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UNIFORMLY–ALMOST–EVEN FUNCTIONS WITH PRESCRIBED VALUES III.

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Dedicated to Karl-Heinz Indlekofer on the occasion of his 60th birthday

Abstract. Given integers $0 < a_1 < a_2 < \dots$ and bounded complex numbers $b_1, b_2, \dots$, we deal with the problem of the existence and uniqueness of a uniformly–almost–even function f satisfying

$$f(a_n) = b_n, \quad \text{for all} \quad n \in \mathbb{N}.$$

We give necessary and sufficient conditions that there exists at most or at least one function f with this interpolation property.

1. Introduction

A function $f : \mathbb{N} \to \mathbb{C}$ is called r-even, if the equation $f\big(\gcd(n,r)\big) = f(n)$ holds for all integers n; f is called *even*, abbreviated $f \in \mathcal{B}$, if there is some r for which f is r-even. The closure of $\mathcal{B}$ with respect to the *"uniform"* norm $\|f\|_u = \sup_{n\in\mathbb{N}} |f(n)|$ is the complex algebra $\mathcal{B}^u$ of *uniformly-almost-even functions*. Starting with the complex vector-space $\mathcal{D}$ of all periodic functions one obtains similarly the algebra $\mathcal{D}^u$ of *uniformly-almost-periodic functions* (see, for example, [7], IV.1).

In this note the following interpolation problem is dealt with: Let $\{a_n\}_n$ be a strictly increasing sequence of positive integers, and $\{b_n\}_n$ a bounded sequence of complex numbers; when does a uniformly-almost-even function f (resp. a uniformly-almost-periodic function) exist with values

$$(P) \qquad f(a_n) = b_n \quad \text{for} \quad n = 1, 2, \ldots ?$$

When is there at most one such function? [1]

Under more restrictive conditions the problem of the existence of such functions was already treated in [5] and [7], IV.5, Theorems 5.1 and 5.2. The authors used the fact that the Banach algebra $\mathcal{B}^u$ is isomorphic with the algebra of functions continuous on the compact space $\Delta_{\mathcal{B}}$ of maximal ideals, and this space was explicitly given,

$$\Delta_{\mathcal{B}} = \prod_p \{1, p^1, p^2, \ldots, p^\infty\},$$

where the factors are one-point compactifications of the discrete spaces $\{1, p^1, p^2, \ldots\}$, $p \in \mathbb{P}$. Later the second-named author tried to prove this result without using Gelfand's theory (see [6]). However, unfortunately there is a gap in this paper: in the proof that $\{g_{K_c}\}_{c\in\mathbb{N}}$ is a Cauchy-sequence, one case is missing. Schwarz & Spilker [8] used the method of [6] to prove other existence results under different assumptions.

In this paper we prove elementarily, without using Gelfand's theory, uniqueness results (Theorems 1 and 2, Section 2) and existence theorems (Theorems 3 and 4, Section 3).

Notations. $\mathbb{N} = \{1, 2, \ldots\}$ is the set of positive integers, $\mathbb{P} = \{2, 3, 5, \ldots\}$ the set of primes. For $n \in \mathbb{N}$, $p \in \mathbb{P}$, we denote by $\mathrm{o}_p(n)$ the order of p in the factorization of n, so that $p^{\mathrm{o}_p(n)} \mid n$, but $p^{\mathrm{o}_p(n)+1} \nmid n$.

2. Sets of uniqueness

In this section we deal with the (much simpler) problem of uniqueness in our interpolation problem (see equation (P)).

[1] Karl-Heinz Indlekofer investigated uniqueness sets for additive functions; as far as the second-named author remembers correctly, Indlekofer gave a talk on this subject already in Oberwolfach in the year 1978. He returned to this subject in joint papers (see [1], [2]) with Fehér, Stachó, and Timofeev.

Definition. A subset $A = \{a_n : n \in \mathbb{N}\}$ of $\mathbb{N}$ is called a *set of uniqueness* for $\mathcal{B}^u$, if the condition

$$\{f, g \in \mathcal{B}^u, \quad f(a_n) = g(a_n) \quad \text{for all} \quad n \in \mathbb{N}\}$$

implies $f = g$.

Sets of uniqueness for $\mathcal{B}^u$ are characterized by the following theorem.

Theorem 1. *The following properties of the set $A = \{a_n : n \in \mathbb{N}\} \subseteq \mathbb{N}$ are equivalent:*

(1) *A is a set of uniqueness for $\mathcal{B}^u$.*

(2) *For any integers $d, k \in \mathbb{N}$ satisfying $d \mid k!$ there exists an integer $n \in \mathbb{N}$ such that the greatest common divisor $\gcd(a_n, k!)$ equals d.*

Proof.

(**1**) $\Rightarrow$ (**2**): Let $\{a_n : n \in \mathbb{N}\}$ be a set of uniqueness, and let $d, k \in \mathbb{N}$ satisfy $d \mid k!$. Define a $k\,!$-even function $f_1(n)$ for $n \mid k!$ by

$$f_1(n) = \begin{cases} 0, & \text{if } n \mid k!,\ n \neq d, \\ 1, & \text{if } n = d, \end{cases}$$

and, for $n \in \mathbb{N}$, by $f_1(n) = f_1\big(\gcd(n, k!)\big)$. If there were no $n \in \mathbb{N}$ satisfying $\gcd(a_n, k\,!) = d$, then there would be two different solutions f_1 and $f_2 = 0$ for the interpolation problem $f(a_n) = 0$, a contradiction to (1).

(**2**) $\Rightarrow$ (**1**): Assume that there is a function $f \in \mathcal{B}^u$, $f \neq 0$, satisfying $f(a_n) = 0$ for any $n \in \mathbb{N}$. Fix an integer d such that $f(d) \neq 0$, and choose a large k, $k \geq d$, and a $k!$-even function h satisfying $\|f - h\|_u < \frac{1}{2} \cdot |f(d)|$. Because of (2) there is an integer n so that $\gcd(a_n, k\,!) = d$, and so $h(a_n) = h(d)$. This gives the contradiction

$$|f(d)| \leq |f(d) - h(d)| + |h(a_n) - f(a_n)| \leq 2 \cdot \|f - h\|_u < |f(d)|.$$

Examples. The set $(\mathbb{P} + 1) \cup (\mathbb{P} + 2)$, the union of two sets of shifted primes, is a set of uniqueness for $\mathcal{B}^u$.

We verify condition (2). Let positive integers $d, r \in \mathbb{N}$, $d \mid r$ be given. [2]

a) If d is *even*, denote by π a prime $\pi \mid r$. Then the integer

$$n_\pi = \pi^{o_\pi(d)} - 1, \quad \text{if} \quad \pi \mid d, \qquad n_\pi = \pi + 1, \quad \text{if} \quad \pi \nmid d,$$

[2] It would be sufficient to restrict ourselves to numbers r of the form $r = k\,!$.

satisfies

$$\mathrm{o}_\pi(n_\pi) = 0, \quad \mathrm{o}_\pi(n_\pi + 1) = \mathrm{o}_\pi(d).$$

Any solution $n \in \mathbb{N}$ of the system of congruences

$$n \equiv n_\pi \mod \pi^{\mathrm{o}_\pi(r)+1} \quad \text{for every} \quad \pi \mid r$$

satisfies

$$\gcd(n, r) = 1, \quad \gcd(n + 1, r) = d.$$

By the prime number theorem for arithmetic progressions there exists a prime $p \equiv n \mod r$, and for this prime we get $\gcd(p + 1, r) = d$.

b) If d is *odd*, we find a prime p satisfying $\gcd(p + 2, r) = d$, in a similar manner.

The set $(\mathbb{P} + 1) \cup (2\mathbb{P} + 1)$ is also a set of uniqueness for $\mathcal{B}^u$.

The sets $\mathbb{P} + a$, where $a \in \mathbb{N}_0$, the set of squares, the set of squarefree numbers, the set of k-free numbers, the set of factorials and the set of powers of an integer $a \in \mathbb{N}$ are not sets of uniqueness for $\mathcal{B}^u$.

Without proof we give the corresponding result for $\mathcal{D}^u$.

Theorem 2. *A set $\mathcal{A} = \{a_n : n \in \mathbb{N}\} \subseteq \mathbb{N}$ is a set of uniqueness for $\mathcal{D}^u$, if and only if for any $d, r \in \mathbb{N}$, $d \leq r$ there exists an integer n so that $a_n \equiv d \bmod r$.*

Examples. Any strictly monotone sequence $\mathcal{A}$ of integers a_n, which is uniformly distributed modulo r for any $r \in \mathbb{N}$, is a set of uniqueness for $\mathcal{D}^u$. In particular[3]

– every set $\mathcal{A} \subseteq \mathbb{N}$ with density 1,
– the set $\{a_n,\ a_n = [P(n)]\}$, where $P(x)$ is a polynomial in $\mathbb{R}[x]$, and $P(x) - P(0)$ has at least one irrational coefficient,[4]
– the set $a_n = [n^c]$, where $c > 0$, $c \notin \mathbb{Z}$.[5]

The set $(\mathbb{P}+1) \cup (\mathbb{P}+2)$ is not a set of uniqueness for $\mathcal{D}^u$, being disjoint to the residue class 11 mod 30. Also, the sets $\bigcup\limits_{1 \leq n \leq N} (\alpha_n \mathbb{P} + \beta_n)$, $\alpha_n \in \mathbb{N}$, $\beta_n \in \mathbb{N} \cup \{0\}$, the set of B-numbers (a_n is a B-number, if it is representable as a sum of two squares of integers)[6] are not sets of uniqueness for $\mathcal{D}^u$.

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[3] For the definition and simple properties of uniform distribution modulo r, see, for example, Kuipers & Niederreiter [4], Chapter 5, p. 305ff.

[4] See [4], Theorem 1.4, p. 307.

[5] See [4], Exercise 1.10, p. 318.

[6] B-numbers are easily characterized by conditions concerning prime factors $p \equiv 3 \bmod 4$.

3. Existence theorems

Given *finitely* many integers $a_1, a_2, \dots, a_N$ and complex numbers $b_1, b_2, \dots, b_N$, then there is an *even* function $f \in \mathcal{B}$ assuming the values $f(a_n) = b_n$ for $1 \le n \le N$: write $\alpha = a_1 a_2 \cdots a_N$, and define for all divisors $a \mid \alpha$ and for $n \in \mathbb{N}$,

$$f_a(n) = \begin{cases} 1, & \text{if } (n, \alpha) = a, \\ 0, & \text{if } (n, \alpha) \neq a. \end{cases}$$

Then $f = \sum_{1 \le n \le N} b_n \cdot f_{a_n}$ is such a function. By the way (see [6]),

$$f_a(n) = \frac{\varphi(\frac{\alpha}{a})}{\alpha} \sum_{r \mid \alpha} \frac{c_r(a)}{\varphi(r)}\, c_r(n),$$

where $c_r(n) = \sum_{d \mid (r,n)} d\mu\left(\frac{r}{d}\right)$ is the Ramanujan-sum. So, we are only concerned with *infinite* subsets of $\mathbb{N}$.

Theorem 3. *For a strictly increasing sequence $\{a_n\}_{n \in \mathbb{N}}$ of positive integers and a bounded sequence $\{b_n\}_{n \in \mathbb{N}}$ of complex numbers the following two conditions (3) and (4) are equivalent.*

(3) *There exists a function $f \in \mathcal{B}^u$ with the values $f(a_n) = b_n$ $(n \in \mathbb{N})$.*

(4) *If $\{n_k\}_{k \in \mathbb{N}}$ is any strictly increasing sequence of positive integers such that for any $r \in \mathbb{N}$ the sequence $\{\gcd(a_{n_k}, r\,!)\}_{k \in \mathbb{N}}$ is eventually constant, then the limit*

$$\lim_{k \to \infty} b_{n_k} \quad \textit{exists},$$

and, in the case that, with some integer m [not depending on r],

$$\lim_{k \to \infty} \gcd(a_{n_k}, r\,!) = \gcd(a_m, r\,!)$$

for every r, its value is b_m.

Before proving Theorem 3 we reformulate the conditions concerning $\mathrm{o}_p(a_{n_k})$.

Lemma. *For any sequence $\{m_k\}_{k \in \mathbb{N}}$ of positive integers the following results are true.*

(5) *Properties (5a) and (5b) are equivalent.*

(5a) *For every $r \in \mathbb{N}$ the sequence $\{\gcd(m_k, r\,!)\}_{k \in \mathbb{N}}$ is eventually constant.*

(5b) *For every prime p the sequence $\{\mathrm{o}_p(m_k)\}_{k\in\mathbb{N}}$ is eventually constant or tends to infinity.*

(6) *Properties (6a) and (6b) are equivalent.*

(6a) *For every $r\in\mathbb{N}$ the sequence $\{\gcd(m_k,r!)\}_{k\in\mathbb{N}}$ is eventually constant, and there exists an integer $m\in\mathbb{N}$ so that for every r the relation $\gcd(m,r!)=\lim\limits_{k\to\infty}\gcd(m_k,r!)$ holds.*

(6b) *For every prime p the sequence $\{\mathrm{o}_p(m_k)\}_{k\in\mathbb{N}}$ is eventually constant and $\lim\limits_{k\to\infty}\mathrm{o}_p(m_k)\neq 0$ for at most finitely many primes p.*

Proof.

(**5a**) $\Rightarrow$ (**5b**): Let (5a) hold for the sequence $\{m_k\}_k$, and let p be a prime. For any $j\in\mathbb{N}$ there is some $k_j\in\mathbb{N}$ so that $\min\{\mathrm{o}_p(m_k),\mathrm{o}_p((p^j)!)\}$ does not depend on k, if $k>k_j$; say, this minimum is e_j.

If $e_j<\mathrm{o}_p(p^j!)$ for some $j\in\mathbb{N}$, then the sequence $\{\mathrm{o}_p(m_k)\}_k$ is eventually constant.

If $e_j=\mathrm{o}_p(p^j!)$ for every $j\in\mathbb{N}$, then the sequence $\{\mathrm{o}_p(m_k)\}_k$ tends to ∞.

(**5b**) $\Rightarrow$ (**5a**): Fix $r\in\mathbb{N}$. By (5b) there is some integer k_0 with the property, that the sequence $\{\mathrm{o}_p m_k\}_{k>k_0}$ is constant for all primes $p\le r$, or there is some prime $p\le r$ such that $\mathrm{o}_p(m_k)>\mathrm{o}_p(r!)$ for every $k>k_0$. Thus $\min\{\mathrm{o}_p m_k,\mathrm{o}_p(r!)\}$ is independent of $k>k_0$ [there is no prime $p>r$ dividing $r!$], and therefore the sequence $\{\gcd(m_k,r!)\}_{k>k_0}$ is constant.

(**6a**) $\Rightarrow$ (**6b**): Assume that condition (6a) is true for the sequence $\{m_k\}_k$. By (6a) there is an integer m so that for any prime p

$$\min\{\mathrm{o}_p(m),\mathrm{o}_p(r!)\}=\lim_{k\to\infty}\min\{\mathrm{o}_p(m_k),\mathrm{o}_p(r!)\}.$$

According to (5), the sequence $\{\mathrm{o}_p(m_k)\}_k$ is eventually constant or its limit [for $k\to\infty$] is ∞. Put $r=p^j$, where $j>\mathrm{o}_p(m)$; then $\mathrm{o}_p(m)\ge\mathrm{o}_p(m_k)$, if k is large; therefore the case $\lim\limits_{k\to\infty}\mathrm{o}_p(m_k)=\infty$ is impossible. If $p>m$, then $\mathrm{o}_p(m)=0$, and so $\lim\limits_{k\to\infty}\mathrm{o}_p(m_k)=0$.

(**6b**) $\Rightarrow$ (**6a**). Assume that for the sequence $\{m_k\}_k$ condition (6b) is true. Given $r\in\mathbb{N}$, the sequence $\{\gcd(m_k,r!)\}_k$ is eventually constant, by $(5b)\Rightarrow$ $\Rightarrow(5a)$. Write $e_p=\lim\limits_{k\to\infty}\gcd(m_k,r!)$. The number $m=\prod\limits_p p^{e_p}$ is well-defined by (6b), and, for every $r\in\mathbb{N}$,

$$\gcd(m,r!)=\prod_p p^{\min\{e_p,\mathrm{o}_p(r!)\}}=\lim_{k\to\infty}\prod_p p^{\min\{\mathrm{o}_p(m_k),\mathrm{o}_p(r!)\}}=\lim_{k\to\infty}\gcd(m_k,r!).$$

Thus the Lemma is proved.

Proof of Theorem 3.

(**3**) $\Rightarrow$ (**4**). Let a strictly increasing sequence $\{a_n\}$ of positive integers, a bounded sequence $\{b_n\}$ of complex numbers, and a function $f \in \mathcal{B}^u$ be given satisfying the interpolation-property $f(a_n) = b_n$; take a strictly monotone sequence $\{n_k\}_k$ in $\mathbb{N}$, so that

for every $r \in \mathbb{N}$ the sequence $\{\gcd(a_{n_k}, r!)\}_{k\in\mathbb{N}}$ is eventually constant.

$f \in \mathcal{B}^u$ implies that for any given $\varepsilon > 0$ there is some $s \in \mathbb{N}$ and an (s)-even function h approximating f, so that $\|f - h\|_u < \frac{1}{4}\varepsilon$.

a) There is some $k_0 \in \mathbb{N}$, $k_0 = k_0(\varepsilon)$ so that for all $k, \ell > k_0$ the relation $\gcd(a_{n_k}, s!) = \gcd(a_{n_\ell}, s!)$ holds, and so $h(a_{n_k}) = h(a_{n_\ell})$. Therefore we obtain for every $k, \ell > k_0$:

$$|b_{nk} - b_{n_\ell}| \overset{by\ (3)}{=} |f(a_{nk}) - f(a_{n_\ell})| \le$$

$$\le |f(ank) - h(a_{nk})| + |f(a_{n_\ell}) - h(a_{n_\ell})| \le 2 \cdot \|f - h\|_u < \frac{1}{2}\varepsilon,$$

and so $\{b_{nk}\}_k$ is a Cauchy-sequence, and thus convergent.

b) Now, we take for granted that in addition (with some integer m)

$$\gcd(a_m, r!) = \lim_{k\to\infty} \gcd\bigl(a_{nk}, r!\bigr), \quad \text{for every} \ \ r \in \mathbb{N}.$$

Note that $f(a_m) = b_m$, and that the sequence $\{\gcd(a_{nk}, r!)\}_{k\in\mathbb{N}}$ is eventually constant; thus there is some $\ell_0 > k_0$ $[\ell_0 = \ell_0(s)$, and so ℓ_0 depends on $\varepsilon]$ with the property that for any $\ell > \ell_0$

$$\gcd(a_m, s!) = \gcd(a_{n_\ell}, s!), \quad \text{and so, in particular,} \ \ h(a_m) = h(a_{n_\ell}).$$

Therefore we obtain, with some $\ell > \ell_0$,

$$|b_m - \lim_{k\to\infty} b_{nk}| \le |b_m - b_{n_\ell}| + \left|\lim_{k\to\infty} b_{nk} - b_{n_\ell}\right|,$$

and by the inequalities in a) this is

$$\le |f(a_m) - f(a_{n_\ell})| + \frac{1}{2}\varepsilon \le |f(a_m) - h(a_m)| + |f(a_{n_\ell}) - h(a_{n_\ell})| + \frac{1}{2}\varepsilon \le$$

$$\le 2 \cdot \|f - h\|_u + \frac{1}{2}\varepsilon < \varepsilon,$$

and so $\lim\limits_{k\to\infty} b_{n_k} = b_m$.

Now we come to the more difficult part, the proof of the implication.

(**4**) $\Rightarrow$ (**3**). Given sequences $\{a_n\}$ and $\{b_n\}$ as in the theorem; without loss of generality we may assume that the b_n are non-negative real numbers. We have to find a function $f \in \mathcal{B}^u$, so that $f(a_n) = b_n$ for every n.

Define for any positive integers n and k *satisfying* $n \mid k!$ the set

$$M(n,k) := \{m \in \mathbb{N} : \ \gcd(a_m, k!) = n\} = \\ = \left\{ m \in \mathbb{N} : \ a_m \equiv 0 \mod n, \quad \text{and} \quad \gcd\left(\frac{a_m}{n}, \frac{k!}{n}\right) = 1 \right\}.$$

The set $M(n,k)$ is empty if and only if $\gcd(a_m, k!) = n$ is impossible for any m; in particular, if n does not divide any a_m, then $M(n,k) = \emptyset$. We define two $k!$-even functions f_k^+ and f_k^-, first for integers $n \mid k!$, by

$$f_k^+(n) = \begin{cases} \sup\left\{b_m : \ m \in M\big(n,k\big)\right\}, & \text{if} \quad M\big(n,k\big) \neq \emptyset, \\ 0, & \text{if} \quad M\big(n,k\big) = \emptyset, \end{cases}$$

and similarly $f_k^-(n)$, replacing "sup" with "inf", and then obtain $k!$-even functions by the definition

$$f_k^{\pm}(n) = f_k^{\pm}\big(\gcd(n,k!)\big) \quad \text{for any } n \in \mathbb{N}.$$

So,

$$f_k^+(n) = \sup\left\{b_m : \ m \in M\big((n,k!),k\big)\right\}, \quad \text{if} \quad M\big((n,k)\big) \neq \emptyset, \quad \text{otherwise} = 0.$$

It is sufficient to show the equation

$$\lim_{k\to\infty} \|f_k^+ - f_k^-\|_u = 0. \tag{7}$$

The reasons are:

(α) For any $k, n \in \mathbb{N}$ the inequalities

$$f_k^-(n) \le f_{k+1}^-(n) \le f_{k+1}^+(n) \le f_k^+(n)$$

hold. [This implies that $\|f_k^+ - f_k^-\|_u$ is decreasing.]

Without loss of generality $n \mid (k+1)!$. On behalf of $\big[\gcd(a_m,(k+1)!) = n$ implies $\gcd(a_m,k!) = \gcd(n,k!)\,\big]$ we obtain

$$M(n,k+1) \subseteq M\big(\gcd(n,k!),k\big),$$

and this gives the first and last inequality.

(β) The sequence $(f_k^+)_{k\in\mathbb{N}}$ is a Cauchy-sequence in $\mathcal{B}^u$, because of (see (α))

$$\|f_k^+ - f_{k+\ell}^+\|_u \le \|f_k^+ - f_k^-\|_u \text{ for any } k,\ell \in \mathbb{N}.$$

The space $(\mathcal{B}^u, \|\cdot\|_u)$ is complete, therefore

$$f = \lim_{k\to\infty} f_k^+ \text{ exists and is in } \mathcal{B}^u.$$

(γ) The function f defined in (β) does interpolate the prescribed values b_n:

If $k \ge a_n$, then $n \in M(a_n,k)$, therefore $f_k^-(a_n) \le b_n \le f_k^+(a_n)$ [by the definition of f_k^-, f_k^+], and so

$$f(a_n) \overset{(\beta)}{=} \lim_{k\to\infty} f_k^+(a_n) = b_n,$$

[by (7) and the inequalities $f_k^-(a_n) \le b_n \le f_k^+(a_n)$].

So it remains to proof equation (7), $\|f_k^+ - f_k^-\| \to 0$, as $k \to \infty$.

Assume that (7) is wrong. Since the sequence $\big\{\|f_k^+ - f_k^-\|_u\big\}_{k\in\mathbb{N}}$ is decreasing [see (α)], there is some $c > 0$ so that $\|f_k^+ - f_k^-\|_u > c$ for all $k \in \mathbb{N}$. Therefore, for every $k \in \mathbb{N}$ there is some integer $\nu = \nu(k)$ for which $f_k^+(\nu) - f_k^-(\nu) > c$.

By the definition of $f_k^\pm$, for every k there exist integers n_k^+ and n_k^- in $M\big(\gcd(\nu,k!),k\big)$ with the properties

(a) $\gcd(a_{n_k^+},k!) = \gcd(a_{n_k^-},k!) \quad \big[= \gcd(\nu,k!)\,\big]$,

and

(b) $b_{n_k^+} - b_{n_k^-} > c$.

The sequence $\{b_n\}_n$ is bounded; therefore there is[7] a constant b such that for some increasing subsequence $\{k(j)\}_j$ the inequalities

$$b_{n_{k(j)}^-} < b - \frac{1}{3}c < b + \frac{1}{3}c < b_{n_{k(j)}^+}$$

[7] For b, one may take, for example, a point of accumulation of the sequence $\big\{\frac{1}{2}\big(bn_k^+ + bn_k^-\big)\big\}_k$.

hold for every $j \in \mathbb{N}$. It follows that

$$b_{n^+_{k(j)}} - b_{n^-_{k(i)}} > \frac{2}{3}c \quad \text{for any} \quad i, j \in \mathbb{N}.$$

So we got a sequence $k(1) < k(2) < \ldots$ of integers and integers $n^+_{k(j)},\ n^-_{k(j)}$ satisfying

(a) $\quad \gcd(a_{n^+_{k(j)}}, (k(j))!) = \gcd(an^-_{k(j)}, (k(j))!)$,

and

(b') $\quad b_{n^+_{k(j)}} - b_{n^-_{k(i)}} > \dfrac{2}{3}c \quad \forall i, j \in \mathbb{N}.$

Now we consider the set

$$\mathcal{M} = \big\{(d, k(j)) \in \mathbb{N} \times \mathbb{N},\ d \mid k(j)!\ \big\}$$

of pairs of integers, together with a relation "$\prec$" defined for $(d, k(j))$, $(d^*, k(j^*)) \in \mathcal{M}$ by

$$(d, k(j)) \prec (d^*, k(j^*)) \Longleftrightarrow j \leq j^* \ \text{ and } \ \gcd\big(d^*, k(j)!\big) = d.$$

This relation induces a partial ordering $\prec$ on $\mathcal{M}$.

We say that a pair $(d, k(j)) \in \mathcal{M}$ is **"evil"**, if there are indices $n^+_{k(j)},\ n^-_{k(j)}$, so that (a) and (b') are true.

For any $j \in \mathbb{N}$ the pair $(d, k(j))$ is "evil", if $d = \Big(a_{n^+_{k(j)}}, k(j)!\Big)$. So we have shown that for every j there exists an "evil" pair $(d, k(j))$. And, if $(d, k(j)) \prec$ $\prec (d^*, k(j+1))$, and $(d^*, k(j+1))$ is "evil", then $(d, k(j))$ is "evil", too.[8]

In the tree of "evil" pairs there is an infinite [totally ordered] branch $\big(d_{k(j)}, k(j)\big)_{j\in\mathbb{N}}$. The reason is: for every pair $(d_{k(j)}, k(j))$ having infinitely many "evil" successors, there is an "evil" pair $(d_{k(j+1)}, k(j+1)) \succ (d_{k(j)}, k(j))$, which has infinitely many "evil" successors, too (see also the Lemma of D. König, [3], p. 381).

As described some lines before, to every pair $(d_{k(j)}, k(j))$ from this infinite branch of "evil" pairs, there are indices $n^+_{k(j)}, n^-_{k(j)}$, so that for all r satisfying $r \leq k(j)$ we have

$$\gcd(a_{n^+_{k(j)}}, r!) = \gcd(a_{n^-_k(j)}, r!).$$

[8] For every $a \in \mathbb{N}$ the relation $\gcd(a, (k(j+1)!) = d^*$ implies $\gcd(a, k(j)!) =$ $= \gcd(d^*, k(j)!)$. Since $(d, k(j)) \prec (d^*, k(j+1))$, the last gcd-equation gives $\gcd(a, k(j)!) = d$. Then take $n^+_{k(j)} = n^+_{k(j+1)}$, and $n^-_{k(j)} = n^-_{k(j+1)}$.

In the special case $k(1) = 1,\ k(2) = 2, \ldots$ the tree $(\mathcal{M}, \prec)$ looks like this:

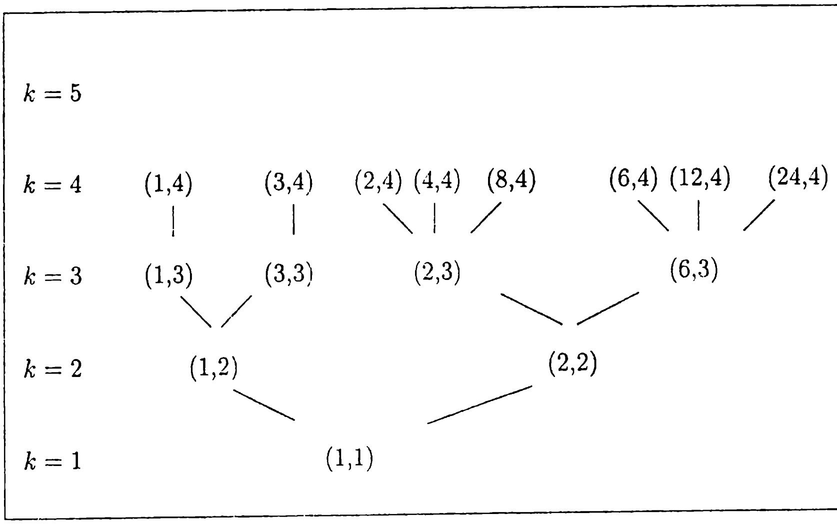

Figure 1. The tree $(\mathcal{M}, \prec)$ [in a special case]

We now distinguish three possible cases and obtain a contradiction in every of these cases.

1. Both of the sequences $\{n^+_{k(j)}\}_j$ and $\{n^-_{k(j)}\}_j$ contain infinitely many different elements.

   Choose from every sequence a strictly increasing subsequence, form the union of these subsequences, and order this union to a strictly increasing sequence $\{n_k\}_{k\in\mathbb{N}}$. According to the definition of "evil", there are arbitrarily large indices n_k, n_ℓ with the property $b_{n_k} - b_{n_\ell} > \frac{1}{3}c$; in particular, $\{b_{n_k}\}_{k\in\mathbb{N}}$ is not a Cauchy-sequence.

   On the other hand, the sequence $\{\gcd(a_{n_k}, r!)\}_{k\in\mathbb{N}}$ is eventually constant for any integer r. According to (4a) the sequence $\{b_{n_k}\}$ is convergent — a contradiction.

2. One of the two sequences, say $\{n^+_{k(j)}\}_j$ has infinitely many elements, the other only finitely many. Choose from $\{n^+_{k(j)}\}_j$ a strictly increasing subsequence $\{n_k\}_k$, and choose from $\{n^-_{k(j)}\}_j$ one value m, which occurs infinitely often. Thus, for any $r \in \mathbb{N}$ and for infinitely many k, say for

$k_1, k_2, \ldots$, the relation $\gcd(a_{n_{k_i}}, r!) = \gcd(a_m, r!)$ holds for $i = 1, 2, \ldots$; according to (4), *"in the case that ... "* we obtain

$$\lim_{i\to\infty} b_{n_{k_i}} = b_m.$$

This is a contradiction to the inequality $b_{nk} - b_m > \frac{1}{3}c$, which is valid for sufficiently large k.

3. If both of the sequences $\{n^+_{k(j)}\}_j$ and $\{n^-_{k(j)}\}_j$ contain only finitely many elements, then choose from every sequence one value which occurs infinitely often, say n^+ and n^-. Then

$$\gcd(a_{n^+}, k(j)!) = \gcd(a_{n^-}, k(j)!) \quad \text{for every} \quad j \in \mathbb{N},$$

therefore $a_{n^+} = a_{n^-}$ and $n^+ = n^-$, contradicting $b_{n^+} - b_{n^-} > \frac{1}{3}c$.

Thus we arrived at a contradiction in any of these three cases, and Theorem 3 is proved.

Corollary. *Let $\{b_n\}_n$ be a convergent sequence of complex numbers and $\{a_n\}_n$ a strictly monotone sequence of positive integers, satisfying at least one of the following three properties:*

α) $a_1 > 1$, and the least prime factor $p_{\min}(a_n)$ of a_n tends to ∞ (see [7], p. 155*);*

β) for all $m < n$ the relation $a_m \nmid a_n$ is true (see [8], Satz 1.2*);*

γ) for every $m < n$ the relation $a_m \mid a_n$ holds.

Then there is a function $f \in \mathcal{B}^u$ with values $f(a_n) = b_n$ for all $n \in \mathbb{N}$.

Proof. For any of these three examples we have to check condition (4). Let $\{n_k\}_k$ be a strictly increasing sequence of indices, for which the sequence $\{\gcd(a_{n_k}, r!)\}_k$ becomes eventually constant for every $r \in \mathbb{N}$. The sequence $\{b_{nk}\}_k$, being a subsequence of a convergent sequence, is convergent.

We are going to show that the assumption in (4), *" in the case that ... "* does not occur for any of these three examples.

Assume that m is an index so that $\gcd(a_m, r!) = \lim_{k\to\infty} \gcd(a_{nk}, r!)$ for every $r \in \mathbb{N}$.

α) Since, for any fixed r, $\lim_{k\to\infty} \gcd(a_{nk}, r!) = 1$ on behalf of the condition $p_{\min}(a_n) \to \infty$, we conclude that $\gcd(a_m, r!) = 1$ for any r, and so $a_m = 1$; but this is impossible.

β) In the second case, for any $p \mid a_m$, we choose an integer $j \geq \max_{p\mid a_m} \mathrm{o}_p(a_m)$ and a large k with the property

$$\gcd(a_m, (p^j)!) = \gcd(a_{nk}, (p^j)!) \text{ for these primes } p \text{ dividing } a_m.$$

Then, for every $p \mid a_m$, we obtain

$$\mathrm{o}_p(a_m) = \min\{\mathrm{o}_p(a_m), \mathrm{o}_p(p^j!)\} = \min\{\mathrm{o}_p(a_{nk}), \mathrm{o}_p(p^j!)\} \leq \mathrm{o}_p(a_{nk}).$$

Therefore a_m divides a_{nk}, and so $n_k \leq m$ [by (β)]. For large k this is a contradiction.

γ) In the third case the relation $a_{nk} \mid a_{n_{k+1}}$ holds for any k, and so the sequence $\{\mathrm{o}_p(a_{nk})\}_k$ is increasing for any prime p. Since $a_{nk} \to \infty$ as $k \to \infty$, the sequence $\{\mathrm{o}_p(a_{nk})\}_k$ is not bounded for at least one prime p. For this prime p we obtain a contradiction to the inequality

$$\lim_{k\to\infty} \min\left\{\mathrm{o}_p(a_{nk}), \mathrm{o}_p(p^j!)\right\} \leq \mathrm{o}_p(a_m), \text{ for any } j \in \mathbb{N}.$$

Finally, without proof, we state an existence theorem for $\mathcal{D}^u$.

Theorem 4. *Let $\{a_n\}_{n\in\mathbb{N}}$ be a strictly increasing sequence of positive integers and $\{b_n\}_{n\in\mathbb{N}}$ a bounded sequence of complex numbers. Then the following two properties are equivalent:*

(8) There is a function $f \in \mathcal{D}^u$ with values $f(a_n) = b_n$ for all $n \in \mathbb{N}$.

(9) If $\{n_k\}_k$ is a strictly increasing sequence of positive integers, with the property, that for any $q \in \mathbb{N}$ there exists an integer $k_q \in \mathbb{N}$, so that $a_{n_k} \equiv a_{n_{k'}} \mod q$ for all $k, k' > k_q$, then

a) the corresponding sequence $\{b_{n_k}\}_k$ is convergent;

b) the limit $\lim_{k\to\infty} b_{n_k}$ equals b_m, if for all $q \in \mathbb{N}$ there is are integers $k_q,\ m \in \mathbb{N}$ satisfying $a_{n_k} \equiv a_m \mod q$ for all $k > k_q$.

The proof of this Theorem is similar to the proof of Theorem 3.

Example. If f is in $\mathcal{D}^u$, then the interpolation-problem $a_n = n$, $b_n = f(a_n)$ has the solution f in $\mathcal{D}^u$. Choosing a function f not in $\mathcal{B}^u$, then this problem does have a solution in $\mathcal{D}^u$, but no solution in $\mathcal{B}^u$.

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