Annales Univ. Sci. Budapest., Sect. Comp. 23 (2004) 41-46

ON THE ABSOLUTE CESÀRO SUMMABILITY

L. Leindler (Szeged, Hungary)

Dedicated to Professor Imre Kátai on his 65th birthday

Abstract. The traditional monotonity assumption on the coefficients of orthogonal series is weakened to locally almost monotonicity condition.

1. Let $\{\varphi_n(x)\}$ be an orthonormal system defined on the finite interval (a, b). We consider the orthogonal series

$$\sum_{n=0}^{\infty} c_n \varphi_n(x) \quad \text{with} \quad \sum_{n=0}^{\infty} c_n^2 < \infty. \tag{1.1}$$

The (C, α)-means of (1.1) are

$$\sigma_n^{(\alpha)}(x) := \frac{1}{A_n^{(\alpha)}} \sum_{\nu=0}^{n} A_{n-\nu}^{(\alpha)} c_\nu \varphi_\nu(x), \quad \alpha > -1,$$

and $A_n^{(\alpha)} := \binom{n+\alpha}{n}$. The series (1.1) is said to be absolute (C, α)-summable or briefly $|C, \alpha|$-summable if

$$\sum_{n=0}^{\infty} \left| \sigma_{n+1}^{(\alpha)}(x) - \sigma_n^{(\alpha)}(x) \right| < \infty.$$

AMS Subject Classification: 40F05, 40G05.

This research was partially supported by the Hungarian Foundation for Scientific Research under Grant T042462.

K. Tandori [2] proved that the *condition*

$$\sum_{m=0}^{\infty}\left\{\sum_{n=2^m+1}^{2^{m+1}} c_n^2\right\}^{1/2} < \infty \tag{1.2}$$

is necessary and sufficient that the series (1.1) *for every orthonormal system* $\{\varphi_n(x)\}$ *should be absolute* $(C,1)$*-summable almost everywhere in* (a,b).

We ([1]) showed that *the condition* (1.2) *is also necessary and sufficient that the series* (1.1) *for every orthonormal system* $\{\varphi_n(x)\}$ *be absolute* (C,α)*-summable with* $\alpha > 1/2$ *almost everywhere.*

In [1] we also gave conditions implying the $\left|C, \frac{1}{2}\right|$- and $|C,\alpha|$-summability with $-1 < \alpha < 1/2$, respectively.

Our result reads as follows

Theorem A. *In order that the series* (1.1) *for every orthonormal system* $\{\varphi_n(x)\}$ *should be* $|C,\alpha|$*-summable almost everywhere:*

(i) *for* $\alpha = 1/2$ *the condition*

$$\sum_{m=1}^{\infty}\sqrt{m}\left\{\sum_{n=2^m+1}^{2^{m+1}} c_n^2\right\}^{1/2} < \infty, \tag{1.3}$$

(ii) *for* $-1 < \alpha < 1/2$ *the condition*

$$\sum_{m=0}^{\infty} 2^{\frac{m}{2}(1-2\alpha)}\left\{\sum_{n=2^m+1}^{2^{m+1}} c_n^2\right\}^{1/2} < \infty \tag{1.4}$$

is sufficient; for monotone coefficients $\{c_n\}$ *these conditions are also necessary if the summability is claimed for the all orthonormal systems.*

In this note was shall verify that the conditions (1.3) and (1.4) not only for monotone decreasing sequences $\{c_n\}$ are necessary, but a wider set of sequences as well, more precisely they are also necessary if the sequence $\{c_n\}$ is only locally almost monotone decreasing.

A nonnegative sequence $\mathbf{c} := \{c_n\}$ is called locally almost monotone decreasing if there exists a constant $K(\mathbf{c})$, depending on the sequence $\mathbf{c}$ only, such that

$$c_n \le K(\mathbf{c})c_m \tag{1.5}$$

holds for all m and $m \leq n \leq 2m$. Such a sequence will be denoted by $\mathbf{c} \in$ $\in LAMS$.

2. To demonstrate the necessity of the conditions (1.3) and (1.4) we establish the following theorem.

Theorem. *If the Rademacher series*

$$\sum_{n=0}^{\infty} c_n r_n(x) \quad (r_n(x) := \text{sign } \sin 2^n \pi x) \tag{2.1}$$

with a locally almost monotone decreasing sequence $\{c_n\}$ *is* $|C, \alpha|$*-summable almost everywhere, then the conditions (1.3) and (1.4) hold according* $\alpha = 1/2$ *or* $-1 < \alpha < 1/2$.

3. To the proof we employ the following known lemma (see e.g. A. Zygmund [3], p. 213).

Lemma. *If* $\sum\limits_{n=0}^{\infty} c_n^2 < \infty$ *then*

$$A \left\{ \sum_{n=0}^{\infty} c_n^2 \right\}^{1/2} \leq \int_0^1 |h(x)| dx \leq B \left\{ \sum_{n=0}^{\infty} c_n^2 \right\}^{1/2}, \tag{3.1}$$

where A and B are positive and finite constants, furthermore $h(x)$ denotes the sum of the series (2.1).

We shall use the following notations and estimations.

$$L_{n,\nu}^{(\alpha)} := \frac{A_{n+1-\nu}^{(\alpha)}}{A_{n+1}^{(\alpha)}} - \frac{A_{n-\nu}^{(\alpha)}}{A_n^{(\alpha)}} = \frac{A_{n-\nu}^{(\alpha)}}{A_n^{(\alpha)}} \frac{\nu\alpha}{(n+1-\nu)(n+1+\alpha)},$$

$$\frac{A_m^{(\alpha)}}{m^\alpha} \leq c(\alpha) \quad (m > 0,\ \alpha > -1), \tag{3.2}$$

$$\left| L_{n,\nu}^{(\alpha)} \right| \geq d(\alpha) \frac{(n+1-\nu)^{\alpha-1}\nu}{n^{\alpha+1}} \quad (\alpha > -1,\ \alpha \neq 0), \tag{3.3}$$

where $c(\alpha)$ and $d(\alpha)$ are positive constants.

4. Let us assume that the series (2.1) is $\left|C, \frac{1}{2}\right|$-summable almost everywhere, and let $\varepsilon = A^2/4$. Then by Egoroff's theorem there exists a constant M and a measurable set $G(\subset (0,1))$ with measure $\mu(G) > 1-\varepsilon$, such that for any $x \in G$

$$\sum_{n=0}^{\infty} \left|\sigma_{n+1}^{(\frac{1}{2})}(x) - \sigma_n^{(\frac{1}{2})}(x)\right| < M \tag{4.1}$$

holds. Denote $CG := (0,1)\backslash G$.

Employing our Lemma, by (3.3) and (4.1), we obtain that

(4.2)

$$\begin{aligned}
M\mu(G) &\geq \sum_{n=0}^{\infty}\left(\int_0^1 - \int_{CG}\right)\left|\sigma_{n+1}^{(\frac{1}{2})}(x) - \sigma_n^{(\frac{1}{2})}(x)\right| dx \geq \\
&\geq \sum_{n=0}^{\infty}\left(\int_0^1 \left|\sigma_{n+1}^{(\frac{1}{2})}(x) - \sigma_n^{(\frac{1}{2})}(x)\right| dx - \right.\\
&\qquad \left. - \mu(CG)^{1/2}\left\{\int_0^1 \left(\sigma_{n+1}^{(\frac{1}{2})}(x) - \sigma_n^{(\frac{1}{2})}(x)\right)^2 dx\right\}^{1/2}\right) \geq \\
&\geq \sum_{n=0}^{\infty}\left(A - \varepsilon^{1/2}\right)\left\{\sum_{\nu=0}^{n}\left(L_{n,\nu}^{(\frac{1}{2})}\right)^2 c_\nu^2 + c_{n+1}^2\left(A_{n+1}^{(\frac{1}{2})}\right)^{-2}\right\}^{1/2} \geq \\
&\geq \frac{A}{2} d(\alpha) \sum_{n=0}^{\infty}\left\{\sum_{\nu=0}^{n} n^{-3}(n+1-\nu)^{-1}\nu^2 c_\nu^2\right\}^{1/2} \geq \\
&\geq \frac{A}{2} d(\alpha) \sum_{m=0}^{\infty}\sum_{n=2^m+1}^{2^{m+1}}\left\{\sum_{\nu=2^m+1}^{n} n^{-3}(n+1-\nu)^{-1}\nu^2 c_\nu^2\right\}^{1/2} =: S_1,
\end{aligned}$$

say. Since $\{c_n\} \in LAMS$, thus

$$S_1 \geq \frac{A}{2^2 K(\mathbf{c})} \sum_{m=0}^{\infty} \frac{c_{2^{m+1}}}{2^{\frac{m+1}{2}}} \sum_{n=2^m+1}^{2^{m+1}} \left\{\sum_{\nu=2^m+1}^{n} (n+1-\nu)^{-1}\right\}^{1/2} \geq$$

$$\geq \frac{A}{2^5 K(\mathbf{c})} \sum_{m=1}^{\infty} 2^{\frac{m+1}{2}} \sqrt{m+1}\, c_{2^{m+1}} \geq \tag{4.3}$$

$$\geq \frac{A}{2^5 K(\mathbf{c})^2} \sum_{m=2}^{\infty} \sqrt{m} \left\{\sum_{\nu=2^m+1}^{2^{m+1}} c_\nu^2\right\}^{1/2}.$$

The estimations (4.2) and (4.3) yield that if the series (2.1) is $\left|C, \frac{1}{2}\right|$-summable and $\{c_n\} \in LAMS$ then the condition (1.3) maintains.

If the series (2.1) is $|C, \alpha|$-summable $\left(-1 < \alpha < \frac{1}{2}\right)$ then a similar arguing leads to

(4.4)
$$M_\mu(G) \geq \sum_{n=0}^{\infty} \left(A - \varepsilon^{1/2}\right) \left\{ \sum_{\nu=0}^{n} \left(L_{n,\nu}^{(\alpha)}\right)^2 c_\nu^2 + c_{n+1}^2 \left(A_{n+1}^{(\alpha)}\right)^{-2} \right\}^{1/2} =: S_2.$$

Now we can omit the sum in the curly bracket, thus

(4.5)
$$\begin{aligned} S_2 &\geq \frac{A}{2c(\alpha)K(\mathbf{c})} \sum_{n=2}^{\infty} \frac{c_n}{n^\alpha} = \frac{A}{2c(\alpha)K(\mathbf{c})} \sum_{m=0}^{\infty} \sum_{n=2^m+1}^{2^{m+1}} \frac{c_n}{n^\alpha} \geq \\ &\geq \frac{A}{2c(\alpha)K(\mathbf{c})} \sum_{m=0}^{\infty} 2^{-(m+1)\alpha} \sum_{n=2^m+1}^{2^{m+1}} c_n \geq \\ &\geq \frac{A}{4c(\alpha)K(\mathbf{c})^2} \sum_{m=0}^{\infty} 2^{-(m+1)\alpha} 2^{\frac{m+1}{2}} \left\{ \sum_{\nu=2^{m+1}+1}^{2^{m+2}} c_\nu^2 \right\}^{1/2} = \\ &= \frac{A}{4c(\alpha)K(\mathbf{c})^2} \sum_{m=1}^{\infty} 2^{\frac{m}{2}(1-2\alpha)} \left\{ \sum_{\nu=2^m+1}^{2^{m+1}} c_\nu^2 \right\}. \end{aligned}$$

It is clear that (4.4) and (4.5) verify that (1.4) holds.

Herewith the proof is complete.

References

[1] **Leindler L.,** Über die absolute Summierbarkeit der Orthogonalreihen, *Acta Sci. Math. (Szeged),* **22** (1961), 243-268.

[2] **Tandori K.,** Über die orthogonalen Funktionen IX. (Absolute Summation), *Acta Sci. Math. (Szeged),* **21** (1960), 292-299.

[3] **Zygmund A.,** *Trigonometric series I.,* University Press, Cambridge, 1959.

(Received March 17, 2003)

L. Leindler
University of Szeged
Aradi vértanúk tere 1.
H-6720 Szeged, Hungary
leindler@math.u-szeged.hu