Annales Univ. Sci. Budapest., Sect. Comp. 22 (2003) 395-402

HIGH ORDER MEAN–VALUE THEOREMS FOR MULTIPLICATIVE FUNCTIONS VIA HALÁSZ'S METHOD

Wen-Bin Zhang (Urbana, IL, USA)

Dedicated to Professor Karl-Heinz Indlekofer
on his 60-th birthday

1. Introduction

Mean-value theorems for multiplicative functions f on $\mathbb{N}$ via Halász's method are classical in probabilistic number theory [8, 7]. The method was first presented in [8]. These theorems give information about mean-values

$$m_f := \lim_{x\to\infty} x^{-1} \sum_{n\le x} f(n) \tag{1.1}$$

of functions f of which the generating function

$$\hat{F}(s) := \sum_{n=1}^{\infty} f(n) n^{-s}$$

satisfies

$$\hat{F}(s) = \frac{A}{s-1} + o\left(\frac{|s|}{\sigma - 1}\right), \quad s = \sigma + it \tag{1.2}$$

uniformly for $\sigma > 1$ and $-\infty < t < \infty$. A shortcoming of these theorems is that they do not convey information about the "higher order" mean-values of functions of which

$$\hat{F}(s) = \frac{A}{(s-1)^\tau} + o\left(\frac{|s|}{(\sigma-1)^\tau}\right) \tag{1.3}$$

with $\tau > 1$. A variety of multiplicative functions f satisfy condition (1.3), but not (1.2). A trivial example is the divisor function $f(n) = d(n)$, which satisfies the three conditions of Halász's general mean-value theorem and (1.3) with $\tau = 2$. In particular, the functions $\omega(d)$ in the theory of sieves of dimension $\kappa > 1$ are of this variety [9]. For these functions f, instead of the mean-value m_f defined by (1.1), one should consider the "higher order" mean-values

$$m_f := \lim_{x\to\infty} \frac{1}{x(\log x)^{\tau-1}} \sum_{n\le x} f(n), \tag{1.4}$$

which we call, when it exists, an order τ mean-value of the function f. Notice that, in classical theory, only order one ($\tau = 1$) mean-value has been discussed.

2. A general mean-value theorem

This lacuna is removed principally in [12]. Actually, a general mean-value theorem for multiplicative functions f defined on a set $\mathcal{N}$ of generalized integers n_j, associated with a set $\mathcal{P}$ of generalized primes p_j (henceforth, g-integers, g-primes, etc.) in Beurling's sense [1, 2], is proved as follows.

Let $f(n_j)$ be a complex-valued function on $\mathcal{N}$ and

$$F(x) := \sum_{n_j\le x} f(n_j).$$

Also, let $\Lambda(n_j)$ be the analog of the classical von Mongoldt function and

$$\psi(x) := \sum_{n_j\le x} \Lambda(n_j)$$

be the Chebyshev function associated with $\mathcal{P}$.

Theorem 1. (i) *Suppose there exist a constant c, real constants α and $\tau > 0$, and a measurable slowly oscillating function $L(u)$ with $|L(u)| = 1$ such that*

$$F(x) = \frac{cx^{1+i\alpha}(\log x)^{\tau-1}}{\Gamma(\tau)(1+i\alpha)} L(\log x) + o\left(x \log^{\tau-1} x\right), \tag{2.1}$$

where $\Gamma(x)$ is Euler's gamma function. Then the asymptotic formula

$$\hat{F}(s)=\frac{c}{(s-1-i\alpha)^{\tau}}L\left(\frac{1}{\sigma-1}\right)+o\left(\frac{|s|}{(\sigma-1)^{\tau}}\right) \tag{2.2}$$

holds as $\sigma\to 1+$ uniformly for $-\infty<t<\infty$.

(ii) *Conversely, suppose*

(1) There exist positive constants δ and K_δ such that

$$\limsup_{\sigma\to 1+}(\sigma-1)\sum_{p_j}\max\left\{1,|f(p_j)|^{1+\delta}\right\}p_j^{-\sigma}\log p_j=K_\delta;$$

(2)

$$\sum_{p_j}\sum_{k\geq 2}|f(p_j^k)|p_j^{-k}<\infty;$$

and

(3)

$$1+\sum_{k=1}^{\infty}f(p_j^k)p_j^{-k(1+it)}\neq 0$$

for every $p_j\in\mathcal{P}$ and $-\infty<t<\infty$.

Furthermore, suppose that the counting function $N(x)$ of g-integers satisfies

$$N(x)=x\sum_{r=1}^{m}A_r(\log x)^{\rho_r-1}+O\left(x(\log x)^{-\gamma}\right) \tag{2.3}$$

with real constants $\rho_1<\rho_2<\ldots<\rho_m$ and $A_1,A_2,\ldots,A_m$ such that $\rho_m=\rho\geq$ $\geq 1,\ \rho_r\neq 0,\ A_m=A>0$ and real constants $\gamma>\gamma_0$. Also, suppose that

$$\psi(x)=\left(\rho-2\sum_{j=1}^{l}\alpha_j\cos(t_j\log x)\right)x+O\left(x(\log x)^{-M}\right) \tag{2.4}$$

holds with positive integers α_j, real constants $t_j,\ j=1,\ldots,l$, and constants $M>M_0$. Here constants γ_0 and M_0 depend on ρ,δ,K_δ and τ only. Then (2.2) with $\tau\geq 1$ entails (2.1).

This theorem has the following corollary, which is a direct extension of Halász's general mean-value theorem [8].

Corollary 1. *Suppose that*

$$N(x) = Ax + O_k\left(x(\log x)^{-k}\right) \tag{2.5}$$

for every $k \in \mathbb{N}$. If f satisfies conditions (1), (2) and (3) in Theorem 1 then (2.2) with $\tau \geq 1$ entails (2.1).

3. High order mean-value theorems

On the basis of Theorem 1, one can characterize further the asymptotic behavior of the order τ mean-values of multiplicative functions with $\tau \geq 1$ [13]. These theorems assume conditions less restrictive than those of their counterparts in classical theory [10, 7] in some sense. In the following context, conditions (1), (2) and (3) of Theorem 1 are quoted as conditions (1), (2) and (3) without indicating Theorem 1 repeatedly. Also, p is used to denote the general g-prime p_j for convenience. This will not cause any confusion.

We first consider mean-value $m_f = 0$. Let

$$\log^+ |x| := \max\{0, \log |x|\}.$$

Theorem 2. *Suppose that (2.3) and (2.4) are satisfied and that f satisfies conditions (1) and (2).*

If

$$\sum_p p^{-1}\left(\frac{\tau}{\rho} - \Re\left(f(p)p^{-it}\right)\right) + \log^+ |t| \tag{3.1}$$

diverges to $+\infty$ uniformly for $-\infty < t < \infty$ then the order τ mean-value $m_f = 0$.

Conversely, suppose further that there exist a subset $\mathcal{P}_0$ of the set $\mathcal{P}$ of all g-primes and a constant $K > 0$ such that

$$\sum_{\substack{p \in \mathcal{P}_0 \\ p \leq x}} p^{-1}\left(\frac{\tau}{\rho} - \Re\left(f(p)p^{-it}\right)\right) \geq -K$$

uniformly for $0 \leq x < \infty$ and $-\infty < t < \infty$ and such that

$$\sum_{\substack{p \notin \mathcal{P}_0 \\ p \leq x}} |f(p)| \ll \frac{x}{\log x}.$$

If the order τ mean-value $m_f = 0$ then either there exist a real number t_0 and a g-prime p_0 such that

$$1 + \sum_{k=1}^{\infty} f(p_0^k) p_0^{-k(1+it_0)} = 0$$

or (3.1) diverges to$+\infty$ uniformly for $-\infty < t < \infty$.

Theorem 2 is a direct extension of the classical Halász-Wirsing theorem [6, 11].

The theorems on nonzero mean-values are more complicated. For simplicity, only the results under the condition (2.5) are presented here as follows.

Theorem 3. *Suppose that (2.5) is satisfied. Let f be a multiplicative function satisfying condition (2) and*

$$1 + \sum_{k=1}^{\infty} f(p^k) p^{-k} \neq 0 \tag{3.2}$$

for all $p \in \mathcal{P}$. Suppose further that there exist a positive constant η such that both series

$$\sum_{|f(p)| \leq \frac{\tau}{\rho} + \eta} p^{-1} \left| f(p) - \frac{\tau}{\rho} \right|^2 \tag{3.3}$$

and series

$$\sum_{|f(p)| > \frac{\tau}{\rho} + \eta} |f(p)| p^{-1} \tag{3.4}$$

converge. Then the order τ mean-value m_f exists for $\tau \geq 1$ and

$$m_f = \frac{(A)^\tau}{\Gamma(\tau)} \prod_p \left(1 - p^{-1}\right)^\tau \left(1 + \sum_{k=1}^{\infty} f(p^k) p^{-k}\right) \neq 0 \tag{3.5}$$

if and only if

$$\sum_p p^{-1} \left(\frac{\tau}{\rho} - f(p)\right) \tag{3.6}$$

converges.

Theorem 3 is a direct extension of the classical Delange-Halász theorem [4, 6, 8].

As an example, let $f(n) = (d(n))^2$, where $d(n)$ is the divisor function of $n \in \mathbb{N}$. The condition (2) and (3.2) are satisfied. Also, (3.6), (3.3) and (3.4) are satisfied with $\rho = 1,\ \tau = 4$ and $\eta = 1$ (the Riemann zeta function has order $\rho = 1$ and $A = 1$). Therefore the order 4 mean-value

$$m_f = \frac{(\Gamma(1))^2}{\Gamma(4)} \prod_p (1-p^{-1})^4 \left(1 + \sum_{k=1}^{\infty} (k+1)^2 p^{-k}\right) = \\ = \frac{1}{6} \prod_p (1-p^{-2}) = \frac{1}{\pi^2},$$

i.e.

$$\lim_{x\to\infty} \frac{1}{x(\log x)^3} \sum_{n\le x} (d(n))^2 = \frac{1}{\pi^2}.$$

This is well-known is elementary number theory.

4. A generalization of theorems of Elliott and Daboussi

The well-known theorems of Elliott [5] and Daboussi [3] can be extended to functions having high order mean-values [14].

Theorem 4. *Suppose that (2.5) is satisfied. Let f be a multiplicative function. Then the order τ mean-value m_f exists and is nonzero for $\tau \ge 1$ and the limit*

$$\lim_{x\to\infty} \frac{1}{x(\log x)^{\tau^\alpha - 1}} \sum_{n_j \le x} |f(n_j)|^\alpha$$

exists with some constant $\alpha > 1$ if and only if the series

$$\sum_p p^{-1}(\tau - f(p)), \qquad \sum_{|f(p)|\le \tau+\eta} p^{-1}|\tau - f(p)|^2,$$

$$\sum_{|f(p)|>\tau+\eta} p^{-1}|f(p)|^\alpha, \quad \sum_p \sum_{k\ge 2} p^{-k}|f(p^k)|^\alpha$$

converge with some constant $\eta > 0$ *and*

$$\sum_{k=0}^{\infty} p^{-k} f(p^k) \neq 0$$

for all $p \in \mathcal{P}$.

In case $\tau = 1$, Theorem 4 is the well-known theorem of Elliott and Daboussi.

The dual of the Turán-Kubilius inequality plays a key role in proofs of the necessity part of the Elliott-Daboussi theorem [7]. However, in case of higher order ($\tau > 1$) mean-values, the dual inequality fails. This is shown by the trivial example $f(n) = d^2(n)$. Hence the proof of the necessity part of Theorem 4 is based on an intrinsic connection between the order τ mean-value m_f and the order $\tau^\alpha - 1$ mean-value of $|f(n_j)|^\alpha$.

References

[1] **Bateman P.T. and Diamond H.G.,** Asymptotic distribution of Beurling's generalized prime numbers, *Studies in Number Theory, Vol. 6,* Math. Assoc, Amer., Prentice-Hall, Englewood Cliffs, N.J., 1969. 152-210.

[2] **Beurling A.,** Analyse de la loi asymptotique de la distribution des nombres premiers généralisés I., *Acta Math.,* **68** (1937), 255-291.

[3] **Daboussi H.,** Sur les fonctions multiplicatives ayant une valeur moyenne non nulle, *Bull. Soc. Math. France,* **109** (1981), 183-205.

[4] **Delange H.,** Sur les fonctions arithmétique multiplicative, *Ann. Scient. Éc. Norm. Sup.* 3^e^ *série,* **78** (1961), 273-304.

[5] **Elliott P.D.T.A.,** Mean value theorems for multiplicative functions bounded in mean α-power, $\alpha > 1$, *J. Australian Math. Soc. (Series A),* **29** (1980), 177-205.

[6] **Elliott P.D.T.A.,** *Probabilistic number theory I. Mean Value Theorems,* Grund. der Math. Wiss. **239**, Springer, 1979.

[7] **Elliott P.D.T.A.,** *Duality in analytic number theory,* Cambridge Tracts in Math. **122**, Cambridge Univ. Press, 1997.

[8] **Halász G.,** Über die Mittewerte multiplikativer zahlentheoretischer Funktionen, *Acta Math. Acad. Sci. Hung.,* **19** (1968), 365-403.

[9] **Halberstam H. and Richert H.-E.,** *Sieve methods*, Academic Press, London, New York, San Francisco, 1974.

[10] **Mauclaire J.-L.,** *Intégration et théorie des nombres,* Herman, Paris, 1986.

[11] **Wirsing E.,** Das asymptotische Verhalten von Summen über multiplikative Funtionen II., *Acta Math. Acad. Sci. Hung.,* **18** (1967), 411-467.

[12] **Zhang W.-B.,** Mean-value theorems for multiplicative functions on Beurling's generalized integers via Halász's method (I), 2002. (submitted)

[13] **Zhang W.-B.,** Mean-value theorems for multiplicative functions on Beurling's generalized integers via Halász method (II), 2002. (submitted)

[14] **Zhang W.-B.,** Mean-value theorems for multiplicative functions on Beurling's generalized integers via Halász's method (III), 2002. (to be submitted)

Wen-Bin Zhang
Department of Mathematics
University of Illinois
1409 West Green Street
Urbana, Illinois 61801, USA

Department of Mathematics
University of the West Indies
Mona, Kinston, Jamaica