THE MAXIMAL ORDER OF A CLASS OF MULTIPLICATIVE ARITHMETICAL FUNCTIONS

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Dedicated to Professor Karl-Heinz Indlekofer
on his sixtieth birthday

Abstract. We prove simple theorems concerning the maximal order of a large class of multiplicative functions. As an application, we determine the maximal orders of certain functions of the type $\sigma_A(n) = \sum_{d \in A(n)} d$, where $A(n)$ is a subset of the set of all positive divisors of $n$, including the divisor-sum function $\sigma(n)$ and its unitary and exponential analogues. We also give the minimal order of a new class of Euler-type functions, including the Euler-function $\phi(n)$ and its unitary analogue.

1. Introduction

Let $\sigma(n)$ and $\phi(n)$ denote, as usual, the sum of all positive divisors of $n$ and the Euler function, respectively. It is well-known, that

\[
\limsup_{n \to \infty} \frac{\sigma(n)}{n \log \log n} = e^\gamma,
\]

\[
\liminf_{n \to \infty} \frac{\phi(n) \log \log n}{n} = e^{-\gamma},
\]

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where $\gamma$ is Euler’s constant. These results go back to the work of T.H. Gronwall [5] and E. Landau [7] and have been established for a number of modified $\sigma$- and $\phi$-functions.

One such modification relates to unitary divisors $d$ of $n$, notation $d \| n$, meaning that $d | n$ and $(d, n/d) = 1$. The corresponding $\sigma$- and $\phi$-functions are defined by $\sigma^*(n) = \sum d$ and $\phi^*(n) = \# \{1 \leq k \leq n; (k, n)_* = 1\}$, where $(k, n)_*$ denotes the largest divisor of $k$ which is a unitary divisor of $n$. These functions are multiplicative and for prime powers $p^\nu$ given by $\sigma^*(p^\nu) = p^{\nu+1}$, $\phi^*(p^\nu) = p^\nu - 1$, see [3, 8]. They are treated, along with other multiplicative functions, in [2] with the result that

$$
\limsup_{n \to \infty} \frac{\sigma^*(n)}{n \log \log n} = \frac{6}{\pi^2} e^\gamma,
$$

while $\phi^*$ gives again (2). (Actually (3) is written incorrectly in [2] with the factor $6/\pi^2$ missing.)

In [4] it is shown that (3) holds also for $\sigma^{(e)}(n)$, the sum of exponential divisors of $n$. (A number $d = \prod p^{\delta_p}$ is called an exponential divisor of $n = \prod p^{\nu_p}$ if $\delta_p | \nu_p$ for all $p$.)

These and a number of similar results from literature refer to rather special functions. Textbooks dealing with the extremal order of arithmetic functions also treat only particular cases, see [6, 1, 11]. It should be mentioned that a useful result concerning the maximal order of a class of prime-independent functions, including the number of all divisors, unitary divisors and exponential divisors, is proved in [10].

In the present paper we develop easily applicable theorems for determining

$$
L = L(f) := \limsup_{n \to \infty} \frac{f(n)}{\log \log n},
$$

where $f$ are nonnegative real-valued multiplicative functions. Essential parameters are

$$
\rho(p) = \rho(f, p) := \sup_{\nu \geq 0} f(p^\nu)
$$

for the primes $p$, and the product

$$
R = R(f) := \prod_p \left(1 - \frac{1}{p}\right) \rho(p).
$$

These theorems can, in particular, be used to obtain the maximal or minimal order, respectively, of generalized $\sigma$- and $\phi$-functions which arise in connection with Narkiewicz-convolutions of arithmetic functions.
2. General results

We formulate the conditions for lower and upper estimates for $L$ separately. Note that $\rho(p) \geq f(p^0) = 1$ for all $p$.

**Theorem 1.** Suppose that $\rho(p) < \infty$ for all primes $p$ and that the product $R$ converges unconditionally (i.e. irrespectively of order), improper limits being allowed, then

$$L \leq e^\gamma R.$$  

A different assumption uses

**Theorem 2.** Suppose that $\rho(p) < \infty$ for all $p$ and that the product $R$ converges, improper limits being allowed, and that

$$\rho(p) \leq 1 + o\left(\frac{\log p}{p}\right),$$

then (4) holds.

**Remark.** Neither does condition (5) plus convergence of $R$ imply unconditional convergence of $R$ nor vice versa.

To establish $e^\gamma R$ also as the lower limit more information is required: The suprema $\rho(p)$ must be sufficiently well approximated at not too large powers of $p$.

**Theorem 3.** Suppose that $\rho(p) < \infty$ for all primes $p$, that for each prime $p$ there is an exponent $e_p = p^{a(p)} \in \mathbb{N}$ such that

$$\prod_p f(p^{e_p}) \rho(p)^{-1} > 0,$$

and that the product $R$ converges, improper limits being allowed. Then

$$L \geq e^\gamma R.$$  

**Corollary 1.** If for all $p$ we have $\rho(p) \leq (1 - 1/p)^{-1}$ and there are $e_p$ such that $f(p^{e_p}) \geq 1 + 1/p$, then

$$L = e^\gamma R.$$  

*In other words: The maximal order of $f(n)$ is $e^\gamma R \log \log n$.*
Formally $R$ becomes infinite if there is a nonempty set $S$ of primes for which $\rho(p) = \infty$. So one might expect that the assumptions of Theorem 3 taken for all $p$ with finite $\rho(p)$ would imply $L = \infty$. Surprisingly enough this is true only for rather thin sets $S$. But note that for $p \in S$ there is no substitute for the $f(p^r)$ approximating $\rho(p)$.

We begin by stating what the above theorems imply if one ignores the numbers with prime factors from a given set $S$ of primes. For any such set define

\[ N(S) := \{ n : n \in \mathbb{N}, p|n \Rightarrow p \in S \}, \quad C(S) := \{ n : n \in \mathbb{N}, p|n \Rightarrow p \not\in S \}. \]

**Corollary 2.** Modify the assumptions of Theorems 1, 2 and 3 by replacing $R$ with $R_S = R_S(f) := \prod_{p \not\in S} \left( 1 - \frac{1}{p} \right) \rho(p)$, $L$ with $L_S = L_S(f) := \limsup_{n \to \infty, n \in C(S)} \frac{f(n)}{\log \log n}$, condition (5) with

\[ (7) \quad \rho(p) \leq 1 + o\left( \frac{\log p}{p} \right) \quad \text{for} \quad p \not\in S, \]

and (6) with

\[ (8) \quad \prod_{p \not\in S} f(p^e) \rho(p)^{-1} > 0. \]

Assume further that

\[ \sum_{p \in S} \frac{1}{p} < \infty. \]

Then

\[ L_S \leq e^\gamma \prod_{p \in S} \left( 1 - \frac{1}{p} \right) \cdot R_S, \quad L_S \geq e^\gamma \prod_{p \in S} \left( 1 - \frac{1}{p} \right) \cdot R_S, \]

respectively. This applies even if $\rho(p) = \infty$ for some or all of the $p \in S$.

**Theorem 4.** Let $S$ be a set of primes such that

\[ (9) \quad \sum_{p \in S} \frac{1}{p} < \infty. \]
If \( \rho(p) = \infty \) exactly for the \( p \in S \), if (8) holds and \( R_S > 0 \), then \( L = \infty \). Condition (9) must not be waived.

In fact there are counter-examples for any set \( S \) for which \( \sum 1/p \) diverges.

3. The proofs

**Proof of Theorem 1.** An arbitrary \( n = \prod p^{\nu_p} \) we write as \( n = n_1n_2 \) with \( n_1 := \prod_{p \leq \log n} p^{\nu_p} \). Mertens’s formula \( \prod_{p \leq x} (1 - 1/p)^{-1} \sim e^\gamma \log x \) and the definition of \( \rho(p) \) imply

\[
f(n_1) = \prod_{p \leq \log n} f(p^{\nu_p}) \leq \prod_{p \leq \log n} \rho(p) = \prod_{p \leq \log n} \left(1 - \frac{1}{p}\right)^{-1} \cdot \prod_{p \leq \log n} \left(1 - \frac{1}{p}\right) \rho(p),
\]

(10)

\[
f(n_1) \leq (1 + o(1)) e^{\gamma R \log \log n} \text{ as } n \to \infty.
\]

Let \( a \) denote the number of prime divisors in \( n_2 \). Then \( a \leq \log n / \log \log n \). There is nothing to prove if \( R = \infty \), so let \( R < \infty \). Using the unconditional convergence

\[
f(n_2) \leq \prod_{p \mid n, \ p > \log n} \left(1 - \frac{1}{p}\right) \rho(p) \cdot \prod_{p \mid n, \ p > \log n} \left(1 - \frac{1}{p}\right)^{-1} \leq \]

(11)

\[
\leq (1 + o(1)) \cdot \left(1 - \frac{1}{\log n}\right)^{-a} = (1 + o(1)) e^{O(1/\log \log n)} \to 1.
\]

Combining (10) and (11) finishes the proof.

**Proof of Theorem 2.** There is no change in the estimation of \( f(n_1) \). For \( n_2 \) we have

\[
f(n_2) \leq \left(1 + o \left(\frac{\log \log n}{\log n}\right)\right)^{\frac{\log n}{\log \log n}} = 1 + o(1).
\]
Proof of Theorem 3. We treat the case of proper convergence only. There is nothing to prove if \( R = 0 \) and the changes for \( R = \infty \) are obvious. For given \( \varepsilon \) take \( P \) so large that

\[
\prod_{p > P} f(p^\alpha) \rho(p)^{-1} \geq 1 - \varepsilon
\]

and choose exponents \( k_p \) for the \( p \leq P \) such that

\[
\prod_{p \leq P} f(p^{k_p}) \geq (1 - \varepsilon) \prod_{p \leq P} \rho(p).
\]

Keeping \( P \) and the \( k_p \) fixed let \( x \) tend to infinity and consider

\[
n(x) := \prod_{p \leq P} \rho^{k_p} \prod_{P < p \leq x} p^{\alpha_p}.
\]

Now on the one hand, using (12) and (13), we see

\[
f(n(x)) \prod_{p \leq x} \left( 1 - \frac{1}{p} \right) \geq (1 - \varepsilon) \prod_{p \leq x} \left( 1 - \frac{1}{p} \right) \rho(p) \cdot \prod_{P < p \leq x} f(p^\alpha) \rho(p)^{-1} \geq
\]

\[
\geq (1 - \varepsilon)^2 (1 + o(1)) R
\]

and with Mertens’s formula again

\[
f(n(x)) \geq (1 - \varepsilon)^2 (1 + o(1)) Re^\gamma \log x.
\]

On the other hand, since \( e_p = p^{\rho(1)} \), we have

\[
\log n(x) \leq \sum_{p \leq P} k_p \log p + \sum_{P < p \leq x} e_p \log p \leq (1 + o(1)) \sum_{p \leq x} \log p = x^{1 + o(1)},
\]

and therefore

\[
\log \log n(x) \leq (1 + o(1)) \log x.
\]

Together with (14) this yields the lower bound

\[
\limsup_{x \to \infty} \frac{f(n(x))}{\log \log n(x)} \geq (1 - \varepsilon)^2 Re^\gamma
\]

with arbitrary \( \varepsilon > 0 \).

Proof of Corollary 1. Apply Theorems 1 (or 2) and 3.
Proof of Corollary 2. To see this one applies the theorems to the multiplicative function $f^*$ defined by $f^*(n) = f(n)$ for $n \in C(S)$ and $f(n) = 1$ for $n \in N(S)$. One finds

$$L(f^*) = L_S(f), \quad R(f^*) = R_S(f) \prod_{p \in S} \left(1 - \frac{1}{p}\right),$$

and (8) implies (6) for $f^*$ because $\prod_{p \in S} (1 - 1/p)$ converges absolutely.

Note also that for any sequence of numbers $n = n_1 n_2$ tending to $\infty$, where $n_1 \in N(S)$, $n_2 \in C(S)$, we have $f^*(n)/\log \log n = f(n_2)/\log \log(n_1 n_2)$, hence $\limsup f^*(n)/\log \log n = 0$ if $n_2$ stays bounded, and

$$\leq \limsup_{n_2} f(n_2)/\log \log n_2$$

otherwise, with equality if $n_1$ is bounded. Thus $L(f^*) = L_S$.

Proof of Theorem 4. I. Assume (9). With any $n_1 \in N(S)$ we have

$$L \geq \limsup_{n_2 \in C(S)} \frac{f(n_1) f(n_2)}{\log \log(n_1 n_2)} = f(n_1) L_S.$$ 

From Corollary 2, as it refers to Theorem 3, we have $L_S > 0$ and $f(n_1)$ can be chosen arbitrarily large.

II. Assume that (9) does not hold. We shall construct a counter-example. The assumption implies that

$$g(x) := \prod_{p \in S, \ p \leq x} \left(1 + \frac{1}{p}\right)$$

tends to $\infty$ as $x \to \infty$. Choose an increasing sequence of numbers $q_j = p_j^{\nu_j}$ with $p_j \in S$ and $\nu_j$ so large that $g(\log q_j) \geq j^j$ for all $j$, and such that every prime $p \in S$ occurs infinitely often in the sequence of the $p_j$. Put $f(q_j) = j$ for all $j \in \mathbb{N}$ and $f(p^{\nu}) = 1 + 1/p$ for all $p^{\nu}$ that are not among the $q_j$. Then, obviously, $\rho(p) = \infty$ for $p \in S$ and $\rho(p) = 1 + 1/p$ for $p \notin S$. The product

$$R_S = \prod_{p \in S} \left(1 - \frac{1}{p}\right) \left(1 + \frac{1}{p}\right)$$

converges absolutely and so does (choosing $\epsilon_p = 1$) $\prod_{p \in S} f(p^{\nu})/\rho(p) = 1$. Any $n \in \mathbb{N}$ can be written as $n = n_1 n_2$, where $n_1$ collects from the canonical
representation of \( n \) those prime powers that occur among the \( q_j \), while the rest compose \( n_2 \). For given \( n \) let 
\[ k := \max\{j; q_j \mid n_1\}. \]
Then \( f(n_1) \leq k! = o(k^k) = o(g(\log q_k)) = o(g(\log n)) \) by construction. Now for any \( n \in \mathbb{N} \)
\[ f(n) = f(n_1)f(n_2) = \prod_{p|n, \ p \notin S} \left(1 + \frac{1}{p}\right) \cdot o(g(\log n)) = \]
\[ = o \left( \prod_{p \leq \log n} \left(1 + \frac{1}{p}\right) \right) \cdot \prod_{p|n, \ p \geq \log n} \left(1 + \frac{1}{p}\right) \leq \]
\[ \leq o(\log \log n) \cdot \left(1 + \frac{1}{\log n}\right) = o(\log \log n), \]
hence \( L = 0 \).

4. Applications

A general frame for generalizations of the \( \sigma \)- and \( \phi \)-functions mentioned in the introduction can be found in Narkiewicz [9]. Assume that for each \( n \) a set \( A(n) \) of divisors of \( n \) is given and consider the \( A \)-convolution defined by

\[ (f \ast_A g)(n) := \sum_{d \in A(n)} f(d)g \left( \frac{n}{d} \right). \]

Properties of convolution (15) and of arithmetical functions related to it have been studied extensively in the literature, see [9, 8]. The system \( A \) is called multiplicative if \( A(n_1n_2) = A(n_1)A(n_2) \) for coprime \( n_1, n_2 \), with elementwise multiplication of the sets, and not all \( A(n) \) empty. Such a divisor system can be described by the sets \( AE_p(\nu) \) of admissible exponents,

\[ AE_p(\nu) := \{\delta; \ p^\delta \in A(p^\nu)\}. \]

The \( A \)-convolution of any two multiplicative functions \( f \) and \( g \) is multiplicative if and only if \( A \) is multiplicative. In particular multiplicativity of \( A \) implies multiplicativity of the modified divisor function

\[ \sigma_A(n) := \sum_{d \in A(n)} d. \]
As natural means to define an Euler-function attached to $A$ we consider the relation

$$\sum_{d \in A(n)} \phi_A(d) = n, \quad n \geq 1.$$  

This need not be solvable; there is, however, the following

**Theorem 5.** If the divisor system $A$ is multiplicative then (16) has a solution if and only if $n \in A(n)$ for all $n \in \mathbb{N}$. In this case the solution $\phi_A$ is unique and is a multiplicative function with $1 \leq \phi_A(n) \leq n$ for all $n \in \mathbb{N}$.

**Proof.** Suppose a solution exists. Then by induction on $\nu$ the recursion

$$\sum_{\delta \in AE_p(\nu)} \phi_A(p^\nu) = p^\nu$$

implies that $1 \leq \phi_A(p^\nu) \leq p^\nu$ and (therefore) $\nu \in AE_p(\nu)$ for all $\nu : p^\nu \in A(p^\nu)$. It follows from the multiplicativity of $A$ that $n \in A(n)$ for all $n$. If, on the other hand, $n \in A(n)$ for all $n$, then (17) can be solved recursively and the multiplicative function defined from the $\phi_A(p^\nu)$ solves (16). This is in fact the only solution since $\phi_A(n) = n - \sum_{d \in A(n) \setminus \{n\}} \phi_A(d)$.

With suitable additional conditions on $A$ we give the maximal and minimal orders of $\sigma_A$ and $\phi_A$, respectively. Extremal orders of such functions have not been investigated in the literature.

Obviously $\sigma_A(n) \leq \sigma(n)$ and if for any $\nu$ we have $p^\nu, p^\nu-1 \in A(p^\nu)$ then $\sigma_A(p^\nu) \geq p^\nu + p^\nu-1$. So Corollary 1 applies to $f(n) = \sigma_A(n)/n$ and gives

**Theorem 6.** Let the system $A$ of divisors be multiplicative and suppose that for each prime $p$ there is an exponent $e_p$ such that

$$p^{e_p}, \; p^{e_p-1} \in A(p^{e_p})$$

and $e_p = p^{o(1)}$. Then

$$\limsup_{n \to \infty} \frac{\sigma_A(n)}{n \log \log n} = e^\gamma \prod_p \left(1 - \frac{1}{p}\right) \sup_{\nu \geq 0} \frac{\sigma_A(p^\nu)}{p^\nu},$$

where the product converges.

**Remarks.** The quotients $\sigma(p^\nu)/p^\nu$ are of the form $\sum \varepsilon_i p^{-1}$, $\varepsilon_i \in \{0,1\}$, and the set of such numbers is compact. Therefore each $\sup_{\nu} (\sigma_A(p^\nu) p^{-\nu})$ is
itself of this form and we have for each prime $p$ a finite or infinite sequence of exponents $a_i$ such that $2 \leq a_1 < a_2 < \ldots$ and

$$
\left(1 - \frac{1}{p}\right) \sup_\nu \sigma_A(p^\nu)/p^\nu = 1 - \frac{1}{p^{a_1}} + \frac{1}{p^{a_2}} - \frac{1}{p^{a_3}} + \ldots.
$$

The formulae (1) and (3) are obvious consequences of Theorem 6. In the standard case $e_p$ is arbitrary, we have $(1 - 1/p)\rho(p) = 1$ for all $p$, hence $R = 1$. With unitary and exponential divisors the only admissible choices are $e_p = 1$ and $e_p = 2$, respectively, and $(1-1/p)\rho(p) = 1-1/p^2$, hence $R = \zeta(2)^{-1} = 6/\pi^2$ in both cases.

We turn to $\phi_A$, assuming again that $A$ is multiplicative and, in view of Theorem 5, that always $\nu \in AE_p(e)$ for some $e = e_p \geq 1$. In order to determine the minimal order of $\phi_A$ consider the function $f(n) := \phi_A(n)/n$. For all $p$ and $\nu \geq 1$ we have

$$
\phi_A(p^\nu) \geq p^\nu - \phi_A(p^{\nu-1}) - \ldots - \phi_A(1) \geq p^\nu - p^{\nu-1} - \ldots - 1,
$$

which gives

$$
f(p^\nu) < p^{\nu-1} \rho(p) \leq p^{\nu-1}.
$$

Note that $\rho(2)$ may equal $\infty$. If moreover $e - 1 \in AE_p(e)$ for some $e = e_p \geq 1$ then, on the other hand, $\phi_A(p^\nu) \leq p^\nu - \phi_A(p^{\nu-1}) \leq p^\nu - p^{\nu-1} + p^{\nu-2} + \ldots + 1$ if $e \geq 2$, and $\phi_A(n) \leq p - 1$ if $e = 1$. Therefore

$$
f(p^\nu) \geq \frac{p(p-1)}{p^2 - 2p + 2},
$$

$$
f(p^\nu)\rho(p)^{-1} \geq \frac{p(p-2)}{p^2 - 2p + 2} = 1 - \frac{2}{p^2 - 2p + 2},
$$

which is positive and yields a convergent product for $p \geq 3$.

Note that for powers of 2 there is no non-trivial lower estimate for $\phi_A(n)/n$ without further conditions on $A$. This is shown by the following example. Let

$$
\mathcal{N} = \{n_1, n_2, \ldots \} \subset \mathbb{N}, \ n_1 < n_2 < \ldots,
$$

and put $AE_2(n) := \{0, 1, \ldots, n\}$ for $n \in \mathcal{N}$ and $AE_2(n) := \{n\}$ for $n \not\in \mathcal{N}$. Then the recursion gives $\phi_A(2^n) = 2^n$ for $n \not\in \mathcal{N}$ but $\phi_A(2^{n_0}) = 2^{n_0-1}$ for the $n \in \mathcal{N}$, where $n_0 = 0$. Hence it is possible to have $\rho(2) = \sup_\nu \phi_A(2^n)/2^n = \sup_\nu (n_j - n_{j-1}) = \infty$.

Thus applying Corollary 1 or Theorem 4 with $S = \{2\}$ we obtain...
Theorem 7. Let $A$ be multiplicative and $n \in A(n)$ for all $n$. Assume that for each prime $p > 2$ there is an exponent $e_p$ such that $p^{e_p-1} \in A(p^{e_p})$ and $e_p = p^{o(1)}$. Then

$$\liminf_{n \to \infty} \frac{\phi_A(n) \log \log n}{n} = e^{-\gamma} \prod_p \left(1 - \frac{1}{p}\right)^{-1} \inf_{\nu} \frac{\phi_A(p^\nu)}{p^\nu}.$$  

The product converges for $p > 2$; the first factor may vanish.

For the standard Euler function $\phi(n)$ and for its unitary analogue $\phi^*(n)$ we regain (2).

For the system of exponential divisors one has $\phi_A(1) = 1$ because of multiplicativity. The recursion $\sum_{\kappa|\nu} \phi_A(p^\kappa) = p^\nu$ is solved by $\phi_A(p^\nu) = \sum_{\kappa|\nu} \mu(\nu/\kappa)p^\kappa$. Again the minimum of $\phi_A(p^\nu)/p^\nu$ is $1 - 1/p$, it is taken for $\nu = e_p = 2$ and once more (2) follows.

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