Re ew = \frac{1}{2} - \sigma$ and apply the residue theorem. The pole at $w = 1 - s$ contributes a residue which is, by Stirling's formula for $\Gamma(s)$, $\ll 1$. The pole at $w = 0$ yields $\zeta(s)$, and we obtain from (2.11)

$$\zeta(s) \ll 1 + \left| \sum_{n \le Y \log^2 T} e^{-n/Y} n^{-\sigma - it} \right| + Y^{\frac{1}{2} - \sigma} \int\limits_{-\log^2 T}^{\log^2 T} \left| \zeta\left(\frac{1}{2} + it + iv\right) \right| dv.$$

Therefore

$$\int\limits_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^4 |\zeta(\sigma + it)|^2 dt \ll T \log^4 T + I_1(T) + I_2(T), \tag{2.12}$$

say, where

$$
\begin{aligned}
I_1(T) &:= \int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 \left|\sum_{n\le Y\log^2 T} e^{-n/Y} n^{-\sigma-it}\right|^2 dt, \\
I_2(T) &:= Y^{1-2\sigma}\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 \left(\int_{-\log^2 T}^{\log^2 T} \left|\zeta\left(\frac{1}{2}+it+iv\right)\right| dv\right)^2 dt.
\end{aligned}
\tag{2.13}
$$

Similarly to (2.4) we obtain

$$
I_1(T) \ll_\varepsilon T^{1+\varepsilon}(1+Y^{3-2\sigma}T^{-1/2}). \tag{2.14}
$$

To $I_2(T)$ we apply Hölder's inequality for integrals and the sharpest bound for the sixth moment of $|\zeta\left(\frac{1}{2}+it\right)|$ (see [7, Chapter 8]), namely $\int_0^T |\zeta\left(\frac{1}{2}+it\right)|^6 dt \ll$ $\ll T^{5/4}\log^C T$, to deduce that

$$
I_2(T) \ll_\varepsilon T^{\frac{5}{4}+\varepsilon}Y^{1-2\sigma}. \tag{2.15}
$$

Now we choose

$$
Y = T^{3/8}.
$$

Then from (2.12)–(2.15) it follows that

$$
\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 |\zeta(\sigma+it)|^2 dt \ll_\varepsilon T^{1+\varepsilon} + T^{\frac{1}{2}+\frac{9-6\sigma}{8}+\varepsilon} \ll_\varepsilon T^{1+\varepsilon}
$$

for $\sigma \ge 5/6$, which yields the first bound in (1.2) and completes the proof of Theorem 1.

3. Proof of Theorem 2

For the proof of Theorem 2 we shall use the approximate functional equation (see [7, Theorem 4.2])

$$
\zeta^2(s) = \sum_{n\le x} d(n)n^{-s} + \chi^2(s)\sum_{n\le y} d(n)n^{s-1} + O(x^{\frac{1}{2}-\sigma}\log t), \tag{3.1}
$$

where $d(n)$ is the number of divisors of n, $0 < \sigma < 1$; $x, y, t > C > 0$ and $4\pi^2 xy = t$. Here

$$\chi(s) = \frac{\zeta(s)}{\zeta(1-s)} = 2^s \pi^{s-1} \Gamma(1-s) \sin\left(\frac{1}{2}\pi s\right) \asymp t^{\frac{1}{2}-\sigma} \quad (t \ge t_0 > 0)$$

is the expression appearing in the functional equation for $\zeta(s)$. In (3.1) we suppose that $T \le t \le 2T$, $T \le x \le 2T$. Then we obtain

$$|\zeta(\sigma + it)|^4 \ll \log^2 T + \\ + \left|\sum_{n \le x} d(n) n^{-s}\right|^2 + T^{1-2\sigma} \left|\sum_{n \le \frac{t^2}{4\pi^2 x}} d(n) n^{s-1}\right|^2 .$$

Both sums on the right-hand side are split into $O(\log T)$ subsums with $N < < n \le N' \le 2N$, $N \ll T$. Setting $S(u) := \sum_{N < n \le u} d(n) n^{-it}$ we have, by partial summation,

$$\sum_{N < n \le N'} d(n) n^{-\sigma - it} = S(N')(N')^{-\sigma} + \sigma \int_N^{N'} S(u) u^{-\sigma-1} du,$$

$$\sum_{N < n \le N'} d(n) n^{\sigma - 1 - it} = S(N')(N')^{\sigma-1} + \sigma \int_N^{N'} S(u) u^{\sigma-2} du.$$

This gives

$$\left|\sum_{n \le x} d(n) n^{-s}\right|^2 + T^{1-2\sigma} \left|\sum_{n \le \frac{t^2}{4\pi^2 x}} d(n) n^{s-1}\right|^2 \ll \\ \ll \log T \max_N \left(N^{-2\sigma} \max_{N \le u \le N'} |S(u)|^2\right) \left(1 + \left(\frac{T}{N}\right)^{1-2\sigma}\right) \ll \\ \ll \log T \max_N N^{-2\sigma} \max_{N \le u \le N'} |S(u)|^2,$$

since $N \ll T, \sigma \geq \frac{1}{2}$. In the case when $N \leq Y$ (see (2.5)) we have, by (2.2),

$$\int_T^{2T} N^{-2\sigma}|S(u)|^2 \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 dt \ll_\varepsilon$$
$$\ll_\varepsilon T^{1+\varepsilon}N^{1-2\sigma}(1+N^2T^{-1/2}) \max_{N<n\leq N'} d^2(n) \ll_\varepsilon$$
$$\ll_\varepsilon T^{1+\varepsilon}(1+Y^{3-2\sigma}T^{-1/2}) \ll_\varepsilon T^{1+\varepsilon},$$

since $d(n) \ll_\varepsilon n^\varepsilon$.

In the case when $Y < N \ll T$ we shall estimate

$$\bar{S}(u) = \sum_{N<n\leq u} d(n)n^{it} = \sum_{n\leq u} d(n)n^{it} - \sum_{n\leq N} d(n)n^{it}$$

by estimating

$$\sum(u,t) := \sum_{n\leq u} d(n)n^{it} \qquad (N < u \leq N' \leq 2N).$$

By applying the familiar hyperbola method we have

$$\sum(u,t) = \sum_{mn\leq u} (mn)^{it} =$$
$$= 2\sum_{m\leq\sqrt{u}} m^{it} \sum_{n\leq u/m} n^{it} - \left(\sum_{m\leq\sqrt{u}} m^{it}\right)^2 =$$
$$= 2S_1(u,t) - S_2^2(u,t),$$

say. To estimate $S_1(u,t)$, we split the inner sum over n into $O(\log T)$ subsums

$$S_3(u_1,t) := \sum_{u_1<n\leq u_1'\leq 2u_1} n^{it} \qquad (u_1 \ll u/m).$$

Then, since $\ell \geq k$ for any exponent pair (k,ℓ),

$$S_3(u,t) \ll T^k\left(\frac{u}{m}\right)^{\ell-k},$$

which yields

$$S_1(u,t) \ll T^k \log T \cdot N^{\ell-k} \sum_{m\ll\sqrt{N}} m^{k-\ell} \ll T^k N^{\frac{1}{2}(\ell-k+1)} \log T.$$

In a similar vein it follows that

$$S_2(u,t) \ll T^k N^{\frac{1}{2}(\ell-k)} \log T,$$

and thus for $N \gg Y$ we obtain, for $\sigma \ge \frac{1}{2}(\ell-k+1)$,

$$\begin{aligned} N^{-\sigma}|S(u)| &\ll N^{-\sigma}\log^2 T(T^k N^{\frac{1}{2}(\ell-k+1)} + T^{2k}N^{\ell-k}) \ll \\ &\ll (T^k T^{\frac{\ell-k+1-2\sigma}{12-8\sigma}} + T^{2k}T^{\frac{\ell-k-\sigma}{6-4\sigma}})\log^2 T \ll \\ &\ll \log^2 T \end{aligned}$$

for

$$\sigma \ \ge \ \max\left(\frac{11k+\ell+1}{8k+2}, \frac{11k+\ell}{8k+1}\right) = \frac{11k+\ell+1}{8k+2}$$

if $3k+\ell<1$, which we supposed. This proves (1.3). Finally we consider the exponent pair (see [3, p. 39])

$$(k,\ell) = \left(\frac{16}{120Q-32}, \frac{120Q-16q-63}{120Q-32}\right) \quad (Q=2^q,\ q\ge 2).$$

The optimal value for q is in our case found to be $q=3$, giving

(3.2)

$$(k,\ell) = \left(\frac{16}{928}, \frac{849}{928}\right) = \frac{11k+\ell+1}{8k+2} = \frac{1953}{1984} = 0.984375 \ \left(> \frac{1}{2}(\ell-k+1)\right).$$

This completes the proof of Theorem 2, and with a more careful choice of the exponent pair the value (3.2) could be improved a little (namely by the use of the algorithm of [3, Chapter 5]). It is an open problem to find $\sigma_0 = \sigma_0(j)$ (<1) such that

$$\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 |\zeta(\sigma+it)|^{2j} dt \ll_{j,\varepsilon} T^{1+\varepsilon}$$

for $j \in \mathbb{N}$ satisfying $j \ge 3$ and $\sigma > \sigma_0$. By using the method outlined at the end of Section 2, one would obtain the value

$$\sigma_0 \le \frac{\frac{1}{2}+6j\mu(\frac{1}{2})}{1+4j\mu(\frac{1}{2})}.$$

But the right-hand side does not exceed unity if and only if

$$\mu\left(\frac{1}{2}\right) \le \frac{1}{4j},$$

which is not known to hold unless $j = 1$, and this case we already considered. Thus the approach based on the use of exponent pairs seems more appropriate already in the case $j = 2$.

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