Annales Univ. Sci. Budapest., Sect. Comp. 22 (2003) 299-315

DISCRETE APPROXIMATION ON THE SPHERE

M. Pap (Pécs, Hungary)
F. Schipp (Budapest, Hungary)

Dedicated to Professor Karl-Heinz Indlekofer on his 60th birthday

Abstract. In this paper we construct continuous and discrete approximation processes on the sphere S^2 using spherical functions. These processes are generated by summations of Laplace-Fourier series. Replacing the continuous invariant measure on the sphere by a convenient discrete measure we get a discrete approximation process closely connected with the continuous approximation. It turns out that the surface-measure can be obtained as the limit of the discrete measure in question. Among others it will be proved that the discrete and the continuous L^p norms are equivalent on the space of corresponding spherical functions. This is an analogue of Marcinkiewicz-theorem regarding to the equivalence of the discrete and continuous L^p norms for trigonometric polynomials. The clue of proof is the de la Valée-Poussin type summmation method for spherical functions.

1. Introduction

Spherical harmonics play an inportant role in Fourier analysis. They are restrictions to the sphere $x_1^2 + x_2^2 + ... + x_n^2 = 1$ of homogeneous polynomials that are solutions of the n-dimensional Laplace equation (see [3, p.445]).

There is another way to introduce them, namely for example in the case $n = 3$ they are matrix elements t^{ℓ}_{jk} of unitary irreducibile representations of the matrix group $SU(2)$ ([7, p.278]) (see (1.2)).

For $k = 0$ we obtain the classical spherical functions. The functions $\{\sqrt{2\ell+1}t^{\ell}_{j0} : \ell = 0, 1, ..., -\ell \le j \le \ell\}$ constitute an orthonormal system with respect to the invariant measure (given by formula (1.6)) on three dimensional unit sphere S^2 and the corresponding Fourier series is convergent in $L^2(S^2)$.

Taking the classical spherical functions we will construct an approximation process with the nodal points defined by (2.1). We will show that the approximation function can be written as a discrete integral (see (2.6)). The advantage of the integral notation is that it brings to light the formal similarity between the n-th approximation process of the function f defined on S^2 and the n-th partial sum of the Laplace-Fourier series (1.8).

It is natural to ask for conditions under which the spherical approximation process will tend to f?

In the one variable case there is a rich bibliography where the approximation properties of algebraic and trigonometric polynomials are studied, for example P. Erdős and P. Turán in [4], R. Askey in [2], G, Szegő in [5] etc. From the point of view of the numerical method the discrete approximation process plays an important role. An example of such a discrete process is the trigonometric interpolation studied in Zygmund's book [6, vol.II]. Another method which is based on the discrete $\theta-$summation is applied in control theory (see [9]).

In this paper we construct continuous and discrete approximation processes on the sphere S^2 using spherical functions. These processes are generated by summations of Laplace-Fourier series. Replacing the continuous invariant measure on the sphere by a convenient discrete measure we get a discrete approximation process closely connected with the continuous approximation. It turns out that the surface-measure can be obtained as the limit of the discrete measure in question. Among others it will be proved that the discrete and the continuous L^p norms are equivalent on the space of corresponding spherical functions. This is an analogue of Marcinkiewicz-theorem regarding to the equivalence of the discrete and continuous L^p norms for trigonometric polynomials. The clue of the proof is the de la Valée-Poussin type summmation method for spherical functions.

2. Spherical functions

In this section we will summarize some results connected with spherical functions. The spherical functions can be expressed by irreductible representations of the group $SU[2]$, where

$$SU(2) = \{g \in SL(2) : g^* = g^{-1}\}$$

is the set of second order unitary matrices. If $g \in SU(2)$, then it can be represented in the following form:

$$g = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}, \quad |\alpha|^2 + |\beta|^2 = 1, \quad \alpha, \beta \in \mathbb{C}.$$

Every element from $SU(2)$ can be represented with the so called Euler angles, namely there exist $\theta \in (0,\pi), \varphi \in [0, 2\pi), \psi \in [-2\pi, 2\pi)$ so that:

$$g = \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix} \begin{pmatrix} \cos(\theta/2) & i\sin(\theta/2) \\ i\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{-i\psi/2} \end{pmatrix} :=$$

$$:= k(\varphi)a(\theta)k(\psi),$$

where $|\alpha| = \cos(\theta/2)$, $\mathrm{Arg}\alpha = (\varphi+\psi)/2$, $\mathrm{Arg}\beta = (\varphi - \psi + \pi)/2$.

Let $\ell \in \mathbb{N}$ and $I_\ell := \{-\ell, -\ell+1, ..., \ell\}$. Denote by X^ℓ the space of all homogeneous complex polynomials of degree 2ℓ in two variables. This is $2\ell+1$ dimensional and is spanned by the polynomials

$$e_k^\ell(z_1, z_2) = \frac{z_1^{\ell+k} z_2^{\ell-k}}{\sqrt{(\ell+k)!(\ell-k)!}},$$

where $k \in I_\ell$.

Denote by

$$[t_{jk}^\ell]_{j,k\in I_\ell} = T^\ell$$

the matrix of this representation regarding to the base $\{e_k^\ell : k \in I_\ell\}$ of the group $SU[2]$.

If g has the form $g(\theta) = a(\theta)$ let define

$$P_{jk}^\ell(\cos\theta) := t_{jk}^\ell(a(\theta)) = \tag{2.1}$$

$$= \sqrt{\frac{(\ell-j)!}{(\ell+k)!(\ell-k)!(\ell+j)!}} 2^{j-l} i^{j-k} (\cos(\theta/2))^{j+k} (\sin(\theta/2))^{j-k} \cdot$$

$$\frac{d^{\ell+j}}{dy^{\ell+j}} [(y-1)^{\ell+k}(y+1)^{\ell-k}]|_{y=\cos\theta}.$$

If $g = k(\varphi)a(\theta)k(\psi) \in SU(2)$, then the correspondent t_{jk}^ℓ has the following form

$$t_{jk}^\ell(g(\theta,\varphi,\psi)) = e^{-i(j\varphi+k\psi)} P_{jk}^\ell(\cos\theta), \tag{2.2}$$

where (θ,φ,ψ) are the Euler angles.

For $k=0$ we obtain $t^{\ell}_{j0}(g)=e^{-ij\varphi}P^{\ell}_{j0}(\cos\theta):=Y_{\ell j}(\varphi,\theta)$, $\ell\in\mathbb{N}$, $j\in I_\ell$, which are called spherical functions.

Let denote by S^2 the three dimensional unit sphere.

The normalized spherical functions

$$\sqrt{2\ell+1}\,t^{\ell}_{j0}(\varphi,\theta),\quad \ell\in\mathbb{N}, j\in I_\ell$$

form an orthonormal system regarding to the scalar product generated by the following continuous measure on the unit sphere

$$\int_{S^2} f(x)d\mu(x):=\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} f(\varphi,\theta)\sin\theta d\theta d\varphi. \tag{2.3}$$

i.e.

$$\sqrt{(2\ell+1)(2\ell'+1)}\int_{S^2} t^{\ell}_{m0}(g)\overline{t^{\ell'}_{m'0}}(g)d\mu(g)= \tag{2.4}$$

$$=\frac{\sqrt{(2\ell+1)(2\ell'+1)}}{4\pi}\int_0^{2\pi}\int_0^{\pi} e^{i(m-m')\varphi}P^{\ell}_{m0}(\cos\theta)\overline{P^{\ell'}_{m'0}(\cos\theta)}\sin\theta d\theta d\varphi=$$

$$=\delta_{mm'}\delta_{\ell\ell'}.$$

Moreover, every function f from $L^2(S^2)$ can be represented in the following form

$$f(\varphi,\theta)=\sum_{\ell=0}^{\infty}(2\ell+1)\sum_{k=-\ell}^{k=\ell}C_{\ell k}t^{\ell}_{k0}(\varphi,\theta), \tag{2.5}$$

where

$$C_{\ell k}=\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} f(\varphi,\theta)\overline{t^{\ell}_{k0}(\varphi,\theta)}\sin\theta d\theta d\varphi \tag{2.6}$$

are the Laplace-Fourier coefficients and the series being convergent in $L^2(S^2)$ with respect to the measure on S^2. For $m=m'$ the relation (2.4) implies

$$\frac{\sqrt{(2\ell+1)(2\ell'+1)}}{2}\int_0^{\pi} P^{\ell}_{m0}(\cos\theta)\overline{P^{\ell'}_{m0}(\cos\theta)}\sin\theta d\theta=\delta_{\ell\ell'} \tag{2.7}$$

and by substituting $\cos\theta = x$ we obtain

$$\int_{-1}^{1} P_{m0}^{\ell}(x)\overline{P_{m0}^{\ell'}(x)}dx = \frac{2}{2\ell+1}\delta_{\ell\ell'}. \tag{2.8}$$

Let $N \in \mathbb{N}^*$, $\ell \in \{0, ..., N-1\}$, $m \in I_\ell$, then there are

$$\sum_{\ell=0}^{N-1}(2\ell+1) = N^2$$

spherical functions of order less than N. For more details connected to spherical functions see [7]. In what follows we need the following theorem (see [5] p.47).

Theorem A. *Let denote by $\lambda_k^N \in (-1,1)$, $k \in \{1, ..., N\}$ the roots of Legendre polynomials P_N of order N, and for $j = 1, ..., N$ let*

$$\ell_j^N(x) := \frac{(x-\lambda_1^N)...(x-\lambda_{j-1}^N)(x-\lambda_{j+1}^N)...(x-\lambda_N^N)}{(\lambda_j^N-\lambda_1^N)...(\lambda_j^N-\lambda_{j-1}^N)(\lambda_j^N-\lambda_{j+1}^N)...(\lambda_j^N-\lambda_N^N)},$$

be the corresponding fundamental polynomials of Lagrange interpolation. Denote by

$$\mathcal{A}_k^N := \int_{-1}^{1} \ell_k^N(x)dx \quad (1 \le k \le N), \tag{2.9}$$

the corresponding Cristoffel-numbers. Then for every polynomial f of order less than $2N$,

$$\int_{-1}^{1} f(x)dx = \sum_{k=1}^{N} f(\lambda_k^N)\mathcal{A}_k^N.$$

3. Discretisation

In what follows, taking as starting point that the spherical functions form an orthonormal system regarding to the continuous measure given by (2.3), we will give the set of nodal points in $[0,\pi]\times[0,2\pi]$ and the discrete measure regarding to the orthonormality property is also valid. Let denote by

$$X=\{z_{kj}=(\theta_k,\varphi_j)=\left(\arccos\lambda_k^N,\ \ \frac{2\pi j}{2N+1}\right):k=\overline{1,N},j=\overline{0,2N}\}\tag{3.1}$$

the set of nodal points, and

$$\mu_N(z_{kj}):=\frac{\mathcal{A}_k^N}{2(2N+1)}.$$

Let define the following discrete integral on the set of nodal points X

$$\int_X f d\mu_N:=\sum_{k=1}^{N}\sum_{j=0}^{2N}f(z_{kj})\mu_N(z_{kj})=\sum_{k=1}^{N}\sum_{j=0}^{2N}f(\theta_k,\varphi_j)\frac{\mathcal{A}_k^N}{2(2N+1)}.\tag{3.2}$$

Theorem 3.1. *Let $N\in\mathbb{N}$, $N\geq 1$, then the finite collection of normalized spherical functions*

$$\{\sqrt{2\ell+1}t_{m0}^{\ell}:S^2\to\mathbb{C}\,|\,m\in I_\ell,\ \ \ell\in\{0,...,N-1\}\}$$

form an orthonormal system on the set of nodal points X regarding to the discrete integral defined by (3.2), i.e

$$\sqrt{2\ell+1}\sqrt{2\ell'+1}\int_X t_{m0}^{\ell}\overline{t_{p0}^{\ell'}}d\mu_N=\delta_{\ell\ell'}\delta_{mp}\tag{3.3}$$

$$(\ell,\ell'<N,\ \ m\in I_\ell,\ \ p\in I_{\ell'}).$$

Proof. Using the definition of the discrete measure and properties of the roots of order $2N+1$ of the unity, the left side of (3.3) is equal to

$$\sqrt{2\ell+1}\sqrt{2\ell'+1}\int_X t_{m0}^{\ell}\overline{t_{p0}^{\ell'}}d\mu_N=$$

$$=\sqrt{2\ell+1}\sqrt{2\ell'+1}\sum_{k=1}^{N}\sum_{j=0}^{2N}t_{m0}^{\ell}(\theta_k,\varphi_j)\overline{t_{p0}^{\ell'}(\theta_k,\varphi_j)}\frac{\mathcal{A}_k^N}{2(2N+1)}=$$

$$= \sqrt{2\ell+1}\sqrt{2\ell'+1}\sum_{k=1}^{N} P^{\ell}_{m0}(\lambda^N_k)\overline{P^{\ell'}_{p0}(\lambda^N_k)}\frac{\mathcal{A}^N_k}{2(2N+1)}\sum_{j=0}^{2N} e^{i(m-p)\frac{2\pi j}{2N+1}} =$$

$$(3.4)\qquad = \begin{cases} 0, & m \neq p, \\ \frac{1}{2}\sqrt{2\ell+1}\sqrt{2\ell'+1}\sum\limits_{k=1}^{N} P^{\ell}_{m0}(\lambda^N_k)\overline{P^{\ell'}_{m0}(\lambda^N_k)}\mathcal{A}^N_k, & m = p. \end{cases}$$

Taking into account that $\mathrm{gr} P^{\ell}_{m0}P^{\ell'}_{m0} = \ell+\ell' < 2N$, we can apply Theorem A and using the relation (2.8) we obtain

$$\frac{1}{2}\sqrt{2\ell+1}\sqrt{2\ell'+1}\sum_{k=1}^{N} P^{\ell}_{m0}(\lambda^N_k)\overline{P^{\ell'}_{m0}(\lambda^N_k)}\mathcal{A}^N_k =$$

$$= \frac{1}{2}\sqrt{2\ell+1}\sqrt{2\ell'+1}\int_{-1}^{1} P^{\ell}_{m0}(x)\overline{P^{\ell'}_{m0}(x)}dx =$$

$$= \frac{1}{2}\sqrt{2\ell+1}\sqrt{2\ell'+1}\frac{2}{2\ell+1}\delta_{\ell\ell'} = \begin{cases} 0, & \ell \neq \ell', \\ 1, & \ell = \ell'. \end{cases}$$

In [8] P. Barone proved that the discrete harmonics transform operator of the function f evaluated on the lattice $\mathcal{L}$ (similar defined as X) has orthonormal columns.

Next we will prove that the discrete integral defined by (3.2) tends to the invariant measure on $SU(2)$ given by (2.3), namely

Theorem 3.2. *For all $f \in C(S^2)$,*

$$\lim_{N\to\infty}\int_X f d\mu_N = \int_{S^2} f d\mu.$$

Proof. Let denote by $U = C(S^2)$, and introduce the bounded linear functionals $A_N(f) = \int\limits_X f d\mu_N$, $A(f) = \int\limits_{S^2} f d\mu$. Theorem 3.2 is a consequence of the Banach-Steinhaus theorem. We will check that all conditions of this theorem are satisfied for the functionals $A_N : U \to \mathbb{C}$ and $A : U \to \mathbb{C}$. Let denote by Z the set of all spherical functions. It can be proved that Z is a dense subset of $C(S^2)$ on the base of the Stone-Weierstrass theorem, because of the points of U are separated by the functions in Z. Namely if $g, h \in SU[2]$

so that $g \neq h$, then $t^1_{10}(g) \neq t^1_{10}(h)$. It is enough to prove that the product of two spherical functions can be expressed as a linear combination of spherical functions. This follows from properties of representations. It is well known that the direct product of two representations T^ℓ and $T^{\ell'}$ can be expressed as $\sum\limits_{k=\ell-\ell'}^{\ell+\ell'} T^k$. This implies that

$$t^\ell_{mm'} t^{\ell'}_{nn'} = \sum_{k,j,j'} \overline{C(\ell,\ell',k,m,n,j)} C(\ell,\ell',k,m',n',j') t^k_{jj'},$$

where $C(\ell,\ell',k,m,n,j)$ are the so called Clebsh-Gordon coefficients. Because of orthonormality of elements $t^\ell_{mm'}$ these coefficients can be computed in the following way:

$$\overline{C(\ell,\ell',k,m,n,j)} C(\ell,\ell',k,m',n',j') =$$

$$= (2k+1)\frac{1}{16\pi^2} \int_{-2\pi}^{2\pi} \int_0^{2\pi} \int_0^{\pi} t^\ell_{mm'} t^{\ell'}_{nn'} \overline{t^k_{jj'}} \sin\theta d\theta d\varphi d\psi.$$

Taking into account that $t^\ell_{mm'}(g) = e^{i(m\varphi + m'\psi)} P^\ell_{mm'}(\cos\theta)$, the above integral is different from zero only when $m + n = j$ and $m' + n' = j'$. From this we get that if $m' = n' = 0$, then $j' = 0$ and we obtain that

$$t^\ell_{m0} t^{\ell'}_{n0} = \sum_{k,j,j'} \overline{C(\ell,\ell',k,m,n,j)} C(\ell,\ell',k,0,0,0) t^k_{j0}.$$

From [5 p.48 (3.4.5)] it follows that A_N is a bounded linear operator, namely

$$||A_N|| = \sum_{k=1}^{N} \sum_{j=0}^{2N} \frac{|\mathcal{A}^N_k|}{2(2N+1)} = \sum_{k=1}^{N} \frac{|\mathcal{A}^N_k|}{2} = 1 < \infty.$$

From the orthonormality property it follows that for all $z = t^l_{m0} \in Z$, for all N so that $l < N \quad A_N(z) - A(z) = 0$, consequently $\lim\limits_{N\to\infty} |A_N(z) - A(z)| = 0$, $z \in Z$. Applying the Banach-Steinhaus theorem we get that

$$|A_N(f) - A(f)| \to 0, \quad \text{for all} \quad f \in C(S^2), \quad N \to \infty.$$

In fact Theorem 3.2 means that the limit of the 0-th disccrete Laplace-Fourier coefficient is equal by the 0-th Laplace-Fourier coefficient. In an analogous way can be proved that in general the discrete Laplace-Fourier coefficients of the function from $C(S^2)$ tend to the corresponding Laplace-Fourier coefficients of f.

4. (C,α) kernel of Laplace-Fourier series

Let denote $g = a(\theta)k(\varphi)$, $h = a(\theta')k(\varphi')$. Let $n < N$ and denote by

$$(I_{N,n}f)(g) = (I_{N,n}f)(\theta,\varphi) := \sum_{\ell=0}^{n}(2\ell+1)\sum_{k=-\ell}^{\ell} c_{\ell k}^{N} t_{k0}^{\ell}(\theta,\varphi), \tag{4.1}$$

the n-th partial sum of discrete Laplace-Fourier series of f, where $c_{\ell k}^{N}$ is given by

$$c_{\ell k}^{N} = \int_X \overline{t_{k0}^{\ell}} f d\mu_N = \sum_{m=1}^{N}\sum_{j=0}^{2N} f(\theta_m,\varphi_j)\overline{t_{k0}^{\ell}(\theta_m,\varphi_j)}\frac{\mathcal{A}_m^N}{2(2N+1)}. \tag{4.2}$$

$I_{N,n}f$ is n-th partial sum of the discrete Fourier-Laplace series of the function f defined on the unit sphere S^2. We can observe that

$$(I_{N,n}f)(\theta,\varphi) = \tag{4.3}$$

$$= \sum_{\ell=0}^{n}(2\ell+1)\sum_{k=-\ell}^{\ell}\left(\sum_{m=1}^{N}\sum_{j=0}^{2N} f(\theta_m,\varphi_j)\overline{t_{k0}^{\ell}(\theta_m,\varphi_j)}\frac{\mathcal{A}_m^N}{2(2N+1)}\right) t_{k0}^{\ell}(\theta,\varphi) =$$

$$= \int_X f(\theta',\varphi')\left(\sum_{l=0}^{n}(2\ell+1)\sum_{k=-\ell}^{\ell}\overline{t_{k0}^{\ell}(\theta',\varphi')}t_{k0}^{\ell}(\theta,\varphi)\right)d\mu_N.$$

Let denote by

$$\chi^{\ell}(\theta,\theta',\varphi,\varphi') := \sum_{k=-\ell}^{\ell}\overline{t_{k0}^{\ell}(\theta',\varphi')}t_{k0}^{\ell}(\theta,\varphi). \tag{4.4}$$

Taking into account that the representation T^{ℓ} of $SU(2)$ is unitary (see [7] p.284), and using that every element $u := h^{-1}g \in SU(2)$ can be represented as

$$u = vk(t)v^{-1}, v \in SU(2),$$

we obtain that

$$(4.5)\qquad \chi^\ell(\theta,\theta',\varphi,\varphi') = \chi^\ell(h^{-1}g) = \mathrm{spur}(T^\ell(h^{-1}g)) = \mathrm{spur}T^\ell(vk(t)v^{-1}) =$$

$$= \mathrm{spur}T^\ell(v)T^\ell(k(t))T^\ell(v^{-1}) = \frac{\sin(\ell+1/2)t}{\sin(t/2)}.$$

Denote by

$$(4.6)\qquad D_n(h^{-1}g) := \sum_{\ell=0}^{n}(2\ell+1)\chi^\ell(h^{-1}g)$$

the kernel function. Then the discrete Fourier-Laplace sum can be expressed in the following way:

$$(4.7)\qquad (I_{N,n}f)(g) = (I_{N,n}f)(\theta,\varphi) = \int_X f(h)D_n(h^{-1}g)d\mu_N(h).$$

It can be seen the analogy between the $(I_{N,n}f)$ and the partial sum of the series (2.5):

$$(4.8)\qquad (S_nf)(g) = \int_{S^2} f(h)D_n(h^{-1}g)d\mu(h).$$

Let denote by

$$(4.9)\qquad K_n^\alpha := \frac{1}{A_n^\alpha}\sum_{\ell=0}^{n}A_{n-\ell}^\alpha(2\ell+1)\chi^\ell, \qquad A_n^\alpha := \frac{(\alpha+1)(\alpha+2)...(\alpha+n)}{n!},$$

the (C,α) kernels of the Laplace-Fourier series. It is easy to show that

$$K_n^\alpha := \frac{1}{A_n^\alpha}\sum_{\ell=0}^{n}A_{n-\ell}^{\alpha-1}D_\ell,$$

consequently K_n^α is a (C,α) kernel and similarly to the trigonometric case (see [6] vol.I, chap.III.) K_n^α corresponds to a (C,α) summation method. From (1.14) of [1] we get that for $\alpha = 2$

$$(4.10)\qquad K_n^2 := \frac{1}{A_n^2}\sum_{\ell=0}^{n}A_{n-\ell}^2(2\ell+1)\chi^\ell \geq 0.$$

Using the orthonormality property (2.4), (3.3) and the definition of χ^k it is easy to check that

$$(4.11) \qquad \int_X K_n^2(h^{-1}g)d\mu_N(h) = \int_{S^2} K_n^2(h^{-1}g)d\mu(h) = 1.$$

The last two properties show that K_n^2 has two important properties of Fejér kernel. Let introduce the analogue of de la Valée-Poussin kernel denoted by

$$(4.12) \qquad M_n := \frac{1}{n^2}(A_{3n}^2 K_{3n}^2 - 2A_{2n}^2 K_{2n}^2 + A_n^2 K_n^2).$$

Note that the partial sum of order n of M_n is equal to

$$(4.13) \qquad S_n[M_n] = D_n.$$

Let denote by $\mathcal{T}_n = \text{span}\{t_{k0}^{\ell}, \ell \in \{0, 1, ..., n\}, k \in I_\ell\}$. From the orthonormality property of spherical finctions and (4.13) follows that

$$\int_{S^2} f(h)M_n(h^{-1}g)d\mu(h) = f(g), \quad \text{for all} \quad f \in \mathcal{T}_n.$$

Denote by
(4.15)

$$(\sigma_n f)(g) := \int_{S^2} f(h)K_n^2(h^{-1}g)d\mu(h), \quad (V_n f)(g) := \int_{S^2} f(h)M_n(h^{-1}g)d\mu(h)$$

and

$$(\sigma_{n,N} f)(g) := \int_X f(h)K_n^2(h^{-1}g)d\mu_N(h),$$

$$(4.16) \qquad (V_{n,N} f)(g) := \int_{S^2} f(h)M_n(h^{-1}g)d\mu_N(h)$$

$$(f \in C(S^2), \quad g \in SU[2]).$$

the continuous and discrete summmation processes corresponding to the K_n^2 kernels and M_n.

Theorem 4.1 *For all $f \in C(S^2)$,* *1)*

$$||\sigma_n f - f|| \to 0, \quad if \quad n \to \infty, \tag{4.17}$$

2)

$$||V_n f - f|| \to 0 \quad if \quad n \to \infty \tag{4.18}$$

3)

$$||\sigma_{n,N} f - f|| \to 0, \quad if \quad N, n \to \infty, \quad so\ that \quad n < N, \tag{4.19}$$

4)

$$||V_{n,N} f - f|| \to 0 \quad if \quad n \to \infty, \quad so\ that \quad 3n < N, \tag{4.20}$$

where the norm is the maximum norm.

Proof. In the proof of Theorem 3.2 we have already showed that the set Z of all spherical functions is dense in $C(S^2)$. From (4.10) and (4.11) we obtain that $||\sigma_n f|| \leq ||f||$ for all $f \in C(S^2)$, consequently the linear operators $\sigma_n : C(S^2) \to \mathbb{C}$ are uniformly bounded. Let denote by $\sigma : C(S^2) \to \mathbb{C}$, $\sigma f := f$ the identity operator. Because of orthonormality property of spherical functions we get that for every $z = t^\ell_{k0}$ from Z, if $n \to \infty$ then $\sigma_n t^\ell_{k0} = \frac{A^2_{n-\ell}}{A^2_n} t^\ell_{k0} \to t^\ell_{k0}$, consequently for all $z \in Z$

$$\lim_{n \to \infty} ||\sigma_n z - \sigma z|| = 0.$$

Applying the Banach-Steinhaus theorem we get that (4.17) is true.

From (4.10) and (4.11) we obtain that

$$||V_n f|| \leq \frac{1}{n^2}(A^2_{3n} + 2A^2_{2n} + A^2_n)||f|| \leq 25||f||.$$

Consequently $V_n : C(S^2) \to \mathbb{C}$ are also uniformly bounded. If $l < n$, then $V_n t^\ell_{k0} = t^\ell_{k0}$. In analogous way as before we get that (4.18) is true. The proof of (4.19) and (4.20) is similar.

As in the case of one variable trigonometric interpolation (see [6] vol.II, Chap.5) arises the question if we could compare the behaviour of $(I_{N,n} f)$ and $(S_n f)$?

The behaviour of $(I_{N,n} f)$ and $(S_n f)$ in norm will be the same if we could prove the equivalence of the norms genereted by the continuous $(d\mu)$

and discrete ($d\mu_N$) measures. In one variable case the basic tool to prove the equivalence of the norms is the so called Marczinkievicz inequality (see [6] vol.II, p.30). In what follows we will prove Marczinkievicz type inequality regarding to the spherical polynomials in the system t^{ℓ}_{m0}.

5. Marcinkiewicz type inequalities

Theorem 5.1. *Let n, N be positive integers so that $3n < N$. Consider a linear combination of spherical functions of order n of the following form*

$$S(g) = \sum_{l=0}^{n}(2l+1)\sum_{k=-l}^{l} c^N_{lk} t^l_{k0}(\theta, \varphi). \tag{5.1}$$

Then there exist constants A and A_p, depending only on p, such that

$$\left(\int_{S^2} |S(g)|^p d\mu_N\right)^{1/p} \leq A\left(\int_{S^2} |S(g)|^p d\mu\right)^{1/p} \qquad (1 \leq p \leq +\infty) \tag{5.2}$$

and

$$\left(\int_{S^2} |S(g)|^p d\mu\right)^{1/p} \leq A_p\left(\int_{S^2} |S(g)|^p d\mu_N\right)^{1/p} \qquad (1 < p < +\infty). \tag{5.3}$$

Proof. Let S be a linear combination of spherical functions of order n, then

$$S_n[S](x) = S(x) = \int_{S^2} S(y)D_n(y^{-1}x)dy. \tag{5.4}$$

We can substitute D_n in formula (5.4) by any linear combination of spherical functions of which partial sum of order n is D_n. One of these functions is M_n defined by formula (4.12).

We observe that

$$A^2_{3n} + 2A^2_{2n} + A^2_n = \tag{5.5}$$

$$= \frac{1}{2}[(3n+1)(3n+2) + 2(2n+1)(2n+2) + (n+1)(n+2)] = (3n+2)^2.$$

Replacing in (5.4) D_n by M_n we have

$$|S(x)| = \tag{5.6}$$

$$= \left| \int_{S^2} S(y) M_n(y^{-1}x) dy \right| = \left| \int_{S^2} S(y) \frac{1}{n^2} (A_{3n}^2 K_{3N}^2 - 2A_{2n}^2 K_{2n}^2 + A_n^2 K_n^2) dy \right| \leq$$

$$\leq \frac{1}{n^2} A_{3n}^2 \int_{S^2} |S(y)| K_{3n}^2 dy + \frac{1}{n^2} 2A_{2n}^2 \int_{S^2} |S(y)| K_{2n}^2 dy + \frac{1}{n^2} A_n^2 \int_{S^2} |S(y)| K_n^2 dy.$$

Let $n \geq 2$ and Φ be a nondecreasing convex function, and $p(x) \geq 0$, then

$$\Phi\left(\int pg\right) \leq \int p\Phi(g). \tag{5.7}$$

Because K_n^2 is a positive kernel from (4.10), (4.11) and from the properties of Φ we get

$$\Phi\left(\frac{1}{4^2}|S(x)|\right) \leq \Phi\left(\frac{n^2}{(3n+2)^2}|S(x)|\right) \leq \frac{1}{(3n+2)^2} A_{3n}^2 \Phi\left(\int_{S^2} |S(y)| K_{3n}^2 dy\right) +$$

(5.8)

$$+\frac{1}{(3n+2)^2} 2A_{2n}^2 \Phi\left(\int_{S^2} |S(y)| K_{2n}^2 dy\right) + \frac{1}{(3n+2)^2} A_n^2 \Phi\left(\int_{S^2} |S(y)| K_n^2 dy\right) \leq$$

$$\leq \frac{1}{(3n+2)^2} A_{3n}^2 \int_{S^2} \Phi(|S(y)|) K_{3n}^2 dy + \frac{1}{(3n+2)^2} 2A_{2n}^2 \int_{S^2} \Phi(|S(y)|) K_{2n}^2 dy +$$

$$+ \frac{1}{(3n+2)^2} A_n^2 \int_{S^2} \Phi(|S(y)|) K_n^2 dy.$$

Considering the discrete integral of the last inequality we obtain

$$\int_X \Phi\left(\frac{1}{4^2}|S(x)|\right) d\mu_N(x) \leq \frac{1}{(3n+2)^2} A_{3n}^2 \int_X \left[\int_{S^2} \Phi(|S(y)|) K_{3n}^2 d\mu y\right] d\mu_N(x) +$$

$$+\frac{1}{(3n+2)^2}2A_{2n}^2\int\limits_X\left[\int\limits_{S^2}\Phi(|S(y)|)K_{2N}^2 d\mu y\right]d\mu_N(x)+$$

$$+\frac{1}{(3n+2)^2}A_n^2\int\limits_X\left[\int\limits_{S^2}\Phi(|S(y)|)K_n^2 d\mu y\right]d\mu_N(x). \tag{5.9}$$

Interchanging the order of the integration and taking into account (4.11) we get

$$\int\limits_X\Phi\left(\frac{1}{4^2}|S(x)|\right)d\mu_N(x)\leq\frac{1}{(3n+2)^2}A_{3n}^2\int\limits_{S^2}\Phi(|S(y)|)d\mu y+ \tag{5.10}$$

$$+\frac{1}{(3n+2)^2}2A_{2n}^2\int\limits_{S^2}\Phi(|S(y)|)d\mu y+\frac{1}{(3n+2)^2}A_n^2\int\limits_{S^2}\Phi(|S(y)|)d\mu y=$$

$$=\int\limits_{S^2}\Phi(|S(y)|)d\mu y$$

If we consider $\Phi(u)=u^p$ from (5.10) results that (5.2) is true and the absolute constant is $A=4^2$.

Applying the representation theorem of Riesz we have: if $1/p+1/p'=1$, then

$$\left(\int\limits_{S^2}|S|^p d\mu\right)^{1/p}=\sup_{||g||_{p'}=1}\left|\int\limits_{S^2}Sgd\mu\right|=\sup_{||g||_{p'}=1}\left|\int SS_n[g]d\mu\right|= \tag{5.11}$$

$$=\sup_{||g||_{p'}=1}\left|\int SS_n[g]d\mu_N\right|\leq\sup_{||g||_{p'}=1}\left(\int|S|^p d\mu_N\right)^{1/p}\left(\int|S_n[g]|^{p'}d\mu_N\right)^{1/p'}=$$

$$=\left(\int|S|^p d\mu_N\right)^{1/p}\sup_{||g||_{p'}=1}\left(\int|S_n[g]|^{p'}d\mu_N\right)^{1/p'}\leq A_p\left(\int|S|^p d\mu_N\right).$$

Using (5.2) we obtain

$$\left(\int|S_n[g]|^{p'}d\mu_N\right)^{1/p'}\leq A\left(\int|S_n[g]|^{p'}d\mu\right)^{1/p'}. \tag{5.12}$$

It is easy to see that there exists a number $R_{p'}$ independent from n and g so that

$$\sup_{||g||_{p'}=1}\left(\int |S_n[g]|^{p'}d\mu\right)^{1/p'} < R_{p'}. \tag{5.13}$$

Let denote $A_p := AR_{p'}$, for this constant (5.3) is true.

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M. Pap
Inst. of Math. and Inf.
University of Pécs
Ifjúság u. 6
H-7624 Pécs, Hungary
e-mail: papm@ttk.pte.hu

F. Schipp
Department of Numerical Analysis
Eötvös Loránd University
Pázmány P. sét. 1/C.
H-1117 Budapest, Hungary
Computer and Automation Inst. of HAS
Kende u. 13-17.
H-1502 Budapest, Hungary
e-mail: schipp@ludens.elte.hu