UNIFORMLy–ALMOST–EVEN FUNCTIONS WITH PRESCRIBED VALUES III.

J.-C. Schlage-Puchta (Freiburg im Breisgau, Germany)
W. Schwarz (Frankfurt am Main, Germany)
J. Spilker (Freiburg im Breisgau, Germany)

Dedicated to Karl-Heinz Indlekofer
on the occasion of his 60th birthday

Abstract. Given integers $0 < a_1 < a_2 < \ldots$ and bounded complex numbers $b_1, b_2, \ldots$, we deal with the problem of the existence and uniqueness of a uniformly–almost–even function $f$ satisfying

$$f(a_n) = b_n, \quad \text{for all } n \in \mathbb{N}.$$  

We give necessary and sufficient conditions that there exists at most or at least one function $f$ with this interpolation property.

1. Introduction

A function $f : \mathbb{N} \to \mathbb{C}$ is called $r$-even, if the equation $f(\gcd(n, r)) = f(n)$ holds for all integers $n$; $f$ is called even, abbreviated $f \in \mathcal{B}$, if there is some $r$ for which $f$ is $r$-even. The closure of $\mathcal{B}$ with respect to the “uniform” norm $\|f\|_u = \sup_{n \in \mathbb{N}} |f(n)|$ is the complex algebra $\mathcal{B}^u$ of uniformly-almost-even functions. Starting with the complex vector-space $\mathcal{D}$ of all periodic functions one obtains similarly the algebra $\mathcal{D}^u$ of uniformly-almost-periodic functions (see, for example, [7], IV.1).

In this note the following interpolation problem is dealt with: Let $\{a_n\}_n$ be a strictly increasing sequence of positive integers, and $\{b_n\}_n$ a bounded
sequence of complex numbers; when does a uniformly-almost-even function $f$
(resp. a uniformly-almost-periodic function) exist with values

$$(P) \quad f(a_n) = b_n \text{ for } n = 1, 2, \ldots ?$$

When is there at most one such function? \footnote{Karl-Heinz Indlekofer investigated uniqueness sets for additive functions; as far as the second-named author remembers correctly, Indlekofer gave a talk on this subject already in Oberwolfach in the year 1978. He returned to this subject in joint papers (see [1], [2]) with Fehér, Stachó, and Timofeev.}

Under more restrictive conditions the problem of the existence of such functions was already treated in [5] and [7], IV.5, Theorems 5.1 and 5.2. The authors used the fact that the Banach algebra $B^n$ is isomorphic with the algebra of functions continuous on the compact space $\Delta_B$ of maximal ideals, and this space was explicitly given,

$$\Delta_B = \prod_p \{1, p^1, p^2, \ldots, p^\infty\},$$

where the factors are one-point compactifications of the discrete spaces $\{1, p^1, p^2, \ldots\}, p \in \mathbb{P}$. Later the second-named author tried to prove this result without using Gelfand’s theory (see [6]). However, unfortunately there is a gap in this paper: in the proof that $\{g_{N_k}\}_{k \in \mathbb{N}}$ is a Cauchy-sequence, one case is missing. Schwarz & Spilker [8] used the method of [6] to prove other existence results under different assumptions.

In this paper we prove elementarily, without using Gelfand’s theory, uniqueness results (Theorems 1 and 2, Section 2) and existence theorems (Theorems 3 and 4, Section 3).

Notations. $\mathbb{N} = \{1, 2, \ldots\}$ is the set of positive integers, $\mathbb{P} = \{2, 3, 5, \ldots\}$ the set of primes. For $n \in \mathbb{N}$, $p \in \mathbb{P}$, we denote by $o_p(n)$ the order of $p$ in the factorization of $n$, so that $p^{o_p(n)} | n$, but $p^{o_p(n)+1} / n$.

2. Sets of uniqueness

In this section we deal with the (much simpler) problem of uniqueness in our interpolation problem (see equation $(P)$).
**Definition.** A subset $A = \{a_n : n \in \mathbb{N}\}$ of $\mathbb{N}$ is called a set of uniqueness for $\mathcal{B}^u$, if the condition

$$\{f, g \in \mathcal{B}^u, \ f(a_n) = g(a_n) \text{ for all } n \in \mathbb{N}\}$$

implies $f = g$.

Sets of uniqueness for $\mathcal{B}^u$ are characterized by the following theorem.

**Theorem 1.** The following properties of the set $A = \{a_n : n \in \mathbb{N}\} \subseteq \mathbb{N}$ are equivalent:

1. $A$ is a set of uniqueness for $\mathcal{B}^u$.
2. For any integers $d, k \in \mathbb{N}$ satisfying $d \mid k!$ there exists an integer $n \in \mathbb{N}$ such that the greatest common divisor $\gcd(a_n, k!) = d$.

**Proof.**

1. $\Rightarrow$ 2: Let $\{a_n : n \in \mathbb{N}\}$ be a set of uniqueness, and let $d, k \in \mathbb{N}$ satisfy $d \mid k!$. Define a $k!$-even function $f_1(n)$ for $n \mid k!$ by

$$f_1(n) = \begin{cases} 0, & \text{if } n \nmid k!, n \neq d, \\ 1, & \text{if } n = d, \end{cases}$$

and, for $n \in \mathbb{N}$, by $f_1(n) = f_1(\gcd(n, k!))$. If there were no $n \in \mathbb{N}$ satisfying $\gcd(a_n, k!) = d$, then there would be two different solutions $f_1$ and $f_2 = 0$ for the interpolation problem $f(a_n) = 0$, a contradiction to (1).

2. $\Rightarrow$ 1: Assume that there is a function $f \in \mathcal{B}^u$, $f \neq 0$, satisfying $f(a_n) = 0$ for any $n \in \mathbb{N}$. Fix an integer $d$ such that $f(d) \neq 0$, and choose a large $k$, $k \geq d$, and a $k!$-even function $h$ satisfying $\|f - h\|_u < \frac{1}{2} |f(d)|$. Because of (2) there is an integer $n$ so that $\gcd(a_n, k!) = d$, and so $h(a_n) = h(d)$. This gives the contradiction

$$|f(d)| \leq |f(d) - h(d)| + |h(a_n) - f(a_n)| \leq 2 \cdot \|f - h\|_u < |f(d)|.$$ 

**Examples.** The set $(\mathbb{P} + 1) \cup (\mathbb{P} + 2)$, the union of two sets of shifted primes, is a set of uniqueness for $\mathcal{B}^u$.

We verify condition (2). Let positive integers $d, r \in \mathbb{N}$, $d \mid r$ be given. 2

a) If $d$ is even, denote by $\pi$ a prime $\pi \mid r$. Then the integer

$$n_\pi = \pi^{\alpha_\pi(d)} - 1, \text{ if } \pi \mid d, \quad n_\pi = \pi + 1, \text{ if } \pi \nmid d,$$

2 It would be sufficient to restrict ourselves to numbers $r$ of the form $r = k!$.  

satisfies
\[ o_\pi(n_\pi) = 0, \quad o_\pi(n_\pi + 1) = o_\pi(d). \]
Any solution \( n \in \mathbb{N} \) of the system of congruences
\[ n \equiv n_\pi \mod \pi^{o_\pi(r)+1} \text{ for every } \pi \mid r \]
satisfies \( \gcd(n, r) = 1, \quad \gcd(n + 1, r) = d. \)

By the prime number theorem for arithmetic progressions there exists a prime \( p \equiv n \mod r \), and for this prime we get \( \gcd(p + 1, r) = d \).

b) If \( d \) is odd, we find a prime \( p \) satisfying \( \gcd(p + 2, r) = d \), in a similar manner.

The set \((\mathbb{P} + 1) \cup (2\mathbb{P} + 1)\) is also a set of uniqueness for \( B^w \).

The sets \( \mathbb{P} + a \), where \( a \in \mathbb{N}_0 \), the set of squares, the set of squarefree numbers, the set of \( k \)-free numbers, the set of factorials and the set of powers of an integer \( a \in \mathbb{N} \) are not sets of uniqueness for \( B^w \).

Without proof we give the corresponding result for \( D^u \).

**Theorem 2.** A set \( A = \{a_n : n \in \mathbb{N}\} \subseteq \mathbb{N} \) is a set of uniqueness for \( D^u \), if and only if for any \( d, r \in \mathbb{N} \), \( d \leq r \) there exists an integer \( n \) so that \( a_n \equiv d \mod r \).

**Examples.** Any strictly monotone sequence \( A \) of integers \( a_n \), which is uniformly distributed modulo \( r \) for any \( r \in \mathbb{N} \), is a set of uniqueness for \( D^u \). In particular \(^3\)
- every set \( A \subseteq \mathbb{N} \) with density 1,
- the set \( \{a_n, \ a_n = [P(n)]\} \), where \( P(x) \) is a polynomial in \( \mathbb{R}[x] \), and \( P(x) - \frac{1}{P(0)} \) has at least one irrational coefficient,\(^4\)
- the set \( a_n = [n^c] \), where \( c > 0, \ c \not\in \mathbb{Z} \).\(^5\)

The set \((\mathbb{P}+1)\cup(2\mathbb{P}+1)\) is not a set of uniqueness for \( D^u \), being disjoint to the residue class 11 mod 30. Also, the sets \( \bigcup_{1 \leq n \leq N} (\alpha_n \mathbb{P} + \beta_n), \ \alpha_n \in \mathbb{N}, \ \beta_n \in \mathbb{N} \cup \{0\}, \)
the set of \( B \)-numbers \( (a_n) \) is a \( B \)-number, if it is representable as a sum of two squares of integers\(^6\) are not sets of uniqueness for \( D^u \).

\(^3\) For the definition and simple properties of uniform distribution modulo \( r \), see, for example, Kuipers & Niederreiter [4], Chapter 5, p. 305ff.
\(^4\) See [4], Theorem 1.4, p. 307.
\(^5\) See [4], Exercise 1.10, p. 318.
\(^6\) \( B \)-numbers are easily characterized by conditions concerning prime factors \( p \equiv 3 \mod 4 \).
3. Existence theorems

Given finitely many integers $a_1, a_2, \ldots, a_N$ and complex numbers $b_1, b_2, \ldots, b_N$, then there is an even function $f \in \mathcal{B}$ assuming the values $f(a_n) = b_n$ for $1 \leq n \leq N$: write $\alpha = a_1 a_2 \cdots a_N$, and define for all divisors $a | \alpha$ and for $n \in \mathbb{N}$,

$$f_a(n) = \begin{cases} 1, & \text{if } (n, \alpha) = a, \\ 0, & \text{if } (n, \alpha) \neq a. \end{cases}$$

Then $f = \sum_{1 \leq n \leq N} b_n \cdot f_a(n)$ is such a function. By the way (see [6]),

$$f_a(n) = \frac{\varphi(\alpha)}{\alpha} \sum_{r | \alpha} c_r(a) \varphi(r) \mu\left(\frac{r}{\alpha}\right),$$

where $c_r(n) = \sum_{d | (r, n)} d \mu\left(\frac{r}{d}\right)$ is the Ramanujan-sum. So, we are only concerned with infinite subsets of $\mathbb{N}$.

**Theorem 3.** For a strictly increasing sequence $\{a_n\}_{n \in \mathbb{N}}$ of positive integers and a bounded sequence $\{b_n\}_{n \in \mathbb{N}}$ of complex numbers the following two conditions (3) and (4) are equivalent.

(3) There exists a function $f \in \mathcal{B}^u$ with the values $f(a_n) = b_n$ ($n \in \mathbb{N}$).

(4) If $\{n_k\}_{k \in \mathbb{N}}$ is any strictly increasing sequence of positive integers such that for any $r \in \mathbb{N}$ the sequence $\{\gcd(a_{n_k}, r!}\}_{k \in \mathbb{N}}$ is eventually constant, then the limit

$$\lim_{k \to \infty} b_{n_k}$$

exists, and, in the case that, with some integer $m$ (not depending on $r$),

$$\lim_{k \to \infty} \gcd(a_{n_k}, r!) = \gcd(a_m, r!)$$

for every $r$, its value is $b_m$.

Before proving Theorem 3 we reformulate the conditions concerning $\alpha_{p(a_{n_k})}$.

**Lemma.** For any sequence $\{m_k\}_{k \in \mathbb{N}}$ of positive integers the following results are true.

(5) Properties (5a) and (5b) are equivalent.

(5a) For every $r \in \mathbb{N}$ the sequence $\{\gcd(m_k, r!)\}_{k \in \mathbb{N}}$ is eventually constant.
(5b) For every prime $p$ the sequence $\{o_p(m_k)\}_{k \in \mathbb{N}}$ is eventually constant or tends to infinity.

(6) Properties (6a) and (6b) are equivalent.

(6a) For every $r \in \mathbb{N}$ the sequence $\{\gcd(m_k, r!\} \in \mathbb{N}$ is eventually constant, and there exists an integer $m \in \mathbb{N}$ so that for every $r$ the relation $\gcd(m, r!) = \lim_{k \to \infty} \gcd(m_k, r!)$ holds.

(6b) For every prime $p$ the sequence $\{o_p(m_k)\}_{k \in \mathbb{N}}$ is eventually constant and $\lim_{k \to \infty} o_p(m_k) \neq 0$ for at most finitely many primes $p$.

Proof.

(5a) $\Rightarrow$ (5b): Let (5a) hold for the sequence $\{m_k\}_k$, and let $p$ be a prime. For any $j \in \mathbb{N}$ there is some $k_j \in \mathbb{N}$ so that $\min \{o_p(m_k), o_p(p^j)\}$ does not depend on $k$, if $k > k_j$; say, this minimum is $e_j$.

If $e_j = o_p(p^j)$ for some $j \in \mathbb{N}$, then the sequence $\{o_p(m_k)\}_k$ is eventually constant.

If $e_j < o_p(p^j)$ for every $j \in \mathbb{N}$, then the sequence $\{o_p(m_k)\}_k$ tends to $\infty$.

(5b) $\Rightarrow$ (5a): Fix $r \in \mathbb{N}$. By (5b) there is some integer $k_0$ with the property, that the sequence $\{o_p(m_k)\}_{k > k_0}$ is constant for all primes $p \leq r$, or there is some prime $p \leq r$ such that $o_p(m_k) > o_p(r!)$ for every $k > k_0$. Thus $\min\{o_p(m_k), o_p(r!\} \in \mathbb{N}$ is independent of $k > k_0$ [there is no prime $p > r$ dividing $r!$], and therefore the sequence $\{\gcd(m_k, r!\}_{k > k_0}$ is constant.

(6a) $\Rightarrow$ (6b): Assume that condition (6a) is true for the sequence $\{m_k\}_k$. By (6a) there is an integer $m$ so that for any prime $p$

$$\min\{o_p(m), o_p(r!\} = \lim_{k \to \infty} \min\{o_p(m_k), o_p(r!\}.$$ 

According to (5), the sequence $\{o_p(m_k)\}_k$ is eventually constant or its limit [for $k \to \infty$] is $\infty$. Put $r = p^j$, where $j > o_p(m)$: then $o_p(m) \geq o_p(m_k)$, if $k$ is large: therefore the case $\lim_{k \to \infty} o_p(m_k) = \infty$ is impossible. If $p > m$, then $o_p(m) = 0$, and so $\lim_{k \to \infty} o_p(m_k) = 0$.

(6b) $\Rightarrow$ (6a). Assume that for the sequence $\{m_k\}_k$ condition (6b) is true. Given $r \in \mathbb{N}$, the sequence $\{\gcd(m_k, r!\}_{k \in \mathbb{N}}$ is eventually constant, by (5b) $\Rightarrow$ (5a). Write $d_r = \lim_{k \to \infty} \gcd(m_k, r!)$. The number $m = \prod_p p^{e_r}$ is well-defined by (6b), and, for every $r \in \mathbb{N}$,

$$\gcd(m, r!) = \prod_p p^{\min\{e_r, o_p(r!)\}} = \lim_{k \to \infty} \prod_p p^{\min\{o_p(m_k), o_p(r!\} = \lim_{k \to \infty} \gcd(m_k, r!)$$.


Thus the Lemma is proved.

**Proof of Theorem 3.**

(3) $\Rightarrow$ (4). Let a strictly increasing sequence $\{a_n\}$ of positive integers, a bounded sequence $\{b_n\}$ of complex numbers, and a function $f \in \mathcal{B}^u$ be given satisfying the interpolation-property $f(a_n) = b_n$; take a strictly monotone sequence $\{n_k\}_k$ in $\mathbb{N}$, so that for every $r \in \mathbb{N}$ the sequence $\{\gcd(a_{n_k}, r!)\}_k$ is eventually constant.

$f \in \mathcal{B}^u$ implies that for any given $\varepsilon > 0$ there is some $s \in \mathbb{N}$ and an $(s)$-even function $h$ approximating $f$, so that $\|f - h\|_u < \frac{1}{2}\varepsilon$.

a) There is some $k_0 \in \mathbb{N}$, $k_0 = k_0(\varepsilon)$ so that for all $k, \ell > k_0$ the relation $\gcd(a_{n_k}, r!) = \lim_{k \to \infty} \gcd(a_{n_k}, r!)$ holds, and so $h(a_{n_k}) = h(a_{n_\ell})$. Therefore we obtain for every $k, \ell > k_0$:

$$|b_{n_k} - b_{n_\ell}| \overset{(3)}{=} |f(a_{n_k}) - f(a_{n_\ell})| \leq |f(ank) - h(a_{n_k})| + |f(a_{n_\ell}) - h(a_{n_\ell})| \leq 2 \cdot \|f - h\|_u < \frac{1}{2}\varepsilon,$$

and so $\{b_{n_k}\}_k$ is a Cauchy-sequence, and thus convergent.

b) Now, we take for granted that in addition (with some integer $m$)

$$\gcd(a_m, r!) = \lim_{k \to \infty} \gcd(a_{n_k}, r!), \quad \text{for every } r \in \mathbb{N}.$$  

Note that $f(a_m) = b_m$, and that the sequence $\{\gcd(a_{n_k}, r!)\}_k \in \mathbb{N}$ is eventually constant; thus there is some $\ell_0 > k_0$ [$\ell_0 = \ell_0(s)$, and so $\ell_0$ depends on $\varepsilon$] with the property that for any $\ell > \ell_0$

$$\gcd(a_m, s!) = \gcd(a_{n_\ell}, s!), \quad \text{and so, in particular, } h(a_m) = h(a_{n_\ell}).$$

Therefore we obtain, with some $\ell > \ell_0$,

$$|b_m - \lim_{k \to \infty} b_{n_k}| \leq |b_m - b_{n_\ell}| + \left|\lim_{k \to \infty} b_{n_k} - b_{n_\ell}\right|,$$

and by the inequalities in a) this is

$$\leq |f(a_m) - f(a_{n_\ell})| + \frac{1}{2}\varepsilon \leq |f(a_m) - h(a_m)| + |f(a_{n_\ell}) - h(a_{n_\ell})| + \frac{1}{2}\varepsilon \leq 2 \cdot \|f - h\|_u + \frac{1}{2}\varepsilon < \varepsilon,$$
and so \( \lim_{k \to \infty} b_{nk} = b_n \).

Now we come to the more difficult part, the proof of the implication.

(4) \( \Rightarrow \) (3). Given sequences \( \{a_n\} \) and \( \{b_n\} \) as in the theorem; without loss of generality we may assume that the \( b_n \) are non-negative real numbers. We have to find a function \( f \in B^n \), so that \( f(a_n) = b_n \) for every \( n \).

Define for any positive integers \( n \) and \( k \) satisfying \( n \mid k! \) the set

\[
M(n, k) := \{ m \in \mathbb{N} : \gcd(a_m, k!) = n \} = \left\{ m \in \mathbb{N} : a_m \equiv 0 \mod n, \text{ and } \gcd\left( \frac{a_m k!}{n} \right) = 1 \right\}.
\]

The set \( M(n, k) \) is empty if and only if \( \gcd(a_m, k!) = n \) is impossible for any \( m \); in particular, if \( n \) does not divide any \( a_m \), then \( M(n, k) = \emptyset \). We define two \( k! \)-even functions \( f^+_k \) and \( f^-_k \), first for integers \( n \mid k! \), by

\[
f^+_k(n) = \begin{cases} 
\sup \{ b_m : m \in M(n, k) \}, & \text{if } M(n, k) \neq \emptyset, \\
0, & \text{if } M(n, k) = \emptyset,
\end{cases}
\]

and similarly \( f^-_k(n) \), replacing “sup” with “inf”, and then obtain \( k! \)-even functions by the definition

\[
f^\pm_k(n) = f^\pm_k(\gcd(n, k!)) \text{ for any } n \in \mathbb{N}.
\]

So,

\[
f^+_k(n) = \sup \{ b_m : m \in M((n, k!), k) \}, \text{ if } M((n, k)) \neq \emptyset, \text{ otherwise } = 0.
\]

It is sufficient to show the equation

\[
(7) \quad \lim_{k \to \infty} \| f^+_k - f^-_k \|_u = 0.
\]

The reasons are:

(a) For any \( k, n \in \mathbb{N} \) the inequalities

\[
f^-_k(n) \leq f^-_{k+1}(n) \leq f^+_k(n) \leq f^+_k(n)
\]

hold. [This implies that \( \| f^+_k - f^-_k \|_u \) is decreasing.]
Without loss of generality \( n \mid (k + 1)! \). On behalf of \( \gcd(a_n, (k + 1)!) = n \) implies \( \gcd(a_m, k!) = \gcd(n, k!) \) we obtain
\[
M(n, k + 1) \subseteq M(\gcd(n, k!), k),
\]
and this gives the first and last inequality.

(\( \beta \)) The sequence \( (f_k^n)_{k \in \mathbb{N}} \) is a Cauchy-sequence in \( B^u \), because of (see (\( \alpha \))
\[
\|f_k^n - f_{k+\ell}^n\|_u \leq \|f_k^n - f_k^+\|_u \quad \text{for any } k, \ell \in \mathbb{N}.
\]
The space \( (B^u, \| \cdot \|_u) \) is complete, therefore
\[
f = \lim_{k \to \infty} f_k^n \text{ exists and is in } B^u.
\]

(\( \gamma \)) The function \( f \) defined in (\( \beta \)) does interpolate the prescribed values \( b_n \):

If \( k \geq a_n \), then \( n \in M(a_n, k) \), therefore \( f_k^-(a_n) \leq b_n \leq f_k^+(a_n) \) [by the definition of \( f_k^- \), \( f_k^+ \)], and so
\[
f(a_n) \overset{\text{(\( \beta \))}}{=} \lim_{k \to \infty} f_k^+(a_n) = b_n,
\]
[by (7) and the inequalities \( f_k^-(a_n) \leq b_n \leq f_k^+(a_n) \)].

So it remains to proof equation (7), \( \| f_k^n - f_k^- \| \to 0 \), as \( k \to \infty \).

**Assume** that (7) is wrong. Since the sequence \( \{ \| f_k^n - f_k^- \|_u \}_{k \in \mathbb{N}} \) is decreasing [see (\( \alpha \))], there is some \( c > 0 \) so that \( \| f_k^n - f_k^- \|_u > c \) for all \( k \in \mathbb{N} \).

Therefore, for every \( k \in \mathbb{N} \) there is some integer \( \nu = \nu(k) \) for which \( f_k^+(\nu) - f_k^- (\nu) > c \).

By the definition of \( f_k^\pm \), for every \( k \) there exist integers \( n_k^+ \) and \( n_k^- \) in \( M(\gcd(\nu, k!), k) \) with the properties

\( \text{(a)} \) \( \gcd(a_{n_k^+}, k!) = \gcd(a_{n_k^-}, k!) \quad [= \gcd(\nu, k!)] \),

and

\( \text{(b)} \) \( b_{n_k^+} - b_{n_k^-} > c \).

The sequence \( \{ b_n \}_{n} \) is bounded; therefore there is\(^7\) a constant \( b \) such that for some increasing subsequence \( \{ k(j) \}_{j} \) the inequalities
\[
b_{n_{k(j)}^-} < b - \frac{1}{3}c < b + \frac{1}{3}c < b_{n_{k(j)}^+}
\]
\(^7\) For \( b \), one may take, for example, a point of accumulation of the sequence \( \{ \frac{1}{2} (bn_k^+ + bn_k^-) \}_{k}. \)
hold for every \( j \in \mathbb{N} \). It follows that
\[
b_{n_k} - b_{n_{k(i)}} > \frac{2}{3} c \quad \text{for any } i, j \in \mathbb{N}.
\]
So we got a sequence \( k(1) < k(2) < \ldots \) of integers and integers \( n_{k(j)}^+, n_{k(j)}^- \) satisfying
(a) \( \gcd(a_{n_{k(j)}}^+,(k(j))!) = \gcd(a_{n_{k(j)}}^-(k(j))!) \),
(b) \( b_{n_{k(j)}}^+ - b_{n_{k(j)}}^- > \frac{2}{3} c \) \( \forall i, j \in \mathbb{N} \).

Now we consider the set
\[
\mathcal{M} = \{ (d,k(j)) \in \mathbb{N} \times \mathbb{N}, \ d \mid k(j)! \}
\]
of pairs of integers, together with a relation “\( \prec \)” defined for \( (d,k(j)), (d^*,k(j^*)) \in \mathcal{M} \) by
\[
(d,k(j)) \prec (d^*,k(j^*)) \iff j \leq j^* \quad \text{and} \quad \gcd(d^*,k(j^)) = d.
\]
This relation induces a partial ordering \( \prec \) on \( \mathcal{M} \).
We say that a pair \( (d,k(j)) \in \mathcal{M} \) is “evil”, if there are indices \( n_{k(j)}^+, n_{k(j)}^- \), so that (a) and (b) are true.

For any \( j \in \mathbb{N} \) the pair \( (d,k(j)) \) is “evil”, if \( d = (a_{n_{k(j)}}^+,k(j)!)) \). So we have shown that for every \( j \) there exists an “evil” pair \( (d,k(j)) \). And, if \( (d,k(j)) \prec (d^*,k(j+1)) \), and \( (d^*,k(j+1)) \) is “evil”, then \( (d,k(j)) \) is “evil”, too.\(^8\)

In the tree of “evil” pairs there is an infinite [totally ordered] branch \( (d_{k(j)},k(j)) \), \( j \in \mathbb{N} \). The reason is: for every pair \( (d_{k(j)},k(j)) \) having infinitely many “evil” successors, there is an “evil” pair \( (d_{k(j+1)},k(j+1)) \), which has infinitely many “evil” successors, too (see also the Lemma of D. König, [3], p. 381).

As described some lines before, to every pair \( (d_{k(j)},k(j)) \) from this infinite branch of “evil” pairs, there are indices \( n_{k(j)}^+, n_{k(j)}^- \), so that for all \( r \) satisfying \( r \leq k(j) \) we have
\[
\gcd(a_{n_{k(j)}}^+,r!) = \gcd(a_{n_{k(j)}}^-,r!).
\]

\(^8\) For every \( a \in \mathbb{N} \) the relation \( \gcd(a,(k(j+1))!) = d^* \) implies \( \gcd(a,(k(j))!) = \gcd(d^*,k(j)!). \) Since \( (d,k(j)) \prec (d^*,k(j+1)) \), the last \( \gcd \)-equation gives \( \gcd(a,k(j))! = d \). Then take \( n_{k(j)}^+ = n_{k(j+1)}^+ \), and \( n_{k(j)}^- = n_{k(j+1)}^- \).
In the special case $k(1) = 1$, $k(2) = 2$, \ldots the tree $(\mathcal{M}, \prec)$ looks like this:

\[\begin{array}{c}
\begin{array}{cccccccc}
\text{k = 5} \\
\text{k = 4} & (1,4) & (3,4) & (2,4) & (4,4) & (8,4) & (6,4) & (12,4) & (24,4) \\
\text{k = 3} & (1,3) & (3,3) & (2,3) & & (6,3) \\
\text{k = 2} & (1,2) & & & & (2,2) \\
\text{k = 1} & & & (1,1)
\end{array}
\end{array}\]

\textbf{Figure 1.} The tree $(\mathcal{M}, \prec)$ [in a special case]

We now distinguish three possible cases and obtain a contradiction in every of these cases.

1. Both of the sequences $\{n_{k(j)}^+\}_j$ and $\{n_{k(j)}^-\}_j$ contain infinitely many different elements.

Choose from every sequence a strictly increasing subsequence, form the union of these subsequences, and order this union to a strictly increasing sequence $\{n_k\}_{k \in \mathbb{N}}$. According to the definition of "evil", there are arbitrarily large indices $n_k, n_\ell$ with the property $b_{n_k} - b_{n_\ell} > \frac{1}{2}c$; in particular, $\{b_{n_k}\}_{k \in \mathbb{N}}$ is not a Cauchy-sequence.

On the other hand, the sequence $\{\gcd(a_{n_k}, r)!\}_{k \in \mathbb{N}}$ is eventually constant for any integer $r$. According to (4a) the sequence $\{b_{n_k}\}$ is convergent — a contradiction.

2. One of the two sequences, say $\{n_{k(j)}^+\}_j$ has infinitely many elements, the other only finitely many. Choose from $\{n_{k(j)}^+\}_j$ a strictly increasing subsequence $\{n_k\}_k$, and choose from $\{n_{k(j)}^-\}_j$ one value $m$, which occurs infinitely often. Thus, for any $r \in \mathbb{N}$ and for infinitely many $k$, say for
\[ k_1, k_2, \ldots, \] the relation \( \gcd(a_{n_{k_i}}, r!) = \gcd(a_m, r!) \) holds for \( i = 1, 2, \ldots; \)

according to (4), "in the case that \ldots" we obtain

\[ \lim_{i \to \infty} b_{n_{k_i}} = b_m. \]

This is a contradiction to the inequality \( b_{n_k} - b_m > \frac{1}{3}c \), which is valid for sufficiently large \( k \).

3. If both of the sequences \( \{n^+_{k(j)}\}_j \) and \( \{n^-_{k(j)}\}_j \) contain only finitely many elements, then choose from every sequence one value which occurs infinitely often, say \( n^+ \) and \( n^- \). Then

\[ \gcd(a_{n^+}, k(j)!) = \gcd(a_{n^-}, k(j)!) \text{ for every } j \in \mathbb{N}, \]

therefore \( a_{n^+} = a_{n^-} \) and \( n^+ = n^- \), contradicting \( b_{n^+} - b_{n^-} > \frac{1}{3}c \).

Thus we arrived at a contradiction in any of these three cases, and Theorem 3 is proved.

**Corollary.** Let \( \{b_n\}_n \) be a convergent sequence of complex numbers and \( \{a_n\}_n \) a strictly monotone sequence of positive integers, satisfying at least one of the following three properties:

\[ \alpha \) \( a_1 > 1, \) and the least prime factor \( p_{\text{min}}(a_n) \) of \( a_n \) tends to \( \infty \) (see [7], p. 155);

\[ \beta \) for all \( m < n \) the relation \( a_m \mid a_n \) is true (see [8], Satz 1.2);

\[ \gamma \) for every \( m < n \) the relation \( a_m \mid a_n \) holds.

Then there is a function \( f \in B^u \) with values \( f(a_n) = b_n \) for all \( n \in \mathbb{N} \).

**Proof.** For any of these three examples we have to check condition (4). Let \( \{n_k\}_k \) be a strictly increasing sequence of indices, for which the sequence \( \{\gcd(a_{n_k}, r!\}\}_k \) becomes eventually constant for every \( r \in \mathbb{N} \). The sequence \( \{b_{n_k}\}_k \), being a subsequence of a convergent sequence, is convergent.

We are going to show that the assumption in (4), "in the case that \ldots" does not occur for any of these three examples.

Assume that \( m \) is an index so that \( \gcd(a_{m}, r!) = \lim_{k \to \infty} \gcd(a_{n_k}, r!) \) for every \( r \in \mathbb{N} \).

\( \alpha \) Since, for any fixed \( r, \lim_{k \to \infty} \gcd(a_{n_k}, r!) = 1 \) on behalf of the condition \( p_{\text{min}}(a_n) \to \infty \), we conclude that \( \gcd(a_{m}, r!) = 1 \) for any \( r \), and so \( a_m = 1 \); but this is impossible.

\( \beta \) In the second case, for any \( p \mid a_m \), we choose an integer \( j \geq \max_{p \mid a_m} \) and a large \( k \) with the property

\[ \gcd(a_{m}, (p^j)!\) = \gcd(a_{n_k}, (p^j)!) \text{ for these primes } p \text{ dividing } a_m. \]
Then, for every $p \mid a_m$, we obtain

$$o_p(a_m) = \min\{o_p(a_m), o_p(p^j!)\} = \min\{o_p(a_{n_k}), o_p(p^j!)\} \leq o_p(a_{n_k}).$$

Therefore $a_m$ divides $a_{n_k}$, and so $n_k \leq m$ [by (β)]. For large $k$ this is a contradiction.

γ) In the third case the relation $a_{n_k} \mid a_{n_{k+1}}$ holds for any $k$, and so the sequence $\{o_p(a_{n_k})\}_k$ is increasing for any prime $p$. Since $a_{n_k} \to \infty$ as $k \to \infty$, the sequence $\{o_p(a_{n_k})\}_k$ is not bounded for at least one prime $p$. For this prime $p$ we obtain a contradiction to the inequality

$$\lim_{k \to \infty} \min \{o_p(a_{n_k}), o_p(p^j!)\} \leq o_p(a_m), \text{ for any } j \in \mathbb{N}.$$ 

Finally, without proof, we state an existence theorem for $D^u$.

**Theorem 4.** Let $\{a_n\}_{n \in \mathbb{N}}$ be a strictly increasing sequence of positive integers and $\{b_n\}_{n \in \mathbb{N}}$ a bounded sequence of complex numbers. Then the following two properties are equivalent:

8) There is a function $f \in D^u$ with values $f(a_n) = b_n$ for all $n \in \mathbb{N}$.

9) If $\{n_k\}_k$ is a strictly increasing sequence of positive integers, with the property, that for any $q \in \mathbb{N}$ there exists an integer $k_q \in \mathbb{N}$, so that $a_{n_k} \equiv a_{n_{k'}} \mod q$ for all $k, k' > k_q$, then

a) the corresponding sequence $\{b_{n_k}\}_k$ is convergent;

b) the limit $\lim_{k \to \infty} b_{n_k}$ equals $b_m$, if for all $q \in \mathbb{N}$ there is an integer $k_q, m \in \mathbb{N}$ satisfying $a_{n_k} \equiv a_m \mod q$ for all $k > k_q$.

The proof of this Theorem is similar to the proof of Theorem 3.

**Example.** If $f$ is in $D^u$, then the interpolation-problem $a_n = n, b_n = f(a_n)$ has the solution $f$ in $D^u$. Choosing a function $f$ not in $B^u$, then this problem does have a solution in $D^u$, but no solution in $B^u$.

**References**


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J.-C. Schlage-Puchta
Mathematisches Institut
Universität Freiburg
Eckerstraße 1
D-79104 Freiburg im Breisgau
Germany
jcp@arcade.mathematik.uni-freiburg.de

W. Schwarz
Fachbereich Mathematik
J.W. Goethe Universität
Robert Mayer Str. 10
D-60054 Frankfurt am Main
Germany
schwarz@math.uni-frankfurt.de

J. Spilker
Mathematisches Institut
Universität Freiburg
Eckerstraße 1
D-79104 Freiburg im Breisgau, Germany
Juergen.Spilker@math.uni-freiburg.de