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ON THE MEAN VALUES OF MULTIPLICATIVE FUNCTIONS OVER RATIONAL NUMBERS

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Abstract. Two mean value theorems for the multiplicative functions of rational arguments are proved. One of them may be viewed as an analogue of the Delange's theorem well-known in the theory of arithmetical functions.

Definitions and results

A complex valued function $f(m/n)$, defined on the set of positive rational numbers, is called multiplicative, if for all irreducible fractions m/n

$$f\left(\frac{m}{n}\right) = f(m) \cdot f\left(\frac{1}{n}\right)$$

holds with $g(n) = f(n)$, $h(n) = f(1/n)$ being arithmetical multiplicative functions. Using the unique expression of m/n as a product of powers of primes (positive and negative ones) we may write

$$f\left(\frac{m}{n}\right) = \prod_{p^\nu || \frac{m}{n}} f(p^\nu);$$

here $p^\nu || \frac{m}{n}$ for $\nu > 0$ means that $p^\nu || m$ and for $\nu < 0$ it means that $p^{-\nu} || n$.

Let for $x \geq 1$ $\alpha = \alpha_x$, $\beta = \beta_x$ and $0 < \alpha < \beta$. With the system of intervals

$$I_x = (\alpha, \beta), \quad 0 < \alpha < \beta, \quad x \geq 1,$$

we define the following subsets of rational numbers

$$\mathcal{F}_x = \left\{ \frac{m}{n} \in I_x : n \le x \right\};$$

here and in the following all fractions are supposed to be irreducible.

We are interested in the asymptotical behavior of the sums

$$S(f,x) = \sum_{\frac{m}{n} \in \mathcal{F}_x} f\left(\frac{m}{n}\right),$$

where f is a multiplicative function.

This problem was considered in particular cases in the papers [4], [7]. It is our aim to prove in this work the following analogue of the Delange's mean value theorem for the multiplicative functions defined for rational numbers.

Theorem 1. *Let with some* $\xi \in (0,1)$ *and* $\zeta > 0$ *for intervals* $I_x = (\alpha, \beta)$ *the following condition*

$$\max\{x^{-\xi}, \alpha/x\} \ll \beta - \alpha \ll x^{\zeta} \quad (0 < \alpha < \beta) \tag{1}$$

be satisfied. If for a complex valued multiplicative function $f(m/n)$, $|f(m/n)| \le$ ≤ 1, *the series*

$$\sum_p \frac{2 - \mathrm{Re} f(p) - \mathrm{Re} f(p^{-1})}{p} \tag{2}$$

converges, then

$$\frac{1}{\#\mathcal{F}_x} S(f,x) = \Pi_1(x) \cdot \Pi_2(x) \cdot \Pi_3(x) + \mathrm{o}(1) \quad (x \to \infty) \tag{3}$$

holds with $\Pi_i(x)$ *defined as follows*

$$\Pi_1(x) = \prod_{p \le x_*} \left(1 - \frac{2}{p+1}\right) \sum_{|\nu| \ge 0} \frac{f(p^\nu)}{p^{|\nu|}}, \qquad x_* = \min\{(\beta - \alpha)x, x\},$$

$$\Pi_2(x) = \prod_{x_* < p \le x} \left(1 - \frac{1}{p}\right)\left(1 + \frac{f(p^{-1})}{p}\right),$$

$$\Pi_3(x) = \prod_{x_* < p,\ p \in P_*} \left(1 - \frac{1}{p}\right)\left(1 + \frac{f(p)}{p}\right),$$

where $P_* = \{p :$ *there exists* $m/n \in \mathcal{F}_x, p || m\}$.

With some stronger constraints on intervals the asymptotics may be written in more simple form.

Corollary. *Let for every* $\epsilon > 0$

$$x^{-\epsilon} \ll \beta - \alpha < \beta \ll x^{\epsilon}$$

hold. If the condition (2) is satisfied, then

$$\frac{1}{\#\mathcal{F}_x} S(f,x) = \prod_{p \le x} \left(1 - \frac{2}{p+1}\right) \sum_{|\nu| \ge 0} \frac{f(p^\nu)}{p^{|\nu|}} + o(1) \quad (x \to \infty).$$

The proof of Theorem 1 is based on the following statement.

Theorem 2. *Let with some* $\xi \in (0,1)$ *and* $\zeta > 0$ *for intervals* I_x *the conditions (1) be satisfied. Then for the complex valued multiplicative functions* $f(m/n)$ *with the conditions*

$$\left| f\left(\frac{m}{n}\right)\right| \le 1, \quad f(p^\nu) = 1 \quad \text{as} \quad |\nu| \ge 1,\ p \ge y,\ y = \exp\left\{c_1 \frac{\log x}{\log_2 x}\right\}$$

the asymptotics

$$\frac{1}{\#\mathcal{F}_x} S(f,x) = \prod_{p \le y} \left(1 - \frac{2}{p+1}\right) \sum_{|\nu| \ge 0} \frac{f(p^\nu)}{p^{|\nu|}} + o(\log^{-B} x) \quad (x \to \infty) \tag{4}$$

holds uniformly with an arbitrary constant $B > 0$.

This theorem is formulated in a slightly different form in [4]. A sketch of the proof is given in that paper, too. We present here the proof in details.

Proofs

We start with the proof of the Theorem 2 and then, using it prove the Theorem 1.

Proof of Theorem 2. We denote

$$S_n = \sum_{\substack{\alpha n < m < \beta n \\ (m,n)=1}} f(m).$$

Let θ be a real number, $\xi < \theta < 1$. Then for $n \le x^\theta$

$$S_n \ll (\beta - \alpha) x^\theta.$$

Hence, from the obvious equality

$$S(f, x) = \sum_{n \le x} f(n^{-1}) S_n$$

we obtain

$$S(f, x) = \sum_{x^\theta < n \le x} f(n^{-1}) S_n + O((\beta - \alpha) x^{2\theta}). \tag{5}$$

For the natural number n we denote

$$n^* = \prod_{\substack{p^\nu \| n \\ p \le y}} p^\nu, \quad n^{**} = \frac{n}{n^*}.$$

Now

$$S_n = \sum_{\substack{\alpha n < m < \beta n \\ (m, n^*) = 1}} f(m) + O\left(\sum_{\substack{\alpha n < m < \beta n \\ (m, n^{**}) > 1}} 1 \right).$$

The sum under the O-sign can be estimated as

$$\ll \sum_{p | n^{**}} \sum_{\substack{\alpha n < m < \beta n \\ p | m}} 1 \ll \frac{(\beta - \alpha) x}{y} \cdot \frac{\log x}{\log y}.$$

Hence,

$$S_n = S_n^* + O\left(\frac{(\beta - \alpha) x}{y} \cdot \frac{\log x}{\log y} \right), \quad S_n^* = \sum_{\substack{\alpha n < m < \beta n \\ (m, n^*) = 1}} f(m). \tag{6}$$

We estimate now the sum

$$S_n^* = \sum_{\substack{\alpha n < m < \beta n \\ (m, n^*) = 1}} f(m) = \sum_{\substack{\alpha n < m < \beta n \\ (m, n^*) = 1}} f(m^*)$$

for $x^\theta < n \le x$. Let us choose a small number $\tau \in (0,1)$ and taking $w = x^\tau$ split the sum into two parts S^*_{n1}, S^*_{n2} according to whether which of the two conditions $m^* \le w$ or $w < m^*$ is satisfied. Hence,

$$(7)\qquad S^*_n = S^*_{n1} + S^*_{n2}, \quad S^*_{n1} = \sum_{\substack{\alpha n < m < \beta n\\ (m,n^*)=1\\ m^* \le w}} f(m^*), \quad S^*_{n2} = \sum_{\substack{\alpha n < m < \beta n\\ (m,n^*)=1\\ w < m^*}} f(m^*).$$

We shall use the notations $p^-(m),\ p^+(m)$ for the smallest, respectively, largest prime divisor of $m,\ m > 1$, i. e.

$$p^-(m) = \max\{q : \text{ if } p < q, \quad \text{then } p\not| \ m\},$$
$$p^+(m) = \min\{q : \text{ if } p > q, \quad \text{then } p\not| \ m\},$$

here p, q are primes. We set formally $p^-(1) = +\infty,\ p^+(1) = 0$. We estimate S^*_{n2} as follows

$$(8)\qquad \begin{aligned} |S^*_{n2}| &\le \left| \sum_{\substack{w < m < \beta n\\ (m,n^*)=1\\ p^+(m) \le y}} f(m) \# \left\{ l : \frac{\alpha n}{m} < l < \frac{\beta n}{m}, p^-(l) > y \right\} \right| \ll \\ &\ll \sum_{\substack{w < m < \beta n\\ (m,n^*)=1\\ p^+(m) \le y}} \# \left\{ l : \frac{\alpha n}{m} < l < \frac{\beta n}{m}, p^-(l) > y \right\}. \end{aligned}$$

Represent now m in the last sum as $m = m' \cdot m''$, where $w < m' \le w^2$ and $p^+(m') \le p^-(m'')$. For the uniqueness choose m' the largest possible. Hence, the sum in (8) is bounded by

$$\sum_{\substack{w < m' \le w^2\\ (m',n^*)=1\\ p^+(m') \le y}} \ \sum_{\substack{w/m' < m'', (m'',n^*)=1\\ p^+(m') \le p^-(m'')\\ p^+(m'') \le y}} \# \left\{ m'' l : \frac{\alpha n}{m'} < l m'' < \frac{\beta n}{m'}, p^-(l) > y \right\}.$$

For fixed m' and different m'' the sets

$$\left\{ m'' l : \frac{\alpha n}{m'} < l m'' < \frac{\beta n}{m'}, p^-(l) > y \right\}$$

do not intersect themselves. This observation leads to the bound

$$S_{n2}^* \ll \sum_{\substack{w<m'\le w^2\\ (m',n^*)=1\\ p^+(m')\le y}} \#\left\{m : \frac{\alpha n}{m'} < m < \frac{\beta n}{m'}, (m,n^*)=1, p^+(m')\le p^-(m)\right\}.$$

We estimate the summands on the right-hand side using the folowing sieve result (see [1], Theorem 2.5).

Lemma 1. *Let* $2\le s<v<u,\ \ \epsilon_p\in\{0,1\}$ *for all primes* p,

$$P_s = \prod_{p\le s} p^{\epsilon_p},$$

and

$$S(u,v,P_s) = \#\{m : u-v<m\le u, (m,P_s)=1\}.$$

Then

$$S(u,v,P_s) = v\prod_{p|P_s}\left(1-\frac{1}{p}\right)(1+O(R)),$$

uniformly for all sequencies $\{\epsilon_p\}$, *where*

$$R = \exp\left\{-c_2\cdot\frac{\log v}{\log s}\log\left(\frac{\log v}{\log s}\right)\right\} + \exp\{-\log^{1/2} v\}.$$

We apply this Lemma with

$$u=\frac{\beta n}{m'},\quad v=\frac{(\beta-\alpha)n}{m'},\quad s=y=\exp\left\{c_1\frac{\log x}{\log_2 x}\right\},$$

$$P_s = \prod_{\substack{p<p^+(m')\\ or\ \ p|n^*}} p.$$

If τ is chosen such that $\sigma=\theta-\xi-2\tau>0$, then

$$v=\frac{(\beta-\alpha)n}{m'} > x^{\theta-\xi-2\tau} = x^{\sigma}.$$

It is easy to check, that the Lemma 1 is applicable. Because of $\log v>\sigma\log x$, we have for the remainder term R the following bound

$$R \ll \exp\{-c_3\log_2 x\cdot\log_3 x\}.$$

We have then

$$|S_{n2}^*| \ll (\beta-\alpha)n \sum_{\substack{w<m'<w^2\\(m',n^*)=1\\p^+(m')\le y}} \frac{1}{m'} \prod_{\substack{p<p^+(m')\\ or\ p|n^*}} \left(1-\frac{1}{p}\right).$$

Let $q=p^+(m')$; then $m'=q\cdot l,\ p^+(l)\le q$ and

(9) $$|S_{n2}^*| \ll (\beta-\alpha)n \sum_{\substack{q\le y\\(q,n^*)=1}} \frac{1}{q} \prod_{\substack{p<q\\ or\ p|n^*}} \left(1-\frac{1}{p}\right) \sum_{\substack{w/q<l<w^2/q\\(l,n^*)=1\\p^+(l)\le q}} \frac{1}{l}.$$

Due to $q\le y$ we have with some small constant $\epsilon>0$

(10) $$\sum_{\substack{w/q<l<w^2/q\\(l,n^*)=1\\p^+(l)\le q}} \frac{1}{l} \le \sum_{\substack{w^{1-\epsilon}<l\\(l,n^*)=1\\p^+(l)\le q}} \frac{1}{l}.$$

For estimating this last sum we apply the technique used in [5], [3] as well as in [4] for similar sums. With some z,q,t and $\delta\in(0,1)$ consider the sum

$$\sum_{\substack{z<b,p^+(b)\le q\\(b,t)=1}} \frac{1}{b} \le \frac{1}{z^{1-\delta}} \sum_{\substack{z<b,p^+(b)\le q\\(b,t)=1}} \frac{1}{b^\delta} \le \frac{1}{z^{1-\delta}} \prod_{\substack{p\le q\\(p,t)=1}} \left(1+\frac{1}{p^\delta}+\frac{1}{p^{2\delta}}+\dots\right) \le$$

$$\le \exp\left\{-(1-\delta)\log z + \sum_{\substack{p\le q\\(p,t)=1}} \left\{\frac{1}{p}+c_p(\delta)\right\} + \sum_{p\le q}\left(\frac{1}{p^\delta}-\frac{1}{p}\right)\right\},$$

where

$$c_p(\delta)=\sum_{m\ge 2}\frac{1}{p^{m\delta}}.$$

Using the obvious arguments we have

$$\sum_{p\le q}\left(\frac{1}{p^\delta}-\frac{1}{p}\right) \le \sum_{p\le q}\frac{p^{1-\delta}-1}{p} = \sum_{p\le q}\frac{1}{p}\sum_{n=1}^{\infty}\frac{((1-\delta)\log p)^n}{n!} \le$$

$$\le \sum_{n=1}^{\infty}\frac{(1-\delta)^n\log^{n-1}q}{n!}\sum_{p\le q}\frac{\log p}{p} \le 2\sum_{n=1}^{\infty}\frac{(1-\delta)^n\log^n q}{n!} < 2q^{1-\delta}.$$

Hence,

$$\sum_{\substack{z<b,\ p^+(b)\le q\\(b,t)=1}} \frac{1}{b} \le \exp\left\{-(1-\delta)\log z + 2q^{1-\delta} + \sum_{\substack{p\le q\\(p,t)=1}} \left\{\frac{1}{p} + c_p(\delta)\right\}\right\}.$$

We use this bound with $z = w^{1-\epsilon} = x^{c_4}, t = n^*$. Because of $q \le y =$ $= \exp\{c_1 \log x / \log_2 x\}$ taking

$$1-\delta = \frac{1}{c_1}\frac{\log_2 x}{\log x}\log_3 x$$

we get

$$\sum_{\substack{w^{1-\epsilon}<l\\(l,n^*)=1\\p^+(l)\le q}} \frac{1}{l} \ll \exp\left\{\sum_{\substack{p\le q\\(p,n^*)=1}} \frac{1}{p} - c_5\log_2 x\log_3 x\right\}. \tag{11}$$

We obtain now from (9), (10) and (11)

$$|S^*_{n2}| \ll$$

$$\ll (\beta-\alpha)n\exp\big\{-c_5\log_2 x\log_3 x\big\} \sum_{\substack{q\le y\\(q,n^*)=1}} \frac{1}{q} \prod_{\substack{p<q\ or\\p|n^*}} \left(1-\frac{1}{p}\right)\exp\left\{\sum_{\substack{p\le q\\(p,n^*)=1}} \frac{1}{p}\right\}.$$

Using

$$\prod_{\substack{p\le q\ or\\p|n^*}} \left(1-\frac{1}{p}\right)\exp\left\{\sum_{\substack{p\le q\\(p,n^*)=1}} \frac{1}{p}\right\} \ll \prod_{p|n^*}\left(1-\frac{1}{p}\right),$$

we get

$$|S^*_{n2}| \ll (\beta-\alpha)n\exp\big\{-c_5\log_2 x\log_3 x\big\}\prod_{p|n^*}\left(1-\frac{1}{p}\right)\sum_{\substack{q\le y\\(q,n^*)=1}} \frac{1}{q},$$

and, finally,

$$|S^*_{n2}| \ll (\beta-\alpha)n\exp\big\{-c_6\log_2 x\log_3 x\big\}. \tag{12}$$

Let us now for $x^\theta < n \le x$ investigate the sum

$$S_{n1}^* = \sum_{\substack{\alpha n < m < \beta n \\ (m,n^*)=1 \\ m^* \le w}} f(m^*).$$

We have

$$S_{n1}^* = \sum_{\substack{l<w \\ (l,n^*)=1 \\ p^+(l)\le y}} f(l)\#\left\{k : \frac{\alpha n}{l} < k < \frac{\beta n}{l}, p^-(k) > y\right\}.$$

Now we apply the sieve result formulated in Lemma 1 above with $u = \beta n/l,\ v = (\beta-\alpha)n/l, s = y$ and $P_s = \prod_{p\le s} p$. The inequality $v > x^{\theta-\xi}/w > x^\epsilon$ for some $\epsilon > 0$ ensures that the remainder term is bounded uniformly by the same term as above in (12). Then

$$S_{n1}^* =(1+O(R))(\beta-\alpha)n\prod_{p\le y}\left(1-\frac{1}{p}\right)\sum_{\substack{l<w \\ (l,n^*)=1 \\ p^+(l)\le y}} \frac{f(l)}{l},$$

$$R = \exp\{-c_6 \log_2 x \cdot \log_3 x\}.$$

Using for the sum an obvious relation

$$\sum_{\substack{l<w \\ (l,n^*)=1 \\ p^+(l)\le y}} \frac{f(l)}{l} = \sum_{\substack{(l,n^*)=1 \\ p^+(l)\le y}} \frac{f(l)}{l} + O\left(\sum_{\substack{l>w \\ p^+(l)\le y}} \frac{1}{l}\right),$$

one gets expanding the first sum into product of primes and estimating the remainder term (using, for example, (11) with $n^* = 1, q = y$) the following expression

$$\sum_{\substack{l<w \\ (l,n^*)=1 \\ p^+(l)\le y}} \frac{f(l)}{l} = \prod_{\substack{p\le y \\ (p,n^*)=1}} \sum_{\nu\ge 0} \frac{f(p^\nu)}{p^\nu} + O(R),$$

with $R = \exp\{-c_6 \log_2 x \cdot \log_3 x\}$. Now we can deal with the sum S_{n1}^* :
(13)

$$\begin{aligned} S_{n1}^* =&(1+O(R))(\beta-\alpha)n\prod_{p\le y}\left(1-\frac{1}{p}\right)\left(\prod_{\substack{p\le y \\ (p,n^*)=1}} \sum_{\nu\ge 0} \frac{f(p^\nu)}{p^\nu} + O(R)\right) = \\ =&(\beta-\alpha)n\prod_{p\le y}\left(1-\frac{1}{p}\right)\prod_{\substack{p\le y \\ (p,n^*)=1}} \sum_{\nu\ge 0} \frac{f(p^\nu)}{p^\nu} + (\beta-\alpha)nO(R). \end{aligned}$$

We may due to (6), (7), (12) and (13) write now uniformly for $x^\theta < n \le x$

$$S_n = (\beta-\alpha)n\prod_{p\le y}\left(1-\frac{1}{p}\right)\prod_{\substack{p\le y\\(p,n^*)=1}}\sum_{\nu\ge 0}\frac{f(p^\nu)}{p^\nu}+O((\beta-\alpha)xR).$$

With this equality and (5) one gets
(14)

$$S(f,x) = (\beta-\alpha)\prod_{p\le y}\left(1-\frac{1}{p}\right)\sum_{n\le x}nf(n^{-1})\prod_{\substack{p\le y\\(p,n^*)=1}}\sum_{\nu\ge 0}\frac{f(p^\nu)}{p^\nu}+O((\beta-\alpha)x^2R);$$

we extended the range of summation, but this affected only the remainder term.

We proceed further by considering the cases

$$\sum_{\nu\ge 0}\frac{f(2^\nu)}{2^\nu}\ne 0 \quad\text{and}\quad \sum_{\nu\ge 0}\frac{f(2^\nu)}{2^\nu}=0.$$

Consider the non-zero case first. We can rewrite the equality (14) then as

$$S(f,x) = (\beta-\alpha)\prod_{p\le y}\left(1-\frac{1}{p}\right)\sum_{\nu\ge 0}\frac{f(p^\nu)}{p^\nu}\sum_{n\le x}ng(n)+O((\beta-\alpha)x^2R), \tag{15}$$

where $R=\exp\{-c_6\log_2 x\cdot\log_3 x\}$ and $g(n)$ is a multiplicative function defined on the powers of primes as follows:

$$g(p^\nu)=\begin{cases} f(p^{-\nu})=1, & \text{if } p>y,\\ f(p^{-\nu})\left(\sum\limits_{\nu\ge 0}\frac{f(p^\nu)}{p^\nu}\right)^{-1}, & \text{if } p\le y.\end{cases} \tag{16}$$

Note, that $g(p^\nu)$ is bounded on the powers of primes. We proceed using the following Lemma (see [2], Theorem 02).

Lemma 2. *Let $A>0$ and $0<\alpha<1$. Then there exists $y_0=y_0(A,\alpha)$ so that for every complex valued multiplicative function $h(m)$ such that*

$$|h(p)|\le B,\quad \sum_{p,\nu\ge 2}\frac{|h(p^\nu)|}{p^{\alpha\nu}}\le C,\quad h(p)=1\quad \textit{as } p>y\ge y_0,$$

we have uniformly for $z\ge\exp\left\{A^{-1}(B+1)\log y\log_2 y\right\},\ y\ge y_0$,

$$\sum_{n\le z}h(n)=z\prod_{p\le y}\left(1-\frac{1}{p}\right)\sum_{\nu\ge 0}\frac{h(p^\nu)}{p^\nu}+O\left(z\exp\left\{-A\frac{\log z}{\log y}\right\}\right)$$

and the implied constant depends only on A, B and C.

We have $g(p) = 1$ as $p \geq y, y = \exp\{c_1 \log x / \log_2 x\}$. Then with chosen $A > 0$ we may apply Lemma 2 for $h(n) = g(n)$ as $z \geq \exp\left\{\frac{c_7}{A}\log x\right\}$. We have in this range then

$$G(z) = \sum_{n \leq z} g(n) = z \prod_{p \leq y}\left(1 - \frac{1}{p}\right)\sum_{\nu \geq 0}\frac{g(p^\nu)}{p^\nu} + O\left(z \exp\left\{-A\frac{\log z}{\log y}\right\}\right). \tag{17}$$

It is enough for us that we can use (17) for $x/(\log x)^{2A} \leq z \leq x$ with A as large as it is needed the remaining term in (17) still being $O(x\exp\{-A_1 \log_2 x\}), \quad A_1 = A/c_8$.

Using the Cauchy-Buniakowski inequality and the bound for the sum of values of multiplicative function we estimate

$$\left|\sum_{n \leq x/(\log x)^{2A}} ng(n)\right| \leq \left\{\sum_{n \leq x/(\log x)^{2A}} n^2 \sum_{n \leq x/(\log x)^{2A}} |g(n)|^2\right\}^{1/2} \ll$$
$$\ll \frac{x^2}{\log^{2A} x}\log^{c_9} x \ll \frac{x^2}{\log^A x}.$$

Hence,

$$\sum_{n \leq x} ng(n) = \sum_{x/(\log x)^{2A} \leq n \leq x} ng(n) + O\left(\frac{x^2}{\log^A x}\right). \tag{18}$$

Integrating by parts we have now

$$\sum_{x/(\log x)^{2A} \leq n \leq x} ng(n) =$$

$$= \int_{x/(\log x)^{2A}}^{x} z dG(z) = \frac{1}{2}x^2 \prod_{p \leq y}\left(1 - \frac{1}{p}\right)\sum_{\nu \geq 0}\frac{g(p^\nu)}{p^\nu} + O\left(\frac{x^2}{\log^{A_*} x}\right)$$

with $A_* = \min(A, A_1)$.

Using this in (18) we derive from (15) taking into account that the product over primes is bounded by $\log^{c_{10}} x$, we obtain that with an arbitrary constant B

$$S(f, x) = \frac{1}{2}x^2(\beta - \alpha)\prod_{p \leq y}\left(1 - \frac{1}{p}\right)^2 \sum_{|\nu| \geq 0}\frac{f(p^\nu)}{p^{|\nu|}} + O\left((\beta - \alpha)\frac{x^2}{\log^B x}\right). \tag{19}$$

If we take $h\left(\frac{m}{n}\right)=1$, then $S(h,x)=\#\mathcal{F}_x$ and (19) is applicable for the function $f=h$. Hence after some routine calculation we have

$$\#\mathcal{F}_x=\frac{1}{2}x^2(\beta-\alpha)\prod_{p\le y}\left(1-\frac{1}{p}\right)^2\left(1-\frac{2}{p+1}\right)^{-1}+O\left((\beta-\alpha)\frac{x^2}{\log^B x}\right). \tag{20}$$

It is now almost straightforward to derive (4) from (19) and (20).

Let now

$$\sum_{\nu\ge 0}\frac{f(2^\nu)}{2^\nu}=0. \tag{21}$$

We write the sum over n in (14), then

$$\sum_{n\le x}nf(n^{-1})\prod_{\substack{p\le y\\(p,n^*)=1}}\sum_{\nu\ge 0}\frac{f(p^\nu)}{p^\nu}=\prod_{2<p\le y}\sum_{\nu\ge 0}\frac{f(p^\nu)}{p^\nu}\sum_{n\le x}ng^*(n),$$

where $g^*(n)=0$, as $2\not| n$ and $g^*(n)=g(n)$ as $2|n$ with $g(n)$ defined in (16).

Let

$$S=\sum_{n\le x}ng^*(n).$$

Then

$$S(f,x)=(\beta-\alpha)\left(1-\frac{1}{2}\right)\prod_{2<p\le y}\left(1-\frac{1}{p}\right)\sum_{\nu\ge 0}\frac{f(p^\nu)}{p^\nu}S+O((\beta-\alpha)x^2R). \tag{22}$$

We proceed with the expression

$$S=\sum_{n\le x}ng^*(n)=\sum_{2\le 2^m\le x}2^mf\left(2^{-m}\right)\sum_{n\le x/2^m}nh(n) \tag{23}$$

with $h(n)$ being a multiplicative function, $h(2m)=0$ and $h(2m+1)=g(2m+1)$. The Lemma 2 implies that for $z\ge\exp\left\{\frac{c_7}{A}\log x\right\}$

$$H(z)= \tag{24}$$

$$=\sum_{n\le z}h(n)=z\left(1-\frac{1}{2}\right)\prod_{3\le p\le y}\left(1-\frac{1}{p}\right)\sum_{\nu\ge 0}\frac{g(p^\nu)}{p^\nu}+O\left(\exp\left\{-A\frac{\log z}{\log y}\right\}\right).$$

As above this asymptotical equality is needed for $z \geq x/(\log x)^{2A}$, the remaining term still being $O(x \exp\{-A_1 \log_2 x\})$, $A_1 = \dfrac{A}{c_8}$. We split the sum (23) into two parts:

$$S = S_1 + S_2,$$

$$S_1 = \sum_{2^m \leq (\log x)^{2A}} 2^m f\left(2^{-m}\right) \sum_{n \leq x/2^m} nh(n),$$

$$S_2 = \sum_{(\log x)^{2A} \leq 2^m} 2^m f\left(2^{-m}\right) \sum_{n \leq x/2^m} nh(n).$$

Using the Cauchy-Buniakowski inequality

$$\left| \sum_{n \leq u} nh(n) \right| \leq \left(\sum_{n \leq u} n^2 \cdot \sum_{n \leq u} |h(n)|^2 \right)^{1/2} \ll \left(u^3 \cdot u \log^{c_{11}} u\right)^{1/2} \ll u^2 (\log u)^{c_{12}},$$

we have

$$S_2 \ll \sum_{(\log x)^{2A} \leq 2^m} 2^m \frac{x^2}{2^{2m}} (\log x)^{c_{12}} \ll \frac{x^2}{\log^A x}.$$

For S_1 we split the range $n \leq x/2^m$ of the inner sums into two intervals: $n \leq x/(\log x)^{2A}$ and $x/(\log x)^{2A} \leq n \leq x/2^m$. Let according to this partition $S_1 = S_{11} + S_{12}$. Using the Cauchy-Buniakowski inequality as above

$$S_{11} \ll \frac{x^2}{(\log x)^A}.$$

Hence

$$S_1 = \sum_{2^m \leq (\log x)^{2A}} 2^m f\left(2^{-m}\right) \sum_{x/(\log x)^{2A} \leq n \leq x/2^m} nh(n) + O\left(\frac{x^2}{\log^A x}\right) =$$

$$= \sum_{2^m \leq (\log x)^{2A}} 2^m f\left(2^{-m}\right) \int\limits_{x/(\log x)^{2A}}^{x/2^m} z dH(z) + O\left(\frac{x^2}{\log^A x}\right).$$

Using (24) and integrating by parts we derive

$$S_1 =$$

$$= \frac{1}{2} x^2 \left(1 - \frac{1}{2}\right) \prod_{3 \leq p \leq y} \left(1 - \frac{1}{p}\right) \sum_{\nu \geq 0} \frac{g(p^\nu)}{p^\nu} \sum_{2^m \leq (\log x)^{2A}} \frac{f\left(2^{-m}\right)}{2^m} + O\left(\frac{x^2}{\log^A x}\right).$$

If we extend the summation over all m the error introduced will be swallowed by the remainder term. Hence,

$$S_1 = \frac{1}{2}x^2\left(1-\frac{1}{2}\right)\sum_{m\geq 1}\frac{f\left(2^{-m}\right)}{2^m}\prod_{3\leq p\leq y}\left(1-\frac{1}{p}\right)\sum_{\nu\geq 0}\frac{g(p^\nu)}{p^\nu}+O\left(\frac{x^2}{\log^A x}\right),$$

and this asymptotics holds for the whole sum S in (23) as well. Then we derive with this asymptotics after some routine calculations from (25) the equality (4) required for the case (21).

The proof is finally complete.

Proof of Theorem 1. With $y \geq 2$ we define the multiplicative function $f_y(m/n)$ for the powers of primes taking the values

$$f_y(p^\nu) = \begin{cases} f(p^\nu), & \text{if } p < y, \\ 1, & \text{if } p \geq y. \end{cases}$$

With the notation

$$f(p^\nu) = r(p^\nu)\exp\{i\theta(p^\nu)\}, \quad -\pi < \theta(p^\nu) \leq \pi \tag{25}$$

we have

$$f\left(\frac{m}{n}\right) = r\left(\frac{m}{n}\right)\exp\left\{i\theta\left(\frac{m}{n}\right)\right\},$$

where $0 \leq r(m/n) \leq 1$ is a multiplicative, $\theta(m/n)$ an additive function, respectively.

It is our aim to show, that the difference

$$\Delta(y|x) = \frac{1}{\#\mathcal{F}_x}|S(f,x)\exp\{-iA(x)\} - S(f_y,x)\exp\{-iA(y,x)\}| \tag{26}$$

vanishes as $y \leq x$, and $y \to \infty$. Here

$$A(y,x) = \sum_{\substack{p<y \\ p\in\Pi^*}}\frac{\theta(p^\nu)}{p}, \quad A(x) = \sum_{p\in\Pi^*}\frac{\theta(p^\nu)}{p},$$

$$\Pi^* = \left\{p^\nu : |\nu| = 1 \text{ there exists } \frac{m}{n} \in \mathcal{F}_x, p^\nu \| \frac{m}{n}\right\}.$$

Note, that for the set P^* from the formulation of Theorem 1 $P^* \subset \Pi^*$ holds. We start with the following obvious bound

$$\Delta(y|x) \leq \frac{1}{\#\mathcal{F}_x} \sum_{m/n \in \mathcal{F}_x} \left| \exp\{i(A(x) - A(y,x))\} \prod_{\substack{y \leq p \\ p^\nu \| \frac{m}{n}}} f(p^\nu) - 1 \right|.$$

Using the inequality $|u_1 \dots u_l - 1| \leq |u_1 - 1| + \dots + |u_l - 1|$, valid for the complex numbers $|u_j| \leq 1$, the notation

$$\theta_y\left(\frac{m}{n}\right) = \sum_{\substack{y \leq p \\ p^\nu \| \frac{m}{n}}} \theta(p^\nu)$$

and the expression (25) for $f(p^\nu)$ we write

(27)

$$\begin{aligned}\Delta(y|x) \ll & \frac{1}{\#\mathcal{F}_x} \sum_{m/n \in \mathcal{F}_x} \sum_{\substack{y \leq p \\ p^\nu \| \frac{m}{n}}} (1 - r(p^\nu)) + \\ & + \frac{1}{\#\mathcal{F}_x} \sum_{m/n \in \mathcal{F}_x} \left| \exp\left\{ i\theta_y\left(\frac{m}{n}\right) - i(A(x) - A(y,x)) \right\} - 1 \right| = \Delta_1 + \Delta_2.\end{aligned}$$

Changing the order of summation in the first term we obtain

$$\Delta_1 = \frac{1}{\#\mathcal{F}_x} \sum_{y \leq p} \sum_{|\nu| \geq 1} (1 - r(p^\nu)) \# \left\{ \frac{m}{n} : p^\nu \| \frac{m}{n}, \frac{m}{n} \in \mathcal{F}_x \right\}. \tag{28}$$

We proceed with the following bound valid for intervals satisfying conditions (1):

$$\# \left\{ \frac{m}{n} : p^\nu \| \frac{m}{n}, \frac{m}{n} \in \mathcal{F}_x \right\} \ll \# \left\{ \frac{m}{n} : p^{|\nu|} | mn, \frac{m}{n} \in \mathcal{F}_x \right\} \ll \frac{1}{p^{|\nu|}} \#\mathcal{F}_x. \tag{29}$$

This bound is proved, for example, in [6], p.125; for the interested reader we include here the proof.

Let u be a natural number. We consider the set

$$E(x, p^u) = \left\{ \frac{m}{n} \in \mathcal{F}_x : p^u | mn \right\} = \mathcal{D}_1(p^u, x) \cup \mathcal{D}_2(p^u, x),$$

$$\mathcal{D}_1(p^u, x) = \left\{ \frac{m}{n} \in \mathcal{F}_x : p^u | m \right\}, \quad \mathcal{D}_2(p^u, x) = \left\{ \frac{m}{n} \in \mathcal{F}_x : p^u | n \right\}.$$

We have to prove that

$$\#E(x,p^u) \ll \frac{1}{p^{|u|}}\#\mathcal{F}_x. \tag{30}$$

Evidently,

$$\#\mathcal{D}_1(p^u,x) = \left\{\frac{m}{n} : \frac{m}{n} \in \frac{1}{p^u}(\alpha,\beta),\ n \le x,\ (m,n)=(n,p)=1\right\}. \tag{31}$$

For the following we shall use the Lemma ([6], Theorem 1).

Lemma 3. *Let with the real numbers $0 \le \lambda_1 < \lambda_2$, and the natural numbers Q_0, Q_1 and Q_2 having no common prime factors (all numbers may depend on x)*

$$S(x,\lambda_1,\lambda_2,Q_0,Q_1,Q_2) =$$
$$= \#\left\{\frac{m}{n} \in (\lambda_1,\lambda_2) : (m,n) = (m,Q_0Q_1) = (n,Q_0Q_2) = 1\right\}.$$

Then uniformly in Q_0, Q_1, Q_2 and $0 \le \lambda_1 < \lambda_2$, we have

$$S(x,\lambda_1,\lambda_2,Q_0,Q_1,Q_2) = \frac{3}{\pi^2}(\lambda_2-\lambda_1)x^2 \prod_{p|Q_0}\left(1-\frac{2}{p+1}\right) \times$$
$$\times \prod_{p|Q_1Q_2}\left(1-\frac{1}{p+1}\right)\{1+B_\epsilon R(x,Q)\},$$

$$R(x,Q) = 2^{(2+\epsilon)\omega(Q)}\left(\frac{\log x}{x}+\frac{1}{x(\lambda_2-\lambda_1)}\right),$$

where $Q = Q_0Q_1Q_2$, $\omega(Q)$ denotes the number of distinct prime factors of Q, $\epsilon > 0$ is an arbitrary number, and the quantity B_ϵ is a bounded function with the bound depending only on ϵ.

Using this Lemma for (21) we obtain

$$\#\mathcal{D}_1(p^u,x) = \frac{3}{\pi^2}(\beta-\alpha)\frac{x^2}{p^u}\left(1-\frac{1}{p+1}\right)\left\{1+\frac{B\log x}{x}+\frac{Bp^u}{x(\beta-\alpha)}\right\}.$$

Since $p^u \ll \beta x$ we have $\#\mathcal{D}_1(p^u,x) \ll (\beta-\alpha)x^2p^{-u}$, provided that $\beta/(\beta-\alpha) \ll$ $\ll 1$, or, equivalently, $\alpha/(\beta-\alpha) \ll 1$. Let us now consider the case, where the last condition is not satisfied.

Let $c_{13}(\beta-\alpha) < \alpha < c_{14}(\beta-\alpha)x$ with some positive constants c_{13}, c_{14}. We then have

$$
\begin{aligned}
\#\mathcal{D}_1(p^u,x) \le & \sum_{m\le \beta x/p^u} \sum_{\frac{mp^u}{\beta}\le n\le \frac{mp^u}{\alpha}} 1 \le \\
\le & \sum_{m\le \beta x/p^u} \left\{ mp^u \frac{\beta-\alpha}{\alpha\beta} + 1 \right\} \le \frac{\beta-\alpha}{\alpha\beta} p^u \left(\frac{\beta x}{p^u}\right)^2 + \frac{\beta x}{p^u} = \\
= & (\beta-\alpha)\frac{x^2}{p^u}\frac{\beta}{\alpha} + \frac{\beta x}{p^u} \ll (\beta-\alpha)\frac{x^2}{p^u};
\end{aligned}
$$

here we have used the inequalities $\beta/\alpha = 1 + (\beta-\alpha)/\alpha < 1 + 1/c_{13}$, and $\beta x = \alpha x + (\beta-\alpha)x \ll (\beta-\alpha)x^2$. Hence,

$$
\#\mathcal{D}_1(p^u,x) \ll (\beta-\alpha)\frac{x^2}{p^u}
$$

holds provided that $\alpha \ll (\beta-\alpha)x$.

For $\mathcal{D}_2(p^u,x)$ we have

$$
\mathcal{D}_2(p^u,x) = \left\{ \frac{m}{n} : \frac{m}{n} \in p^u(\alpha,\beta),\, n \le \frac{x}{p^u},\, (m,p)=1 \right\}.
$$

Lemma 3 now yields

$$
\begin{aligned}
\#\mathcal{D}_2(p^u,x) = & \frac{3}{\pi^2}(\beta-\alpha)\frac{x^2}{p^u}\left(1-\frac{1}{p+1}\right) \times \\
& \times \left\{ 1 + B\frac{p^u \log(x/p^u)}{x} + \frac{1}{x(\beta-\alpha)} \right\} \ll (\beta-\alpha)\frac{x^2}{p^u}.
\end{aligned}
$$

Using the bounds for $\#\mathcal{D}_1(p^u,x)$, $\#\mathcal{D}_2(p^u,x)$, we obtain (30).

Hence, from (28) and (29) we get

$$
\Delta_1 \ll \sum_{y\le p} \frac{2 - \mathrm{Re}\, f(p) - \mathrm{Re}\, f(p^{-1})}{p} + \sum_{y\le p} \frac{1}{p^2} = \delta(y), \tag{32}
$$

where, due to condition (2), $\delta(y) \to 0$, as $y \to \infty$.

The function θ_y is additive with the values on powers of primes uniformly bounded. Using the inequality $|e^{iu}-1| \le |u|$, and then the Cauchy-Buniakowski inequality, we obtain the bound for Δ_2 in (27)

$$
\begin{aligned}
\Delta_2 &\ll \frac{1}{\#\mathcal{F}_x} \sum_{m/n\in\mathcal{F}_x} \left|\theta_y\left(\frac{m}{n}\right) - (A(x) - A(y,x))\right| \le \\
&\le \left\{ \frac{1}{\#\mathcal{F}_x} \sum_{m/n\in\mathcal{F}_x} \left|\theta_y\left(\frac{m}{n}\right) - (A(x) - A(y,x))\right|^2 \right\}^{1/2}.
\end{aligned} \tag{33}
$$

We proceed with the Kubilius inequality for additive functions, defined for rational numbers (see [8]).

Lemma 4. *Let an additive complex valued function $g(m/n)$ is bounded on powers of primes,*

$$A_g(x) = \sum_{p^\nu\in\Pi^*} \frac{g(p^\nu)}{p}, \quad B_g^2(x) = \sum_{p^\nu\in\Pi^*} \frac{|g(p^\nu)|^2}{p},$$

$$\Pi^* = \left\{p^\nu : |\nu| = 1,\ p^\nu \| \frac{m}{n} \text{ for some } \frac{m}{n} \in \mathcal{F}_x\right\}.$$

Then with the constraints (1) for the intervals (α,β) the following inequality holds

$$\frac{1}{\#\mathcal{F}_x} \sum_{m/n\in\mathcal{F}_x} \left|g\left(\frac{m}{n}\right) - A_g(x)\right|^2 \ll B_g^2(x).$$

Using this inequality in (33) with $g = \theta_y$ we get the bound

$$\Delta_2 \ll \sum_{\substack{y\le p\\ p^\nu\in\Pi^*}} \frac{\theta(p^\nu)^2}{p}.$$

Due to the bound $\theta^2(p^\nu) \ll 1 - \mathrm{Re}\, f(p^\nu)$ (see, for example, this inequality proved in [9], p. 368) and the convergence of (2), we obtain, that Δ_2 vanishes, as $y \to \infty$. From this fact and (27), (32) we have that $\Delta(y|x) \to 0$ as $y \to \infty$.

Then taking into account the form of $\Delta(y|x)$ (see, (26)) we get

$$\frac{1}{\#\mathcal{F}_x} S(f,x) = \exp\{i(A(x) - A(y,x))\} \sum_{m/n\in\mathcal{F}_x} f_y\left(\frac{m}{n}\right) + \mathrm{o}(1) \quad (x\to\infty). \tag{34}$$

As $y = \exp\{c_1 \log x / \log_2 x\}$ we may use for the sum in (34) the asymptotics (4). Hence,

$$\frac{1}{\#\mathcal{F}_x} S(f,x) = \exp\{i(A(x) - A(y,x))\} \prod_{p \le y} \left(1 - \frac{2}{p+1}\right) \sum_{|\nu| \ge 0} \frac{f(p^\nu)}{p^{|\nu|}} + o(1).$$

We are going to replace the factor $\exp\{i(A(x) - A(y,x))\}$ by the appropriate product over primes. We do this as follows:

$$\exp\{i(A(x) - A(y,x))\} = \exp\left\{ i \sum_{\substack{p^\nu \in \Pi^* \\ y \le p}} \frac{\theta(p^\nu)}{p} \right\} =$$
$$= \prod_{\substack{p^\nu \in \Pi^* \\ y \le p}} \left(1 - \frac{1}{p}\right)\left(1 + \frac{f(p^\nu)}{p}\right) L(y,x),$$
$$L(y,x) = \exp\left\{ \sum_{\substack{p^\nu \in \Pi^* \\ y \le p < x}} \frac{1 - f(p^\nu) + i\theta(p^\nu)}{p} + O\left(\sum_{p > y} \frac{1}{p^2}\right) \right\}.$$

We want to show, that $L(y,x) = 1 + o(1)$. It suffices to prove, that the sum under exponent vanishes, as x grows unboundedly. It does indeed, as the following relations show:

$$\sum_{\substack{p^\nu \in \Pi^* \\ y \le p}} \frac{1 - f(p^\nu) + i\theta(p^\nu)}{p} = \sum_{\substack{p^\nu \in \Pi^* \\ y \le p}} \frac{1 - r(p^\nu)\mathrm{e}^{i\theta(p^\nu)} + i\theta(p^\nu)}{p} =$$
$$= \sum_{\substack{p^\nu \in \Pi^* \\ y \le p}} \frac{1 - r(p^\nu)}{p} + i \sum_{\substack{p^\nu \in \Pi^* \\ y \le p}} \theta(p^\nu) \frac{1 - r(p^\nu)}{p} + O\left(\sum_{p > y} \frac{\theta^2(p^\nu)}{p}\right).$$

The convergence of the series (2) and the inequality $\theta^2(p^\nu) \ll 1 - \operatorname{Re} f(p^\nu)$ ensures, that the sum tends to zero, as $x \to \infty$. Now we have

(35)
$$\frac{1}{\#\mathcal{F}_x} S(f,x) = \prod_{\substack{p^\nu \in \Pi^* \\ y \le p}} \left(1 - \frac{1}{p}\right)\left(1 + \frac{f(p^\nu)}{p}\right) \prod_{p \le y} \left(1 - \frac{2}{p+1}\right) \sum_{|\nu| \ge 0} \frac{f(p^\nu)}{p^{|\nu|}} + o(1).$$

For to come to the asymptotics (3) required by the Theorem 1 we need to consider, which primes p or their reciprocals p^{-1} belong to Π^*. Let us show first, that for all $p \le x$ $p^{-1} \in \Pi^*$.

If $m/np \in \mathcal{F}_x$ with $(p, mn) = 1$, then $m/n \in (\alpha p, \beta p)$ and $n \leq x/p$. For to count the number of such m/n we use the Lemma 3 again.

According to the Lemma 3 the number of $m/np \in \mathcal{F}_x$ with $(p, mn) = 1$ equals to

$$\frac{3}{\pi^2}(\beta-\alpha)p\frac{x^2}{p^2}\left(1-\frac{2}{p+1}\right)\left\{1+O\left(\frac{\log(x/p)}{(x/p)}+\frac{1}{x(\beta-\alpha)}\right)\right\}.$$

Due to $(\beta-\alpha)x \to \infty$, the value of this expression is positive if $x/p > c_{15}$ with c_{15} large enough. If $x/p \leq c_{15}$, then $p \geq x/c_{15}$ and there exists $u/p \in (\alpha, \beta)$, supposed that $(\beta-\alpha)p > 2$. This is evidently true for $x \geq x_0$ because of

$$(\beta-\alpha)p \geq \frac{(\beta-\alpha)x}{c_{15}}, \quad (\beta-\alpha)x \to \infty.$$

We proved then that $p^{-1} \in P^*$ for all $p \leq x$.

Let us investigate now which primes p belong to $P^* \subset \Pi^*$. If there exists some $mp/n \in \mathcal{F}_x, (p, mn) = 1$, then $m/n \in (\alpha/p, \beta/p)$ and $n \leq x$. By the Lemma 3 we get, that the number of such rationals equals

$$\frac{3}{\pi^2}\frac{\beta-\alpha}{p}x^2\left(1-\frac{2}{p+1}\right)\left\{1+O\left(\frac{\log x}{x}+\frac{p}{x(\beta-\alpha)}\right)\right\}.$$

It is evident, that this term is positive as $p \leq c_{16}x(\beta-\alpha)$ with an appropriate constant c_{16}. Hence, factors corresponding to primes $p \leq c_{16}x(\beta-\alpha)$ all appear in (35). If we add the factors corresponding to primes in the range $c_{16}x(\beta-\alpha) \leq$ $\leq p \leq x(\beta-\alpha)$, the changes will affect only the remainder term o(1). Then for all p such that $y < p \leq \min((\beta-\alpha)x, x)$ the product

$$\left(1-\frac{1}{p}\right)\left(1+\frac{f(p)}{p}\right)\left(1-\frac{1}{p}\right)\left(1+\frac{f(p^{-1})}{p}\right)$$

appears in (35). If we replace this quantity by

$$\left(1-\frac{2}{p+1}\right)\sum_{|\nu|\geq 0}\frac{f(p^\nu)}{p^\nu},$$

only the remainder term changes. This completes the proof of the theorem.

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