

## MULTIPLICATIVE FUNCTIONS CLOSE TO THE DIVISOR FUNCTION ON SHIFTED PRIMES

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*Dedicated to Professor Imre Kátai  
on his 65th birthday*

**Abstract.** We consider complex-valued multiplicative arithmetical functions which are close to 2 on the set of primes and analyse the asymptotic behaviour of the sum

$$\sum_{p \leq x} f(p+1).$$

In particular we prove

**Corollary 1.** *Let  $f : \mathbb{N} \rightarrow \mathbb{C}$  be multiplicative. Let  $c_1 \geq 4$  and  $c_2 > 0$  be constants such that the inequality*

$$\sum_{|f(p)| \leq c_1} \frac{|f(p)|^2 \log^2 p}{p} + \sum_p \sum_{r \leq 2} \frac{|f(p^r)|^2}{p^r} \leq c_2$$

*holds, and assume that the series*

$$\sum_p \frac{|2 - f(p)|}{p}$$

*converges. Then*

$$\sum_{p \leq x} f(p+1) = x \prod_p \left( 1 - \frac{2}{p} + \left( 1 - \frac{1}{p} \right) \sum_{r=1}^{\infty} \frac{f(p^r)}{p^r} \right) + o(x).$$

**1.** There are many results concerning the mean behaviour of multiplicative arithmetical functions  $f$  on the set  $\{p+1\}$  of shifted primes, especially in the case when the values  $f(p)$  are close to 1 for “almost all” primes  $p$ . As a typical result we mention a theorem of I. Kátai [1].

**Proposition 1** (see [1]). *If  $|f| \leq 1$  and if the series*

$$\sum_p \frac{1-f(p)}{p}$$

*converges then the mean-value*

$$m_p(f) := \lim_{x \rightarrow \infty} \frac{1}{\pi(x)} \sum_{p \leq x} f(p+1)$$

*exists.*

The wish to abandon the restriction on the size of  $f$  led to the investigation of multiplicative functions which belong to the class  $\mathcal{L}^q$ ,  $q \geq 1$ . Here, for  $1 \leq q < \infty$ ,

$$\mathcal{L}^q := \{f : \mathbb{N} \rightarrow \mathbb{C}, \|f\|_q < \infty\}$$

denotes the linear space of arithmetic functions with bounded seminorm

$$\|f\|_q := \left\{ \limsup_{x \rightarrow \infty} x^{-1} \sum_{n \leq x} |f(n)|^q \right\}^{\frac{1}{q}}.$$

Many results in this context are due to K.-H. Indlekofer - N.M. Timofeev ([2], [3], [4]) and A. Hildebrand [5].

The following proposition describes a typical situation.

**Proposition 2** (see [2]). *Suppose that*

$$\limsup_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} |f(n)|^2 < \infty, \quad \sum_{p \leq x} |f(p)|^2 \ll x \log^{-\rho} x$$

*where  $0 < \rho \leq 1$ . If there is a Dirichlet-character  $\chi_d \bmod d$  such that*

$$M(f\chi_d) := \lim_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} f(n)\chi_d(n)$$

*exists and is different from zero, then  $m_p(f)$  exists, too. If the mean-value  $M(|f|)$  exists, then the same is true for the mean-value  $m_p(|f|)$ .*

We shall extend this result to a further class of functions. For this let us consider the *divisor function*  $\tau$ , i.e.  $\tau(n)$  denotes the number of divisors of  $n$ . It is well known that

$$\sum_{n \leq x} \tau(n) = x \log x + (2\gamma - 1)x + R(x)$$

( $\gamma$ : Euler's constant) where  $R(x)$  can easily be estimated by  $O(\sqrt{x})$ , and the question of finding better bounds for  $R(x)$  is referred to as *Dirichlet's divisor problem*.

Concerning the average order of  $\tau$  on the set of shifted primes it is known (see, for example [6], Theorem 3.9) that

$$\sum_{p \leq x} \tau(p+1) = \prod_p \left(1 + \frac{1}{p(p-1)}\right) x + O\left(\frac{x \log \log x}{\log x}\right).$$

This result motivated the present investigations. Observe that  $\tau$  is multiplicative and  $\tau(p) = 2$  for all primes  $p$ . In this paper we deal with multiplicative functions  $f$  the values  $f(p)$  of which are close to 2 for "almost all" primes  $p$ , and analyse the asymptotic behaviour of the sum

$$\sum_{p \leq x} f(p+1).$$

We prove

**Theorem.** *Let  $f$  be a complex-valued multiplicative arithmetical function. Let  $c_1 \geq 4$  and  $c_2$  be positive constants such that the inequality*

$$(1) \quad \sum_{|f(p)| \geq c_1} \frac{|f(p)|^2 \log^2 p}{p} + \sum_p \sum_{r \geq 2} \frac{|f(p^r)|^2}{p^r} \leq c_2$$

*holds. Suppose further that*

$$(2) \quad \lim_{x \rightarrow \infty} \frac{1}{\log x} \sum_{p \leq x} \frac{|f(p) - 2|}{p} \log p = 0.$$

*Then*

$$\sum_{p \leq x} f(p+1) = x \prod_{p \leq x} \left(1 - \frac{2}{p} + \left(1 - \frac{1}{p}\right) \sum_{r=1}^{\infty} \frac{f(p^r)}{p^r}\right) +$$

$$+o\left(x \prod_{p \leq x} \left(1 - \frac{2}{p} + \left(1 - \frac{1}{p}\right) \sum_{r=1}^{\infty} \frac{|f(p^r)|}{p^r}\right)\right)$$

as  $x \rightarrow \infty$ .

Immediate consequences of the Theorem are the following corollaries.

**Corollary 1.** *Let the multiplicative arithmetical function  $f$  satisfy condition (1), and assume that the series*

$$(3) \quad \sum_p \frac{|2 - f(p)|}{p}$$

converges. Then

$$\lim_{x \rightarrow \infty} \frac{1}{x} \sum_{p \leq x} f(p+1) = \prod_p \left(1 - \frac{2}{p} + \left(1 - \frac{1}{p}\right) \sum_{r=1}^{\infty} \frac{f(p^r)}{p^r}\right).$$

**Corollary 2.** *Let  $f$  be multiplicative. Suppose that  $|f(p)| \leq 2$  and the two series*

$$\sum_p \sum_{r \geq 2} \frac{|f(p^r)|^2}{p^r}, \quad \sum_p \frac{2 - f(p)}{p}$$

converge. Then

$$\lim_{x \rightarrow \infty} \frac{1}{x} \sum_{p \leq x} f(p+1) = \prod_p \left(1 - \frac{2}{p} + \left(1 - \frac{1}{p}\right) \sum_{r=1}^{\infty} \frac{f(p^r)}{p^r}\right).$$

**Proof of Corollary 1.** An application of Cauchy's inequality gives

$$\begin{aligned} \sum_{p \leq x} \frac{|f(p) - 2|}{p} \log p &\leq \sum_{p \leq y} \frac{(2 + c_1) \log p}{p} + \sum_{\substack{|f(p)| \leq c_1, \\ p > y}} \frac{|f(p) - 2|}{p} \log x + \\ &+ \left( \sum_{|f(p)| \geq c_1} \frac{4|f(p)|^2}{p} \sum_{p \leq y} \frac{\ln^2 p}{p} \right)^{\frac{1}{2}} + \left( \sum_{\substack{|f(p)| \geq c_1, \\ p \geq y}} \frac{4|f(p)|^2}{p} \sum_{p \leq x} \frac{\log^2 p}{p} \right)^{\frac{1}{2}}. \end{aligned}$$

Letting  $y(x) \rightarrow \infty$  such that  $\frac{\log y(x)}{\log x} \rightarrow 0$  as  $x \rightarrow \infty$ , and making use of the hypothesis (3) and (1) we establish (2). If the series (3) converges then the series

$$\sum_p \frac{|2 - f(p)|}{p}$$

converges, too. Hence both products occurring in the assertion of Theorem are convergent, and this ends the proof of Corollary 1.

**Proof of Corollary 2.** The hypothesis of Corollary 2 implies (1). Using Cauchy's inequality we conclude

$$\sum_{p \leq x} \frac{|f(p) - 2|}{p} \log p \leq 4 \sum_{p \leq y} \frac{\log p}{p} + \left( 2 \sum_{p \geq y} \frac{2 - \operatorname{Re} f(p)}{p} \sum_{p \leq x} \frac{\log^2 p}{p} \right)^{\frac{1}{2}}.$$

In the same way as above we complete the proof of Corollary 2.

**2.** First we establish some preliminary results.

**Lemma 1.** (see [3], Theorem 1) *Let  $f_i$ , ( $i = 1, 2, \dots, k$ ), be complex-valued multiplicative functions satisfying the conditions*

$$(4) \quad A(n) := \sum_{i=1}^k \alpha_i f_i(n) \geq 0 \quad (n = 1, 2, \dots)$$

where  $\alpha_i \in \mathbf{C}$  for  $i = 1, \dots, k$ . Further, assume that there are constants  $c_3 > 0$ ,  $c_4 \geq 0$ ,  $c_5 > 0$  and  $0 < \rho \leq 1$  such that the inequalities

$$(5) \quad \sum_{n \leq x} |f_i(n)|^2 \leq c_3 x (\log x)^{c_4} \quad (i = 1, \dots, k)$$

and

$$(6) \quad \sum_{p \leq x} |f_i(p)|^2 \leq c_5 x (\log x)^{-\rho} \quad (i = 1, \dots, k)$$

hold. Then for  $u = (\log x)^A$  with  $A \geq 8(9c_4 + 8 \cdot 3^4 + 150)$  we have

$$\frac{1}{\pi(x)} \sum_{p \leq x} A(p+1) \ll \frac{\log u}{x} \sum_{\substack{n \leq x, \\ (n-1, P(u))=1}} A(n) + (\log \log x)^{1+\frac{c_4}{2}} (\log x)^{-\frac{\rho}{2}},$$

where  $P(u) := \prod_{p \leq u} p$ . The constant implied in  $\ll$  depends at most on  $c_3$  and  $c_5$ .

**Lemma 2.** *Let  $f$  be multiplicative. Further, let  $c_1 \geq 4$  and  $c_2 > 0$  be constants so that (1) holds, i.e.*

$$\sum_{|f(p)| \geq c_1} \frac{|f(p)|^2 \log^2 p}{p} + \sum_p \sum_{r \geq 2} \frac{|f(p^r)|^2}{p^r} \leq c_2 \quad .$$

Then, with Euler's  $\varphi$ -function,

$$(7) \quad \sum_{n \leq x} \frac{|f(n)|^2}{\varphi(n)} \ll (\log x)^{c_1^2}$$

and

$$(8) \quad \sum_{n \leq x} |f(n)| \ll \frac{x}{\log x} \sum_{n \leq x} \frac{|f(n)|}{n},$$

where the constants implied in  $\ll$  depend at most on  $c_1$  and  $c_2$ .

**Proof.** It is easy to see that

$$\sum_{n \leq x} \frac{|f(n)|^2}{\varphi(n)} \leq \prod_{p \leq x} \left( 1 + \sum_{r=1}^{\infty} \frac{|f(p^r)|^2}{\varphi(p^r)} \right) \ll \exp \left( \sum_{p \leq x} \sum_{r=1}^{\infty} \frac{|f(p^r)|^2}{p^r} \right),$$

and using (1) we obtain (7). As an immediate consequence we deduce

$$\sum_{n \leq x} |f(n)|^2 \leq x \sum_{n \leq x} \frac{|f(n)|^2}{n} \ll x (\log x)^{c_1^2}.$$

Application of the Cauchy inequality and (1) show that

$$\begin{aligned} & \sum_{p^r \leq x} |f(p^r)| \log p^r \leq \\ & \leq c_1 \sum_{p \leq x} \log p + x \left( \sum_p \sum_{r \geq 2} \frac{|f(p^r)|^2}{p^r} \right)^{\frac{1}{2}} \left( \sum_p \sum_{r \geq 2} \frac{(\log p^r)^2}{p^r} \right)^{\frac{1}{2}} + \end{aligned}$$

$$+x \sum_{|f(p)| \geq c_1} \frac{|f(p)|^2}{p} \log p \ll x.$$

Combining the last two estimates gives

$$\begin{aligned} & \sum_{n \leq x} |f(n)| \ll \\ & \ll \sqrt{x} + \frac{1}{\log x} \sum_{n \leq x} |f(n)| \log n \ll \sqrt{x} + \frac{1}{\log x} \sum_{n \leq x} |f(n)| \sum_{p^r \leq \frac{x}{n}} |f(p^r)| \log p^r \ll \\ & \ll \frac{x}{\log x} \sum_{n \leq x} \frac{|f(n)|}{n}. \end{aligned}$$

This completes the proof of Lemma 2.

**Lemma 3.** *Let  $f_1, f_2$  be complex-valued multiplicative functions such that the squares  $f_1^2, f_2^2$  satisfy the condition of Lemma 2. Then*

$$\begin{aligned} & \frac{1}{\pi(x)} \sum_{p \leq x} |f_1(p+1) - f_2(p+1)|^2 \ll \frac{1}{\log x} \sum_{n \leq x} \frac{|f_1(n) - f_2(n)|^2}{\varphi(n)} + \\ & + \frac{1}{\log x} \sum_{p \leq x} \frac{|f_1(p) - f_2(p)|^2}{p} \log p \max_{t \leq x} \frac{1}{\log 2t} \sum_{n \leq t} \frac{|f_2(n)|^2}{\varphi(n)} + \\ & + \frac{1}{(\log x)^{1+\delta}} \sum_{n \leq x} \frac{|f_1(n)|^2 + |f_2(n)|^2}{n} + \frac{1}{(\log x)^\delta}, \end{aligned}$$

where  $\delta > 0$  and the implied constant depends at most on  $c_1, c_2$  and  $\delta$ .

**Proof.** We first apply Lemma 1 with

$$A(n) = |f_1(n) - f_2(n)|^2 = |f_1(n)|^2 - \overline{f_1(n)}f_2(n) - f_1(n)\overline{f_2(n)} + |f_2(n)|^2.$$

The condition (6) of Lemma 1 is satisfied with  $\rho = 1$ , and hypothesis (5) follows directly from (7). So, we deduce from Lemma 1 that

$$(9) \quad \frac{1}{\pi(x)} \sum_{p \leq x} |f_1(p+1) - f_2(p+1)|^2 \ll \frac{\log u}{x} \sum_{\substack{n \leq x, \\ (n-1, P(u))=1}} |f_1(n) - f_2(n)|^2 + (\log x)^{-\frac{1}{2} + \varepsilon}$$

where  $u = \log^A x$ . Since (5) holds for  $f_1$  and  $f_2$  we conclude in the same way as above

$$\begin{aligned} & \sum_{\substack{n \leq x, \\ (n-1, P(u))=1}} |f_1(n) - f_2(n)|^2 \ll \\ & \ll \sqrt{x} + \frac{1}{\log x} \sum_{\substack{np^r \leq x, \\ (np^{r-1}, P(u))=1}} |f_1(n)f_1(p^r) - f_2(n)f_2(p^r)|^2 \log p^r. \end{aligned}$$

Now we split the last sum into three parts  $\sum_1, \sum_2, \sum_3$ , where  $\sum_1$  is the sum over  $np^r \leq x$  with  $p^r \leq y$ ,  $\sum_2$  is the sum over  $np^r \leq x$  such that  $p^r > y$ ,  $r \geq 2$  and  $p \geq y$ ,  $|f_1(p)| \geq c_1$  or  $|f_2(p)| \geq c_1$  and  $\sum_3$  contains the remaining summands, respectively.

By (8) and (1) the sum  $\sum_1$  can be estimated by

$$\begin{aligned} \sum_1 & \ll x \sum_{p^r \leq y} \frac{|f_1(p^r)|^2 + |f_2(p^r)|^2}{p^r} \log p^r \frac{1}{\log \frac{x}{y}} \sum_{n \leq x} \frac{|f_1(n)|^2 + |f_2(n)|^2}{n} \ll \\ & \ll x \frac{\log y}{\log \frac{x}{y}} \sum_{n \leq x} \frac{|f_1(n)|^2 + |f_2(n)|^2}{n}. \end{aligned}$$

From the Cauchy inequality we deduce that

$$\begin{aligned} \sum_2 & \ll x \sum_{n \leq x} \frac{|f_1(n)|^2 + |f_2(n)|^2}{n} \left( \frac{1}{\log y} \sum_{i=1}^2 \sum_{|f_i(p)| \geq c_1} \frac{|f_i(p)|^2}{p} \log^2 p + \right. \\ & \left. + \left( \sum_{r \geq 2} \sum_p \frac{|f_1(p^r)|^4 + |f_2(p^r)|^4}{p^r} \right)^{\frac{1}{2}} \left( \sum_{r \geq 2} \sum_{p^r \geq y} \frac{(\log p^r)^2}{p^r} \right)^{\frac{1}{2}} \right). \end{aligned}$$

Putting  $y = \exp(\log^\alpha x)$ ,  $0 < \alpha < \frac{1}{2}$ , we obtain

$$\sum_1 + \sum_2 \ll \frac{x}{(\log x)^\alpha} \sum_{n \leq x} \frac{|f_1(n)|^2 + |f_2(n)|^2}{n}.$$

The contribution of  $\sum_3$  is bounded by

$$\sum_3 \ll \sum_{31} + \sum_{32} + \sum_{33},$$



where

$$\begin{aligned}\sum_{31} &= \sum_{n \leq \frac{x}{y}} |f_1(n) - f_2(n)|^2 \sum_{\substack{p \leq \frac{x}{n}, \\ (np-1, P(u))=1}} \log p, \\ \sum_{32} &= \sum_{n \leq t} |f_2(n)|^2 \sum_{\substack{p \leq \frac{x}{n}, \\ (np-1, P(u))=1}} \log p\end{aligned}$$

and

$$\sum_{33} = \sum_{\substack{y < p \leq x/t, \\ f_i(p) \leq c_1, \\ i=1,2}} |f_1(p) - f_2(p)| \log p \sum_{\substack{n \leq x/p, \\ (np-1, P(u))=1}} |f_2(n)|^2,$$

respectively. Here  $t$  is given by  $\log t = \log u \cdot \log y$ . To estimate  $\sum_{31}$  and  $\sum_{32}$  we employ Selberg's sieve (see [6]). Choosing  $z = u$ ,  $\xi^2 = \sqrt[4]{\frac{x}{n}} \geq \sqrt[4]{y} > u$  in Theorem 6.2 of [6] we obtain

$$(10) \quad \left| \left\{ p \leq \frac{x}{n}, (np-1, P(u)) = 1 \right\} \right| \ll \frac{x}{\varphi(n) \log u \log \frac{x}{n}}.$$

Hence

$$\sum_{31} + \sum_{32} \ll \frac{x}{\log u} \sum_{n \leq x} \frac{|f_1(n) - f_2(n)|^2}{\varphi(n)} + \frac{x}{\log u} \sum_{n \leq t} \frac{|f_2(n)|^2}{\varphi(n)}.$$

Lemma 2 shows that the second summand is  $O\left(\frac{x}{\log u} (\log t)^{c_1^2}\right)$ . Turning to (9) we arrive at

$$(11) \quad \begin{aligned} \frac{1}{\pi(x)} \sum_{p \leq x} |f_1(p+1) - f_2(p+1)|^2 &\ll \frac{1}{\log x} \sum_{n \leq x} \frac{|f_1(n) - f_2(n)|^2}{\varphi(n)} + \frac{\log u}{x \log x} \sum_{33} + \\ &+ x (\log x)^{\alpha c_1^2 - 1} (\log u)^{c_1^2 - 1} + \frac{\log u}{(\log x)^{1+\alpha}} \sum_{n \leq x} \frac{|f_1(n)|^2 + |f_2(n)|^2}{n}. \end{aligned}$$

Let  $0 < \alpha < \frac{1}{2c_1^2}$ . We are now ready to give an upper bound for  $\sum_{33}$ . First

$$\sum_{\substack{n \leq x/p, \\ (np-1, P(u))=1}} |f_2(n)|^2 \ll \sqrt{\frac{x}{p}} + \frac{1}{\log \frac{x}{p}} \sum_{\substack{q^r n \leq x/p, \\ (q^r np-1, P(u))=1}} |f_2(q^r)|^2 |f_2(n)|^2 \log q^r.$$

The three parts of the last sum where  $q^r \leq y$  or  $q^r \geq y$ ,  $r \geq 2$  or  $q > y$ ,  $|f(q)| \geq c_1$ , respectively, can be estimated in the same way as  $\sum_1$  and  $\sum_2$ . The contribution of these parts are

$$\ll \frac{x \log y}{p \log \frac{x}{py}} \sum_{n \leq x/p} \frac{|f_2(n)|^2}{n} + \frac{x}{p \log y} \sum_{n \leq x/p} \frac{|f_2(n)|^2}{n} \ll \frac{x}{p \log u} \sum_{n \leq x/p} \frac{|f_2(n)|^2}{n}.$$

Recall that  $p \leq x/t$ ,  $\log t = \log u \log y$ ,  $u = \log^A x$ ,  $\log y > \log u$ .

For the remaining summands we use Selberg's sieve as above (see (10)).

Thus

$$\begin{aligned} & \log \frac{x}{p} \cdot \sum_{\substack{n \leq x/p \\ (np-1, P(u))=1}} |f_2(n)|^2 \ll \\ & \ll \frac{x}{p \log u} \sum_{n \leq x/p} \frac{|f_2(n)|^2}{n} + \sum_{n \leq \frac{x}{pu}} |f_2(n)|^2 \sum_{\substack{q \leq \frac{x}{np}, \\ (qn p-1, P(\frac{x}{\sqrt{u}}))=1}} \log q \ll \\ & \ll \frac{x}{p \log u} \sum_{n \leq x/p} \frac{|f_2(n)|^2}{\varphi(n)} \end{aligned}$$

and

$$\sum_{33} \ll \frac{x}{\log u} \sum_{p \leq x} \frac{|f_1(p) - f_2(p)|^2}{p} \log p \frac{1}{\log \frac{x}{p}} \sum_{n \leq x/p} \frac{|f_2(n)|^2}{\varphi(n)}.$$

Combining this estimate with (11) gives the assertion of Lemma 3.

**Lemma 4.** *Let  $f$  be a complex-valued multiplicative function satisfying the condition of Lemma 2, and put*

$$\prod(f, x) := \prod_{p \leq x} \left( 1 + \sum_{r=1}^{\infty} \frac{f(p^r)}{p^r} \right).$$

Then

$$(12) \quad \prod(|f|, x) \asymp \exp \left( \sum_{p \leq x} \frac{|f(p)|}{p} \right).$$

If, in addition,  $f$  satisfies condition (2) then

$$(13) \quad \max_{2 \leq t \leq x} \frac{1}{\log t} \prod(|f|, x) \ll \frac{1}{\log x} \prod(|f|, x).$$

**Proof.** Hypothesis (1) implies the convergence of the product

$$\prod_p \left( 1 + \sum_{r=1}^{\infty} \frac{|f(p^r)|}{p^r} \right) \exp \left( -\frac{|f(p)|}{p} \right).$$

Thus (12) holds. If  $\sqrt{x} \leq y \leq x$  then (1) and (2) yield

$$\frac{\prod(|f|, x)}{\prod(|f|, y)} = \exp \left( \sum_{\substack{y < p \leq x, \\ |f(p)| \leq c_1}} \frac{|f(p)|}{p} \right) (1 + o(1)) = \left( \frac{\log x}{\log y} \right)^2 (1 + o(1)).$$

Hence

$$\frac{\prod(|f|, x)}{\log x} \frac{\log y}{\prod(|f|, y)} \geq \frac{1}{2} \quad \text{and} \quad \frac{\prod(|f|, x)}{\log x} \frac{\log \sqrt{x}}{\prod(|f|, \sqrt{x})} \geq \frac{3}{2}$$

for  $\sqrt{x} \leq y \leq x$  if  $x \geq x_0$ . This implies (13), which ends the proof of Lemma 4.

**Lemma 5.** (see [7], Theorem 5) *Let  $z \leq \sqrt{x}$ . For any positive constant  $A$  there is a constant  $B = B(A)$  such that with  $Q = \sqrt{x}(\log x)^{-B}$*

$$\begin{aligned} & \sum_{d \leq Q} \max_{y \leq x} \max_{(a,d)=1} |\{n : n \leq y, (n, P(z)) = 1, n \equiv a \pmod{d}\}| - \\ & - \frac{1}{\varphi(d)} |\{n : n \leq y, (n, P(z)d) = 1\}| \ll x(\log x)^{-A}. \end{aligned}$$

**Lemma 6.** *Let  $d \leq x^2$ ,  $z \leq \sqrt[4]{x}$ ,  $y \leq \sqrt[4]{x}$ . Then*

$$|\{n : n \leq x, (n, P(z)) = 1, (dn - 1, P(y)) = 1\}| \ll \frac{d}{\varphi(d)} \frac{x}{\log z \log y} + \frac{x \log^3 x}{z}.$$

**Proof.** By Selberg's sieve (see [6], Theorem 6.2) we get

$$|\{n : n \leq x, (n, P(z)) = 1, (dn - 1, P(y)) = 1\}| \ll \prod_{\substack{p \leq y, \\ p|d}} \left( 1 - \frac{1}{p} \right) \sum_{\substack{n \leq x, \\ (n, P(z))=1}} 1 +$$

$$(14) \quad + \sum_{\substack{\delta \leq \xi^2, \\ (\delta, d)=1}} 3^{\omega(\delta)} \left| |\{n : n \leq x, (n, P(z)) = 1, nd \equiv 1 \pmod{\delta}\}| - \right. \\ \left. - \frac{1}{\varphi(\delta)} |\{n : n \leq x, (n, P(z)) = 1\}| \right|,$$

where  $\xi^2 = \sqrt[3]{x}$ . Now,

$$\sum_{\delta \leq \xi^2} \frac{3^{\omega(\delta)}}{\varphi(\delta)} \left| |\{n : n \leq x, (n, P(z)) = 1\}| - |\{n : n \leq x, (n, P(z)\delta) = 1\}| \right| \ll \\ \ll x \sum_{p > z} \frac{1}{p^2} \sum_{\delta \leq \xi^2} \frac{3^{\omega(\delta)}}{\varphi(\delta)} \ll \frac{x(\log x)^3}{z}.$$

Hence, using Cauchy's inequality and Lemma 5, we obtain that the second sum on the right-hand side of (14)

$$\ll \left( x \sum_{\delta \leq \xi^2} \frac{9^{\omega(\delta)}}{\varphi(\delta)} \right)^{\frac{1}{2}} \times \\ \times \left( \sum_{\delta \leq \xi^2} \max_{(a, d)=1} \left| |\{n : n \leq x, (n, P(z)) = 1, n \equiv a \pmod{\delta}\}| - \right. \right. \\ \left. \left. - \frac{1}{\varphi(d)} |\{n : n \leq x, (n, \delta P(z)) = 1\}| \right| \right)^{\frac{1}{2}} + \frac{x(\log x)^3}{z} \ll \frac{x}{\log^A x} + \frac{x(\log x)^3}{z},$$

where  $A$  is an arbitrary positive constant. Now, again by Selberg's sieve,

$$|\{n : n \leq x, (n, P(z)) = 1\}| \ll \frac{x}{\log z}.$$

Substituting these results into (14) yields Lemma 6.

**3. Proof of Theorem.** For  $y \geq 1$  define the multiplicative function  $f_y$  by

$$f_y(p^r) = \begin{cases} f(p^r) & \text{if } p^r = y, \\ r + 1 & \text{if } p^r > y. \end{cases}$$

Assume that  $y = y(x) \rightarrow \infty$  and  $\frac{\log y}{\log x} \rightarrow 0$  as  $x \rightarrow \infty$ . The function  $f_y$  satisfies hypothesis (1). Let  $f_1$  and  $f_2$  be multiplicative functions such that  $f_1(p^r) = \sqrt{f(p^r)}$  and  $f_2(p^r) = \sqrt{f_y(p^r)}$ ,  $r = 1, 2, \dots$ , respectively, where  $\sqrt{z} = \sqrt{|z|} \exp\left(\frac{1}{2}i \arg z\right)$ . The functions  $f_1$  and  $f_2$  satisfy the hypothesis of Lemma 3. Applying Lemma 3 gives

$$\begin{aligned} \frac{1}{\pi(x)} \sum_{p \leq x} |f_1(p+1) - f_2(p+1)|^2 &\ll \frac{1}{\log x} \sum_{n \leq x} \frac{|f_1(n) - f_2(n)|^2}{\varphi(n)} + \\ &+ \frac{1}{\log x} \sum_{p \leq x} \frac{|f_1(p) - f_2(p)|^2}{p} \log p \max_{t \leq x} \frac{1}{\log 2t} \sum_{n \leq x} \frac{|f(n)|}{\varphi(n)} + \\ &+ \frac{1}{(\log x)^\delta} + \frac{1}{(\log x)^{1+\delta}} \sum_{n \leq x} \frac{|f(n)| + |f_y(n)|}{\varphi(n)}. \end{aligned}$$

From (2) we conclude that there is a function  $\varepsilon$  such that  $\varepsilon(x) \downarrow 0$  but  $\varepsilon(x)\sqrt{\log x}$  tends to infinity as  $x \rightarrow \infty$  and

$$(15) \quad \sum_{p \leq x} \frac{|2 - f(p)|}{p} \log p \leq \varepsilon(x) \log x.$$

Let  $\log y \geq \sqrt{\varepsilon(x)} \log x$ . Then

$$(16) \quad \sum_{y < p \leq 2} \frac{||f(p)| - 2|}{p} \leq \frac{1}{\log y} \sum_{p \leq 2} \frac{|f(p) - 2|}{p} \log p \leq \sqrt{\varepsilon(x)}$$

and therefore  $\prod(|f|, x) \asymp \prod(|f_y|, x)$ . Note that  $|\sqrt{f(p)} + \sqrt{2}| \geq \sqrt{2}$ . Then, by (1) and (2) we obtain

$$\begin{aligned} \sum_{p \leq x} \frac{|f_1(p) - f_2(p)|^2}{p} \log p &\leq \frac{1}{2} \sum_{p \leq x} \frac{|f(p) - 2|^2}{p} \log p \leq \frac{c_1 + 2}{2} \varepsilon(x) \log x + \\ &+ \sum_{|f(p)| \geq c_1} \frac{|f(p)|^2}{p} \log p \ll \varepsilon(x) \log x. \end{aligned}$$

Applying Lemma 4 shows that

$$(17) \quad \frac{1}{\pi(x)} \sum_{p \leq x} |f_1(p+1) - f_2(p+1)|^2 \ll$$

$$\ll \frac{1}{\log x} \sum_{n \leq x} \frac{|f_1(n) - f_2(n)|^2}{\varphi(n)} + o\left(\frac{1}{\log x} \prod(|f|, x)\right).$$

Next we prove

$$(18) \quad \sum_{n \leq x} \frac{|f_1(n) - f_2(n)|^2}{\varphi(n)} = o\left(\prod(|f|, x)\right).$$

For this we divide the sum on the left into two parts  $\sum'_1$  and  $\sum'_2$ , where  $\sum'_1$  denotes the sum over  $n \leq x$  such that  $p^r | n$ ,  $r \geq 2$ ,  $p^r \geq y$  or  $p | n$ ,  $p > y$ ,  $|f(p)| \geq c_1$  and  $\sum'_2$  contains the remaining summands, respectively. By (1) we have

$$\sum'_1 \ll \left( \sum_{\substack{p^r \geq y, \\ r \geq 2}} \frac{|f(p^r)| + (r+1)}{p^r} + \sum_{\substack{p^r \geq y, \\ |f(p)| \geq c_1}} \frac{|f(p)|}{p} \right) \prod(|f|, x) = o\left(\prod(|f|, x)\right).$$

Now write each number  $n$  occurring in  $\sum'_2$  in the form  $n = n_1 p_1 \dots p_t$ , where  $n_1$  has only prime divisors  $\leq y$  and  $p_i > y$  with  $|f(p_i)| \leq c_1$ . Then

$$\begin{aligned} |f_1(n) - f_2(n)| &= |f_1(n_1)| \left| f_1(p_1) \dots f_1(p_t) - (\sqrt{2})^t \right| \leq \\ &\leq |f_1(n_1)| c_1^{t-1} \left( |f_1(p_1) - \sqrt{2}| + \dots + |f_1(p_t) - \sqrt{2}| \right). \end{aligned}$$

This gives

$$|f_1(n) - f_2(n)|^2 \leq |f_1(n_1)| c_1^{2t} (|f(p_1) - 2| + \dots + |f(p_t) - 2|),$$

and therefore

$$\sum'_2 \ll c_1^{2 \frac{\log x}{\log y}} \frac{\log x}{\log y} \sum_{y < p \leq x} \frac{|f(p) - 2|}{p} \exp\left(\sum_{p \leq y} \frac{|f(p)|}{p} + \sum_{y < p \leq x} \frac{1}{p}\right).$$

Choosing  $y \geq x\sqrt{\varepsilon(x)}$ ,  $\frac{\log x}{\log y} \leq \frac{1}{4 \log c_1} \log \frac{1}{\varepsilon(x)}$ , we obtain by (15)

$$\sum'_2 \ll c_1^{2 \frac{\log x}{\log y}} \left(\frac{\log x}{\log y}\right)^3 \varepsilon(x) \prod(|f|, x) \ll \sqrt{[3]\varepsilon(x)} \prod(|f|, x).$$

Thus, (18) holds. Using (17) and Lemmas 3 and 4 we see

$$\begin{aligned} \frac{1}{\pi(x)} \sum_{p \leq x} |f_1^2(p+1) - f_2^2(p+1)| &\leq \left( \frac{1}{\pi(x)} \sum_{p \leq x} |f_1(p+1) - f_2(p+1)|^2 \right)^{\frac{1}{2}} \times \\ &\times \left( \frac{1}{\pi(x)} \sum_{p \leq x} |f_1(p+1) + f_2(p+1)|^2 \right)^{\frac{1}{2}} = o\left( \frac{1}{\log x} \prod(|f|, x) \right). \end{aligned}$$

Therefore

$$(19) \quad \frac{1}{\pi(x)} \sum_{p \leq x} f(p+1) = \frac{1}{\pi(x)} \sum_{p \leq x} f_y(p+1) + o\left( \frac{1}{\log x} \prod(|f|, x) \right),$$

where  $y(x) \rightarrow \infty$  and  $\frac{\log y(x)}{\log x} \rightarrow 0$  as  $x \rightarrow \infty$ . The last sum in (19) can be written in the form

$$(20) \quad \sum_{p \leq x} f_y(p+1) = \sum_{\substack{p+1=n_1 n_2 \leq x, \\ n_1 \leq y_1}} f(n_1) \tau(n_2) + O\left( \sum_{\substack{p+1=n_1 n_2 \leq x, \\ n_1 > y_1}} |f(n_1)| \tau(n_2) \right),$$

where  $(n_2, P(y)) = 1$  and where  $n_1$  denotes an even number whose prime divisors are not larger than  $y$ . Let us apply Lemma 3 for  $f_1$  and  $f_2$  where  $f_1^2(n) = f(n)$  and  $f_2^2(n) = 0$  if  $n > 1$ . Then

$$\begin{aligned} \frac{1}{\pi(x)} \sum_{p+1=n_1 \leq x} |f(n_1)| &\ll \frac{1}{\log x} \sum_{n_1 \leq x} \frac{|f(n_1)|}{\varphi(n_1)} + \\ &+ \frac{1}{\log x} \sum_{p \leq y} \frac{|f(p)|}{p} \log p \frac{1}{\log x} \prod(|f|, x) + \frac{1}{(\log x)^{1+\delta}} \prod(|f|, x). \end{aligned}$$

Choose  $\log y \geq \sqrt{\varepsilon(x)} \log x$ . Using Lemma 4 together with (1) and (16) we see that

$$\sum_{n_1 \leq x} \frac{|f(n_1)|}{\varphi(n_1)} \ll \left( \frac{\log y}{\log x} \right)^2 \prod(|f|, x).$$

Hence

$$(21) \quad \frac{1}{\pi(x)} \sum_{p+1=n_1 \leq x} |f(n_1)| \ll \frac{\log y}{\log x} \cdot \frac{1}{\log x} \prod(|f|, x) = o\left( \frac{1}{\log x} \prod(|f|, x) \right)$$

and

$$\begin{aligned} & \frac{1}{\pi(x)} \sum_{\substack{p+1=n_1 n_2 \leq x, \\ n_1 > y_1, n_2 > y}} |f(n_1)| \tau(n_2) \leq \frac{2}{\pi(x)} \sum_{y_1 < n_1 \leq x/y} |f(n_1)| \times \\ & \times \sum_{n_2 \leq \sqrt{\frac{x}{n_1}}} \left( \left| \left\{ n : n \leq \frac{x}{n_2 n_1}, (n_2 n(n_1 n_2 n - 1), P(\sqrt[3]{y})) = 1 \right\} \right| + \sqrt[3]{y} \right). \end{aligned}$$

Applying Selberg's sieve (see [6], Theorem 6.2) gives

$$\left| \left\{ n : n \leq \frac{x}{n_2 n_1}, (n_2 n(n_1 n_2 n - 1), P(\sqrt[3]{y})) = 1 \right\} \right| \ll \frac{x}{\varphi(n_2) \varphi(n_1) \log^2 y}.$$

Therefore

$$(22) \quad \frac{1}{\pi(x)} \sum_{\substack{p+1=n_1 n_2 \leq x, \\ n_1 > y_1, n_2 > y}} |f(n_1)| \tau(n_2) \ll \frac{\log x}{\log^2 y} \sum_{y_1 < n_1 \leq x/y} \frac{|f(n_1)|}{\varphi(n_1)} \sum_{n_2 \leq x} \frac{1}{\varphi(n_2)}.$$

By Cauchy's inequality and the hypothesis (1) we get

$$\begin{aligned} & \sum_{y_1 < n_1 \leq x} \frac{|f(n_1)|}{\varphi(n_1)} \leq \frac{1}{\log y_1} \sum_{y_1 < n_1 \leq x} \frac{|f(n_1)|}{\varphi(n_1)} \log n_1 \leq \\ & \leq \frac{1}{\log y_1} \sum_{p^r \leq 2y} \frac{|f(p^r)|}{\varphi(p^r)} \log p^r \sum_{\frac{y_1}{2} < n_1 \leq x} \frac{|f(n_1)|}{\varphi(n_1)} + \\ & + \frac{1}{\log y_1} \left( \sum_{r \geq 2, p} \frac{|f(p^r)|^2}{\varphi(p^r)} \right)^{\frac{1}{2}} \left( \sum_{\substack{p^r \geq y_1, \\ r \geq 2}} \frac{(\log p^r)^2}{p^r} \right)^{\frac{1}{2}} \sum_{n_1 \leq x} \frac{|f(n_1)|}{\varphi(n_1)}. \end{aligned}$$

Now, let  $\frac{\log y_1}{\log y} \rightarrow \infty$  and  $\frac{\log y_1}{\log x} \rightarrow 0$  as  $x \rightarrow \infty$ . Then

$$\begin{aligned} \sum_{y_1 < n_1 \leq x} \frac{|f(n_1)|}{\varphi(n_1)} & \ll \frac{\log y}{\log y_1} \frac{1}{\log y_1} \sum_{n_1 \leq x} \frac{|f(n_1)|}{\varphi(n_1)} \log n_1 + \frac{1}{\sqrt[3]{y} \log y_1} \sum_{n_1 \leq x} \frac{|f(n_1)|}{\varphi(n_1)} \ll \\ & \ll \left( \frac{\log y}{\log y_1} \right)^2 \sum_{n_1 \leq x} \frac{|f(n_1)|}{\varphi(n_1)} \end{aligned}$$



if  $\sqrt[4]{y} \geq \frac{\log y_1}{(\log y)^2}$ . Thus, by (22),

$$\begin{aligned} \frac{1}{\pi(x)} \sum_{\substack{p+1=n_1 n_2 \leq x, \\ n_1 > y_1, n_2 > y}} |f(n_1)| \tau(n_2) &\ll \frac{\log x}{(\log y_1)^2} \sum_{n_1 \leq x} \frac{|f(n_1)|}{\varphi(n_1)} \sum_{n_2 \leq x} \frac{1}{\varphi(n_2)} \ll \\ &\ll \left( \frac{\log x}{\log y_1} \right)^2 \frac{\log y}{\log x} \frac{1}{\log x} \prod(|f|, x) = o\left( \frac{1}{\log x} \prod(|f|, x) \right) \end{aligned}$$

if  $\frac{\log x}{\log y_1} \leq \sqrt[4]{\frac{\log y}{\log x}}$ . Substituting this result and (21) into (20) shows

$$\begin{aligned} &\frac{1}{\pi(x)} \sum_{p \leq x} f_y(p+1) = \\ &= \frac{1}{\pi(x)} \sum_{n_1 \leq y_1} f(n_1) \sum_{n_2 \leq \sqrt{\frac{x}{n_1}}} \left( 2I(x, n_1 n_2, y) - I(n_2 n_1 \sqrt{\frac{x}{n_1}}, n_1 n_2, y) \right) + \\ &\quad + o\left( \frac{1}{\log x} \prod(|f|, x) \right), \end{aligned}$$

where

$$I(x, n_1 n_2, y) = \left| \left\{ p \leq x, p+1 \equiv 0 \pmod{n_1 n_2}, \left( \frac{p+1}{n_1 n_2}, P(y) \right) = 1 \right\} \right|.$$

Now, let  $Q = Q(x)$  such that  $\frac{\log Q}{\log x} \rightarrow 0$  as  $x \rightarrow \infty$ . By Lemma 6 we have

$$\begin{aligned} &\frac{1}{\pi(x)} \sum_{n_1 \leq y_1} |f(n_1)| \sum_{n_2 \leq \sqrt{\frac{x}{n_1}}} I\left(n_2 n_1 \sqrt{\frac{x}{n_1}}, n_1 n_2, y\right) \leq \\ &\leq \frac{1}{\pi(x)} \sum_{n_1 \leq y_1} |f(n_1)| \sum_{n_2 \leq \sqrt{\frac{x}{n_1}}} \left( \left| \left\{ n : n \leq \sqrt{\frac{x}{n_1}}, (n, P(y)) = 1, \right. \right. \right. \\ &\quad \left. \left. \left. (nn_1 n_2 - 1, P(\sqrt[4]{x})) = 1 \right\} \right| + \sqrt[4]{x} \right) \ll \\ &\ll \frac{1}{\sqrt{x} \log y} \sum_{n_1 \leq y_1} \frac{|f(n_1)| \sqrt{n_1}}{\varphi(n_1)} \sum_{n_2 \leq \sqrt{\frac{x}{n_1}}} \frac{n_2}{\varphi(n_2)} \ll \end{aligned}$$

$$\ll \frac{\log y}{\log x} \frac{1}{\log x} \prod(|f|, x).$$

Therefore

$$\begin{aligned} \frac{1}{\pi(x)} \sum_{p \leq x} f_y(p+1) &= \frac{2}{\pi(x)} \sum_{n_1 \leq y_1} f(n_1) \sum_{n_2 \leq Q^{-1} \sqrt{\frac{x}{n_1}}} I(x, n_1 n_2, y) + \\ (23) \quad &+ O \left( \frac{1}{\pi(x)} \sum_{n_1 \leq y_1} |f(n_1)| \sum_{\sqrt{\frac{x}{n_1}} Q^{-1} \leq n_2 \leq \sqrt{\frac{x}{n_1}}} I(x, n_1 n_2, y) \right) + \\ &+ o \left( \frac{1}{\log x} \prod(|f|, x) \right). \end{aligned}$$

Using Lemma 6 shows that the second term may be estimated by

$$\begin{aligned} &\leq \frac{1}{\pi(x)} \sum_{n_1 \leq y_1} |f(n_1)| \left( \sum_{\sqrt{\frac{x}{n_1}} Q^{-1} \leq n_2 \leq \sqrt{\frac{x}{n_1}}} \left| \left\{ n : n \leq \frac{x}{n_1 n_2}, (n, P(y)) = 1, \right. \right. \right. \\ &\quad \left. \left. \left. (n n_1 n_2 - 1, P(\sqrt[8]{x})) = 1 \right\} \right| + \sqrt[8]{x} \right) \ll \\ &\ll \frac{1}{\log y} \sum_{n_1 \leq y_1} \frac{|f(n_1)|}{\varphi(n_1)} \sum_{\sqrt{\frac{x}{n_1}} Q^{-1} \leq n_2 \leq \sqrt{\frac{x}{n_1}}} \frac{1}{\varphi(n_2)}. \end{aligned}$$

We have

$$\sum_{n_2 \leq x} \frac{1}{\varphi(n_2)} = \sum_{d, (d, P(y))=1} \frac{\mu^2(d)}{d \varphi(d)} \sum_{n_2 \leq x/d} \frac{1}{n_2} = \sum_{n_2 \leq x} \frac{1}{n_2} + O \left( \frac{\log x}{y} \right).$$

Applying the Fundamental Lemma of sieve theory (see [6], Theorem 2.6) we conclude

$$\sum_{n_2 \leq u} 1 = u \prod_{p \leq y} \left( 1 - \frac{1}{p} \right) \left( 1 + O \left( \left( \frac{\log y}{\log u} \right)^2 \right) \right).$$

Then, by partial summation,

$$(24) \quad \sum_{n_2 \leq x} \frac{1}{\varphi(n_2)} = \prod_{p \leq y} \left( 1 - \frac{1}{p} \right) \log x \left( 1 + O \left( \left( \frac{\log y}{\log x} \right)^2 \right) \right).$$

This shows that the first remainder term on the right side in (23) can be estimated by

$$\begin{aligned} &\ll \frac{1}{\log y} \sum_{n_1 \leq y_1} \frac{|f(n_1)|}{\varphi(n_1)} \left( \frac{\log Q}{\log y} + \frac{\log y}{\log x} \right) \ll \\ &\ll \frac{\log y}{\log x} \left( \frac{\log Q}{\log y} + \frac{\log y}{\log x} \right) \frac{1}{\log x} \prod(|f|, x) = o\left( \frac{1}{\log x} \prod(|f|, x) \right). \end{aligned}$$

Hence

$$\begin{aligned} (25) \quad &\frac{1}{\pi(x)} \sum_{p \leq x} f_y(p+1) = \\ &= \frac{2}{\pi(x)} \sum_{n_1 \leq y_1} f(n_1) \sum_{n_2 \leq Q^{-1} \sqrt{\frac{x}{n_1}}} I(x, n_1 n_2, y) + o\left( \frac{1}{\log x} \prod(|f|, x) \right). \end{aligned}$$

Now, Selberg' sieve (see [6], Theorem 7.1) gives

$$\begin{aligned} I(x, n_1 n_2, y) &= \frac{\pi(x)}{\varphi(n_1 n_2)} \prod_{p \leq y} \left( 1 - \frac{\varphi(n_1 n_2)}{\varphi(n_1 n_2 p)} \right) \left( 1 + O\left( \exp\left( -\frac{\log \xi}{\log y} \right) \right) \right) + \\ &+ O\left( \sum_{\substack{d \leq \xi^2, \\ d|P(y)}} 3^{\omega(d)} \left| \pi(x, -1, n_1 n_2 d) - \frac{\text{Lix}}{\varphi(n_1 n_2 d)} \right| \right). \end{aligned}$$

Set  $\xi^2 = \frac{Q\sqrt{n_1}}{\log^A x}$ . Then  $n_1 n_2 \leq \frac{\sqrt{x}}{\log^A x}$ . Applying Cauchy's inequality leads to

$$\begin{aligned} &\sum_{n_1 \leq y_1} |f(n_1)| \sum_{n_2 \leq Q^{-1} \sqrt{\frac{x}{n_1}}} \sum_{\substack{d \leq \xi^2, \\ d|P(y)}} 3^{\omega(d)} \left| \pi(x, -1, n_1 n_2 d) - \frac{\text{Lix}}{\varphi(n_1 n_2 d)} \right| \ll \\ &\ll \left( \sum_{n \leq x} \frac{x}{n} \left( \sum_{n_1 d n_2 = n} |f(n_1)| 3^{\omega(d)} \right)^2 \right)^{\frac{1}{2}} \left( \sum_{d \leq \frac{\sqrt{x}}{\log^A x}} \left| \pi(x, -1, d) - \frac{\text{Lix}}{\varphi(d)} \right| \right)^{\frac{1}{2}}. \end{aligned}$$

The first sum is less than

$$\sum_{n \leq x} \frac{\tau_3(n)}{n} \sum_{n \leq x} \frac{1}{n} \sum_{n \leq x} \frac{|f(n)|^2}{n} \sum_{n \leq x} \frac{9^{\omega(n)}}{n} \ll (\log x) e_1^{13}.$$

The second sum can be estimated by the Vinogradov-Bombieri theorem and is dominated by  $O(x \log^{-(c_1^2+16)} x)$  if  $A \geq A_0$ . Recall that  $n_1$  is an even number. Therefore

$$\begin{aligned} \prod_{p \leq y} \left(1 - \frac{\varphi(n_1 n_2)}{\varphi(n_1 n_2 p)}\right) &= \frac{1}{2} \prod_{2 < p \leq y} \left(1 - \frac{1}{p-1}\right) \prod_{\substack{p|n_1, \\ p \neq 2}} \left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{p-1}\right)^{-1} = \\ &= 2 \frac{\varphi(n_1)}{n_1} \prod_{p \leq y} \left(1 - \frac{1}{p}\right) \prod_{2 < p \leq y} \left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{p-1}\right)^{-1} \prod_{\substack{p|n_1, \\ p \neq 2}} \left(1 + \frac{1}{p-2}\right). \end{aligned}$$

Substituting these results into (25) we arrive at

$$\begin{aligned} \frac{1}{\pi(x)} \sum_{p \leq x} f_y(p+1) &= 4 \prod_{p \neq 2} \left(1 - \frac{1}{p-1}\right) \left(1 - \frac{1}{p}\right)^{-1} \prod_{p \leq y} \left(1 - \frac{1}{p}\right) \sum_{n_1 \leq y_1} \frac{f(n_1)}{n_1} \times \\ &\quad \times \prod_{\substack{p|n_1, \\ p \neq 2}} \left(1 + \frac{1}{p-2}\right) \sum_{n_2 \leq \sqrt{\frac{x}{n_1}} Q^{-1}} \frac{1}{\varphi(n_2)} \times \\ &\quad \times \left(1 + O\left(\exp\left(-\frac{\log \xi}{\log y}\right)\right)\right) \left(1 + O\left(\frac{1}{y}\right)\right) + O\left(\frac{1}{\log x} \prod(|f|, x)\right). \end{aligned}$$

Recall that  $\frac{\log y_1}{\log x} \rightarrow 0$  and  $\frac{\log Q}{\log x} \rightarrow 0$  as  $x \rightarrow \infty$  and

$$\frac{\log \xi}{\log y} \geq \frac{\log Q}{\log y} - \frac{A \log \log x}{\log y}.$$

Let  $y \geq \exp(\sqrt{\log x})$  and  $\frac{\log Q}{\log y} \rightarrow \infty$  as  $x \rightarrow \infty$ . By (24) we see that

$$\begin{aligned} \frac{1}{\pi(x)} \sum_{p \leq x} f_y(p+1) &= \prod_{p \leq y} \left(1 - \frac{2}{p} + \left(1 - \frac{1}{p}\right) \sum_{r=1}^{\infty} \frac{f(p^r)}{p^r}\right) \log x + \\ &\quad + o\left(\frac{1}{\log x} \prod(|f|, x)\right). \end{aligned}$$

Choosing  $y \geq x^{\sqrt{\varepsilon(x)}}$  gives

$$\sum_{y < p \leq x} \frac{|f(p) - 2|}{p} \rightarrow 0$$

as  $x \rightarrow \infty$ , which, together with (19), proves the assertion of the Theorem.

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