Annales Univ. Sci. Budapest., Sect. Comp. 22 (2003) 253-268

EXTREMAL PATTERN–FREE SETS OF POSITIVE INTEGERS

L.G. Lucht (Clausthal-Zellerfeld, Germany)

Dedicated to Karl-Heinz Indlekofer
on the occasion of his sixtieth birthday

Abstract. Let σ be the family of solutions $\{x, y, z\} \subset \mathbb{N}$ to the equation $ax+by = cz$ with given coprime coefficients $a, b \in \mathbb{N}$ and $c \in \mathbb{N}$ sufficiently large. Denote by $\overline{d}(\mathcal{A})$ the upper asymptotic density of the set $\mathcal{A} \subseteq \mathbb{N}$ and by $\overline{d}(\sigma) = \sup\left\{\overline{d}(\mathcal{A}) : \mathcal{A} \subseteq \mathbb{N}, (a\mathcal{A} + b\mathcal{A}) \cap c\mathcal{A} = \emptyset\right\}$ the *upper asymptotic density bound* for σ-free sets $\mathcal{A} \subseteq \mathbb{N}$, i.e. sets containing no solution to the equation $ax + by = cz$. We investigate the fine structure of σ-free sets and determine $\overline{d}(\sigma)$. Define the *upper relative density bound* $\overline{D}(\sigma)$ as the limit superior of the sequence of maximal ratios $|\mathcal{A}|/n$ taken over all σ-free finite sets $\mathcal{A} \subseteq \{1, \dots, n\}$. We show that $\overline{D}(\sigma)$ exceeds $\overline{d}(\sigma)$ by constructing a sequence of σ-free finite sets that is likely to be optimal.

1. Introduction

In two stimulating papers [6] and [7] from 1993 and 1995 Ruzsa reports on methods, results and open problems concerning extremal sets of positive integers containing no solution to a linear equation of the form

$$a_1x_1 + \dots + a_kx_k = b \tag{1.1}$$

The author is grateful to the Institute of Mathematics and Informatics of the University of Debrecen, the Institute of Informatics of the Eötvös Loránd University in Budapest and the Department of Mathematics of the University of Pécs for the invitation to attend the International Conference on *Numbers, Functions, Equations '03* and for support.

with integer coefficients $a_1, \dots, a_k, b$. A general definition of such patterns was already given in [4] (see also Klotz [3]):

A *pattern* σ is a nonempty family of nonempty finite subsets of $\mathbb{N}$. It is called *invariant* if $kS \in \sigma$ and $k + S \in \sigma$ for all $S \in \sigma$ and $k \in \mathbb{N}$. A set $\mathcal{A} \subseteq \mathbb{N}$ is called *σ-free* if no subset of $\mathcal{A}$ belongs to σ.

Clearly many very different problems of number theory can be put into the frame of extremal pattern-free sets, e.g. the problem of the distribution of primes, the Fermat problem, and the problem of progression-free sets. In particular, equation (1.1) corresponds to the system

$$\sigma = \big\{ \{x_1, \dots, x_k\} \subset \mathbb{N} : a_1 x_1 + \cdots + a_k x_k = b \big\}, \tag{1.2}$$

which is invariant if and only if $b = 0$ and $a_1 + \cdots + a_k = 0$. In order to avoid trivial solutions $x_1 = \ldots = x_k$ to the defining equation $a_1 x_1 + \cdots + a_k x_k = 0$ (implying that the only σ-free set is the empty set), the condition that $x_i \neq x_j$ for some indices i, j then has to be added in the definition of σ.

The property of a σ-free set to be extremal depends on the underlying concept of density. The monograph of Ostmann [5] informs about various density concepts. We restrict our consideration to the following: for a set $\mathcal{A}$ of positive integers let $A(n) := \big|\mathcal{A} \cap \{1, \dots, n\}\big|$ denote the number of elements $a \in \mathcal{A}$ with $a \leq n$. Then $\mathcal{A}$ is characterized by the ratios

$$\rho(\mathcal{A}, n) := \frac{A(n)}{n} \qquad (n \in \mathbb{N}),$$

and the *upper asymptotic density* $\overline{d}(\mathcal{A})$ is defined by

$$\overline{d}(\mathcal{A}) := \limsup_{n \to \infty} \rho(\mathcal{A}, n).$$

For a given pattern σ, the density of (infinite) σ-free sets is bounded above by the *upper asymptotic density bound*

$$\overline{d}(\sigma) := \sup \big\{ \overline{d}(\mathcal{A}) : \mathcal{A} \subseteq \mathbb{N} \quad \sigma\text{−free} \big\}. \tag{1.3}$$

On the other hand, the maximum values

$$m(\sigma, n) := \max \big\{ \rho(\mathcal{A}, n) : \mathcal{A} \subseteq \{1, \dots, n\} \quad \sigma\text{−free} \big\}$$

taken over all σ-free subsets of $\{1, \dots, n\}$ lead to the *upper relative density bound*

$$\overline{D}(\sigma) := \limsup_{n \to \infty} m(\sigma, n) \tag{1.4}$$

for (finite) σ-free sets. Since every subset of a σ-free set is again σ-free, it follows that

$$\overline{d}(\sigma) \le \overline{D}(\sigma)\,. \tag{1.5}$$

By replacing the lim sup in (1.3) and (1.4) by the lim inf the corresponding *lower* density bounds $\underline{d}(\sigma)$ and $\underline{D}(\sigma)$ are obtained.

Due to Szemerédi's results [9], [10] on progression-free sets, the above density bounds vanish for invariant systems σ defined by (1.2), whereas they are positive for non-invariant systems. Namely, in the latter case the residue class 1 mod q is σ-free for suitably chosen $q \in \mathbb{N}$. In recent papers Chung and Goldwasser [2], Schoen [8], Baltz and Schoen [1] solve some of Ruzsa's problems by considering non-degenerate systems of the form

$$\sigma = \big\{\{x, y, z\} \subset \mathbb{N} : ax + by = cz\big\} \qquad (a, b, c \in \mathbb{N}) \tag{1.6}$$

with coefficients $a = b = 1$ or $a = c = 1$. In fact, some answers are already contained in the paper [4] from 1976 which seems to have been overlooked.

We recall that the *sum set* $X+Y$ and the *multiple set* λX of sets $X, Y \subseteq \mathbb{R}$ with $\lambda \in \mathbb{R}$ consist of the sums $x + y$ and the products λx, respectively, with $x \in X$ and $y \in Y$. Then, with σ defined in (1.6), an equivalent set-theoretical formulation of the condition on $\mathcal{A}$ to be σ-free is

$$(a\mathcal{A} + b\mathcal{A}) \cap (c\mathcal{A}) = \emptyset.$$

The aim of this note is to investigate the fine structure of extremal σ-free sets, where σ is defined by (1.6) with arbitrary coprime coefficients $a, b, c \in \mathbb{N}$, $a \le b$, and c sufficiently large. This leads to results on the density bounds $\overline{d}(\sigma)$ and $\overline{D}(\sigma)$. The first theorem (compare [4], see also Schoen [8], Theorem 3) determines $\overline{d}(\sigma)$.

Theorem 1. *Let the system σ be defined by (1.6) with $a, b, c \in \mathbb{N}$ satisfying $c \ge 2ab(a+b)$, $a \le b$ and* $\gcd(a, b) = 1$*. Then*

$$\overline{d}(\sigma) = \frac{c^2 - c(a+b)}{c^2 - a(a+b)}. \tag{1.7}$$

The proof of Theorem 1 is based on structural properties of σ-free sets studied in Sections 2 to 4. In fact, they also yield that the lower bound $2ab\,(a+$ $+b)$ for c is sharp in general, which is seen in Sections 2 and 3 for the special case of (1.6) with $a = b = 1$ and $c = 3$.

The next theorem presents a lower bound for $\overline{D}(\sigma)$ and shows that the inequality (1.5) is strict in general.

Theorem 2. *Let the system σ be defined by (1.6) with $a,b,c \in \mathbb{N}$ satisfying $c \geq 2ab(a+b)$, $a \leq b$ and $\gcd(a,b)=1$. Then, with $q := \dfrac{a+b}{c}$, $\alpha := \dfrac{aq}{c}$ and $\beta := \dfrac{bq}{c}$,*

$$\overline{D}(\sigma) \geq \begin{cases} (1-q)\dfrac{1-\alpha^2}{1-\alpha-\alpha^2} & \text{for} \quad a=b, \\ (1-q)\dfrac{1-\beta+\alpha}{1-\beta} & \text{for} \quad a<b. \end{cases} \tag{1.8}$$

In particular, $\overline{D}(\sigma) > \overline{d}(\sigma)$.

The proof of Theorem 2 is based on a construction of finite σ-free sets in Section 5. Baltz and Schoen [1] have obtained equality in (1.8) for $a=b=1$, and it is very likely that this is generally true under the assumptions of Theorem 2.

2. The construction of extremal infinite σ-free sets

For $a \leq b$ and $a+b<c$ we have $q := \dfrac{a+b}{c} < 1$ and observe that $qn < < x,y,z \leq n$ implies $(a+b)\,qn < ax+by \leq (a+b)\,n = cqn < cz \leq cn$. Hence the segment $\mathcal{S} = (qn,n] \cap \mathbb{N}$ yields disjoint sets $a\mathcal{S}+b\mathcal{S}$ and $c\mathcal{S}$ and therefore is σ-free. The idea of constructing infinite σ-free sets consists in taking the union of sufficiently distant segments $\mathcal{S}$. Clearly we proceed by induction: Suppose that $\mathcal{A}_k \subset \mathbb{N}$ for some $k \in \mathbb{N}$ is a non-empty σ-free union of such segments $\mathcal{S}_j$ with $j \leq k$ on the left to the segment $\mathcal{S}_{k+1} := (qn_{k+1}, n_{k+1}] \cap \mathbb{N}$. Then $\mathcal{A}_{k+1} := \mathcal{A}_k \cup \mathcal{S}_{k+1}$ is σ-free if the minimal element of $a\mathcal{S}_{k+1} + b\mathcal{A}_k$ exceeds the maximal element of $c\mathcal{A}_k$. This means $cn_k \leq aqn_{k+1} + b\min\mathcal{A}_k$, which is satisfied if $cn_k \leq aqn_{k+1}$ or, equivalently,

$$n_{k+1} \geq \frac{c}{aq}\,n_k = \frac{c^2}{a\,(a+b)}\,n_k\,.$$

Therefore we set

$$n_k := \left(\frac{c^2}{a(a+b)}\right)^k, \quad \mathcal{S}_k := \left(\frac{a+b}{c}n_k,\ n_k\right] \cap \mathbb{N} \qquad (k \in \mathbb{N}) \tag{2.1}$$

and define

$$\mathcal{A} := \bigcup \{\mathcal{S}_k : k \in \mathbb{N}\}. \tag{2.2}$$

Lemma 1. *For $a, b, c \in \mathbb{N}$ with $a \le b$ and $q := \dfrac{a+b}{c} < 1$, let σ denote the system (1.6). Then the set $\mathcal{A}$ defined by (2.1) and (2.2) is σ-free. Its upper asymptotic density is given by*

$$\overline{d}(\mathcal{A}) = \frac{c^2 - c(a+b)}{c^2 - a(a+b)}.$$

In particular,

$$A(n_k) - \overline{d}(\mathcal{A})\, n_k \ll k.$$

Proof. The first assertion is an immediate consequence of the preceding construction. In order to determine $\overline{d}(\mathcal{A})$ we have to consider the arithmetic means $\rho(\mathcal{A}, n_k)$ for $k \to \infty$. We obtain $A(n_k) = (1-q)\, n_k + A(n_{k-1}) + \mathcal{O}(1)$, from which we derive

$$A(n_k) = (1-q) n_k \sum_{0 \le \kappa < k} \left(\frac{c^2}{a(a+b)} \right)^{\kappa} + \mathcal{O}(k) = \frac{c^2 - c(a+b)}{c^2 - a(a+b)} s_k + \mathcal{O}(k).$$

This completes the proof.

We remark that our construction fails to be optimal for small values of c. For instance, if $a = b = 1$ and $c = 3$, then Lemma 1 only yields $\overline{d}(\sigma) \ge 3/7$, whereas the σ-free set of all odd positive integers has the asymptotic density $1/2$ so that $\overline{d}(\sigma) \ge 1/2$. However, the next sections show that our construction is optimal concerning $\overline{d}(\sigma)$, for all sufficiently large values of c.

3. Gaps

The idea consists in showing that σ-free sets $\mathcal{A}$ with sufficiently large upper asymptotic density $\overline{d}(\mathcal{A})$ necessarily have large gaps. Let $\varepsilon > 0$. Due to the definition of the least upper bound there exists a number $n_0 = n_0(\mathcal{A})$ such that *for all $n \ge n_0$*

$$A(n) \le \left(\overline{d}(\mathcal{A}) + \varepsilon\right) n, \tag{3.1}$$

and there *exist infinitely many* $n \in \mathbb{N}$ satisfying

$$A(n) \geq \big(\overline{d}(\mathcal{A}) - \varepsilon\big) n. \tag{3.2}$$

This gives the trivial part of the next lemma.

Lemma 2. *Let $\mathcal{A} \subseteq \mathbb{N}$ be σ-free, where σ is defined by (1.6) with $a, b, c \in \in \mathbb{N}$, $a \leq b$, $\gcd(a,b) = 1$ and $c > a(a+b)$. Then, for every $\varepsilon > 0$, there is a strictly increasing unbounded sequence of numbers n such that both inequalities (3.1) and (3.2) are valid. Moreover, there exists a constant $K > 0$ depending only on σ and $\mathcal{A}$ such that for all sufficiently large n of this kind the following statements hold:*

(a) If $\overline{d}(\mathcal{A}) > 1 - \dfrac{1}{a+b}$ *then* $\left|\mathcal{A} \cap \left(\dfrac{b}{c}n,\ \dfrac{a+b}{c}n\right]\right| \leq K\varepsilon n.$

(b) If $\overline{d}(\mathcal{A}) > 1 - \dfrac{1}{2b}$ *then* $\mathcal{A} \cap \left(\dfrac{a}{c}n,\ \dfrac{b}{c}n\right] = \emptyset.$

(c) If $\overline{d}(\mathcal{A}) > 1 - \dfrac{1}{2ab}$ *then* $\mathcal{A} \cap \left(\dfrac{1}{c}n,\ \dfrac{a}{c}n\right] = \emptyset.$

The proof of Lemma 2 is based on a simple combinatorial argument.

Lemma 3. *Let $\mathcal{A} \subseteq \mathbb{N}$ be σ-free, where σ is defined by (1.6) with $a, b, c \in \in \mathbb{N}$, $c > a(a+b)$, $a \leq b$ and $\gcd(a,b) = 1$. For an interval $I \subset [0, \infty)$ of length $\lambda(I) < \infty$ and for $\zeta \in \mathcal{A}$ let $I_x, I_y \subset \mathbb{R}$ be determined by $aI_x = c\zeta - bI_y = I$. Then*

$$\big|\mathcal{A} \cap I_x\big| + \big|\mathcal{A} \cap I_y\big| \leq \left(\frac{1}{a} + \frac{1}{b} - \frac{1}{ab}\right)\lambda(I) + 2.$$

Proof. We count the combinations $ax \in I$ and $c\zeta - by \in I$ with $x \in \mathcal{A} \cap I_x$ and $y \in \mathcal{A} \cap I_y$. Since $\mathcal{A}$ is σ-free, they are distinct and belong to the residue classes 0 mod a and $c\zeta$ mod b. Their total number is at most the number of elements of the union of these residue classes contained in I. The assertion now follows from the inclusion-exclusion principle.

Proof of Lemma 2. (a) We may assume that there exists some $\zeta \in \in \mathcal{A} \cap \left(\dfrac{b}{c}n,\ \dfrac{a+b}{c}n\right]$. By choosing $I = \big(c\zeta - bn, an\big]$ in Lemma 3 we obtain with $I_x = \left(\dfrac{c\zeta - bn}{a}, n\right]$ and $I_y = \left[\dfrac{c\zeta - an}{b}, n\right)$ that

$$A(n) - A\left(\frac{c\zeta - bn}{a}\right) + A(n) - A\left(\frac{c\zeta - an}{b}\right) \leq \left(\frac{1}{a} + \frac{1}{b} - \frac{1}{ab}\right)\big((a+b)n - c\zeta\big) + 3.$$

By inserting inequalities (3.1) and (3.2) and rearranging we see that

$$2\big(\overline{d}(\mathcal{A})-\varepsilon\big)n \le$$

$$\le \left(\frac{1}{a}+\frac{1}{b}-\frac{1}{ab}\right)\big((a+b)n-c\zeta\big)+\big(\overline{d}(\mathcal{A})+\varepsilon\big)\left(\frac{c\zeta-bn}{a}+\frac{c\zeta-an}{b}\right)+3n_0$$

or, equivalently,

$$\left(\overline{d}(\mathcal{A})-1+\frac{1}{a+b}\right)\frac{a+b}{ab}\big((a+b)n-c\zeta\big)\le$$

$$\le \varepsilon\left(\left(2-\frac{a}{b}-\frac{b}{a}\right)n+\left(\frac{1}{a}+\frac{1}{b}\right)c\zeta\right)+3n_0.$$

Since $c\zeta \le (a+b)n$, the right hand side is at most $4\varepsilon n+3n_0$, and we conclude from $\overline{d}(\mathcal{A}) > 1-\dfrac{1}{a+b}$ that $(a+b)n-c\zeta \le K\varepsilon n$. Consequently

$$\zeta \ge \frac{a+b}{c}n - K\varepsilon n$$

for all admissible $n \ge \dfrac{n_0}{\varepsilon}$, with some constant $K>0$ depending only on σ and $\mathcal{A}$. This gives (a) and, in addition, shows that $\overline{d}(\mathcal{A})<1$.

(b) Suppose to the contrary that there exists some $\zeta \in \mathcal{A}\cap\big(\frac{a}{c}n,\ \frac{b}{c}n\big]$. We choose $I=\big(0,an\big]$ in Lemma 3 so that $I_x=\big(0,n\big]$ and $I_y=\Big[\frac{c\zeta-an}{b},\frac{c\zeta}{b}\Big)$. This gives

$$A(n)+A\left(\frac{c\zeta}{b}\right)-A\left(\frac{c\zeta-an}{b}\right)\le\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{ab}\right)an+3.$$

On using the trivial estimate

$$A(n)-A\left(\frac{c\zeta}{b}\right)\le n-\frac{c\zeta}{b}$$

and inserting inequalities (3.1), (3.2) we obtain

$$(3.3)\qquad \big(1-\overline{d}(\mathcal{A})\big)\frac{c\zeta-an}{b}\le-\left(2\overline{d}(\mathcal{A})-2+\frac{1}{b}\right)n+3\varepsilon n+3.$$

It follows from $1-\dfrac{1}{2b}<\overline{d}(\mathcal{A})<1$ that the right hand side is negative for all sufficiently large admissible n if ε is small enough. This gives $c\zeta<an$, a contradiction.

(c) Again, suppose to the contrary that there exists some $\zeta \in \mathcal{A} \cap \left(\frac{1}{c}n, \frac{a}{c}n\right]$. For $I = \left(0, n\right]$ Lemma 3 implies $I_x = \left(0, \frac{n}{a}\right]$, $I_y = \left[\frac{c\zeta - n}{b}, \frac{c\zeta}{b}\right)$ and

$$A\left(\frac{c\zeta}{a}\right) - A\left(\frac{c\zeta - n}{a}\right) + A\left(\frac{n}{b}\right) \leq \left(\frac{1}{a} + \frac{1}{b} - \frac{1}{ab}\right) n + 3.$$

The trivial estimates

$$A(n) - A\left(\frac{c\zeta}{a}\right) \leq n - \frac{c\zeta}{a}, \quad A(n) - A\left(\frac{n}{b}\right) \leq n - \frac{n}{b}$$

combined with inequalities (3.1) and (3.2) lead to

$$2\left(\overline{d}(\mathcal{A}) - \varepsilon\right) n \leq \left(2 + \frac{1}{a} - \frac{1}{ab}\right) n - \frac{c\zeta}{a} + \left(\overline{d}(\mathcal{A}) + \varepsilon\right) \frac{c\zeta - n}{a} + n_0 + 3$$

or, equivalently,

$$(3.4) \qquad \left(1 - \overline{d}(\mathcal{A})\right) \frac{c\zeta - n}{a} \leq -\left(2\overline{d}(\mathcal{A}) - 2 + \frac{1}{ab}\right) n + 2\varepsilon n + 3n_0.$$

It follows from $\overline{d}(\mathcal{A}) > 1 - \dfrac{1}{2ab}$ that the right hand side is negative for all sufficiently large admissible n and ε small enough. This gives $c\zeta < n$, a contradiction, which completes the proof of Lemma 2.

Corollary 1. *Under the assumptions of Lemma 2 the upper asymptotic density bound for σ-free sets $\mathcal{A} \subseteq \mathbb{N}$ satisfies*

$$(3.5) \qquad \overline{d}(\sigma) \leq \max\left\{1 - \frac{1}{2ab}, \quad 1 - \frac{a + b - 1}{c - 1}\right\}.$$

In particular, the system σ defined according to (1.6) by $x + y = 3z$ satisfies $\overline{d}(\sigma) = 1/2$.

Proof. We may assume that $\overline{d}(\sigma) > 1 - \dfrac{1}{2ab}$. For $\varepsilon > 0$ sufficiently small, let $\mathcal{A} \subseteq \mathbb{N}$ be a σ-free set satisfying

$$(3.6) \qquad \overline{d}(\sigma) \geq \overline{d}(\mathcal{A}) \geq \overline{d}(\sigma) - \varepsilon > 1 - \frac{1}{2ab},$$

and let numbers $n \in \mathbb{N}$ be chosen sufficiently large such that (3.1) and (3.2) are satisfied. Then Lemma 2 yields the estimate

$$A(n) \leq A\left(\frac{n}{c}\right) + 3K\varepsilon n + n - \frac{a+b}{c}n$$

with some constant $K = K(\sigma, \mathcal{A}) > 0$. Combined with (3.1) and (3.2), it follows that

$$\overline{d}(\sigma) \leq \frac{c-a-b}{c-1} + K'\varepsilon$$

with some constant $K' = K'(\sigma, \mathcal{A}) > 0$, which gives (3.5).

Finally, for $a = b = 1$ and $c = 3$ inequality (3.5) yields $\overline{d}(\sigma) \leq 1/2$, and we already know from the remark closing Section 2 that $\overline{d}(\sigma) \geq 1/2$. Hence $\overline{d}(\sigma) = 1/2$.

4. An injective mapping

Lemma 2 asserts that the contribution of the interval $\left(\frac{n}{c},\ \frac{a+b}{c}n\right]$ to the σ-free set $\mathcal{A}$ is minor if the quotient $\rho(\mathcal{A}, n) = \frac{A(n)}{n}$ is large enough. We observe that the sets $\mathcal{A}$ constructed in the proof of Lemma 1 share this property even in the larger interval $\left(\frac{a(a+b)}{c^2}n,\ \frac{a+b}{c}n\right]$. Estimating the contribution of the remaining subinterval $\left(\frac{a(a+b)}{c^2}n,\ \frac{n}{c}\right]$ to $\mathcal{A}$ requires the following lemma.

Lemma 4. *Let $\mathcal{A} \subseteq \mathbb{N}$ be σ-free, where σ is defined by (1.6) with $a, b, c \in \mathbb{N}$, $a \leq b$, $\gcd(a, b) = 1$ and $c \geq 2ab(a+b)$, and let $c := c' \gcd(b, c)$. If*

$$\overline{d}(\mathcal{A}) > 1 - \frac{1}{2ab},$$

then there exists a complete residue system $\mathcal{E} \subseteq \mathcal{A} \mod a$ of numbers distinct $\mod c'$.

Proof. It suffices to show that every subset $\mathcal{A}_\rho = \{m \in \mathcal{A} : m \equiv \rho \mod a\}$ contains at least a numbers η'_ρ that are distinct modulo c'. Since $\gcd(a, b) = 1$ and $c' > a$, the numbers η_ρ appear as an appropriate selection from the numbers η'_ρ.

Suppose, to the contrary, that one of the sets $\mathcal{A}_\rho$, say $\mathcal{A}_{\rho'}$, contains less than a elements distinct modulo c'. Then the upper asymptotic density $\alpha_{\rho'} = \overline{d}(\mathcal{A}_{\rho'})$ satisfies $\alpha_{\rho'} < \frac{a}{c'}$. Combined with the estimates

$$1 - \frac{1}{2ab} < \overline{d}(\mathcal{A}) \leq \alpha_0 + \alpha_1 + \cdots + \alpha_{a-1} \leq \left(1 - \frac{1}{a}\right) + \alpha_{\rho'}$$

and $\gcd(b, c) \leq b$ it follows that $c < 2a^2 b$, which is a contradiction.

In order to complete the proof of Theorem 1 let $\mathcal{A} \subseteq \mathbb{N}$ be a σ-free set satisfying (3.6), with $\varepsilon > 0$ sufficiently small. Let numbers $n \in \mathbb{N}$ be chosen sufficiently large according to (3.1) and (3.2). We distinguish two cases concerning the intersection of $\mathcal{A}$ with

$$I_z := \left(\frac{a(a+b)}{c^2} n,\ \frac{n}{c} \right].$$

Case 1. $\mathcal{A} \cap I_z = \emptyset$: Then, by Lemma 2,

$$A(n) \leq A\left(\frac{a(a+b)}{c^2} n\right) + \left(1 - \frac{a+b}{c}\right) n + K\varepsilon n. \tag{4.1}$$

By inserting (3.1) and (3.2) we obtain

$$\overline{d}(\mathcal{A}) \left(1 - \frac{a(a+b)}{c^2}\right) \leq 1 - \frac{a+b}{c} + K'\varepsilon$$

with some constant $K' = K'(\sigma, \mathcal{A}) > 0$, which leads to

$$\overline{d}(\sigma) \leq \frac{c^2 - c(a+b)}{c^2 - a(a+b)},$$

as was stated in Theorem 1.

Case 2. $\mathcal{A} \cap I_z \neq \emptyset$: The idea is to count the combinations $cz - b\eta$ and ax in the interval

$$I := \left(\frac{a(a+b)}{c} n,\ n \right]$$

with $z, x \in \mathcal{A}$ for suitably chosen $\eta \in \mathcal{A}$. Observe that $z \in \mathcal{A} \cap I_z$ for $cz \in I$ and $x \in \mathcal{A} \cap I_x$ for $ax \in I$, where

$$I_x := \left(\frac{a+b}{c} n,\ \frac{n}{a} \right].$$

Lemma 4 yields the existence of a fixed subset $\mathcal{E} \subseteq \mathcal{A}$ with the property: For each $z \in \mathcal{A} \cap I_z$ there exists an $\eta = \eta(z) \in \mathcal{E}$ such that

(i) $\varphi(z) := cz - b\eta(z) \equiv 0 \mod a$,

(ii) the mapping $\varphi : \mathcal{A} \cap I_z \to I \cup (I - b\mathcal{E})$ is injective.

Then all numbers $ax, \varphi(z) = cz - b\eta(z)$ with $x \in \mathcal{A} \cap I_x$, $z \in \mathcal{A} \cap I_z$ are distinct, belong to the residue class $0 \mod a$ as well as to $I \cup (I - b\mathcal{E})$. It follows that

$$A\left(\frac{n}{a}\right) - A\left(\frac{a+b}{c}\right) + A\left(\frac{n}{c}\right) - A\left(\frac{a(a+b)}{c^2}n\right) \le \left(\frac{1}{a} - \frac{a+b}{c}\right) n + \max \mathcal{E} + 2.$$

Observing that

$$A(n) - A\left(\frac{n}{a}\right) \le n - \frac{n}{a} \quad \text{and} \quad A\left(\frac{a+b}{n}n\right) \le A\left(\frac{n}{c}\right) + K\varepsilon n$$

we obtain

$$A(n) \le A\left(\frac{a(a+b)}{c^2}n\right) + \left(1 - \frac{a+b}{c}\right) n + K'\varepsilon n$$

with some constant $K' = K'(\sigma, \mathcal{A})$. This is (4.1) again with K replaced by K', and the proof of Theorem 1 is complete.

5. The construction of finite σ-free sets

For the construction of finite σ-free sets $\mathcal{A} \subseteq \{1, \ldots, n\}$ we follow the strategy of Section 2 and use the same notation. For fixed $k \in \mathbb{N}$ let numbers $n_j \in \mathbb{N}$ with $n_1 = n > n_2 > \ldots > n_k$ be chosen such that the σ-free segments

$$\mathcal{S}_j = (qn_j,\ n_j] \cap \mathbb{N} \qquad (j = 1, \ldots, k) \tag{5.1}$$

with $q = \dfrac{a+b}{c} < 1$ are pairwise disjoint. Their union

$$\mathcal{A} = \mathcal{S}_1 \cup \ldots \cup \mathcal{S}_k$$

is σ-free if, for $j = 2, \ldots, k$, the largest element of $c(\mathcal{S}_k \cup \ldots \cup \mathcal{S}_j)$ is less than the smallest element of $a(\mathcal{S}_{j-1} \cup \ldots \cup \mathcal{S}_1) + b\mathcal{A}$. This gives

$$n_1 = n, \quad cn_j \le aqn_{j-1} + bqn_k \qquad (j = 2, \ldots, k). \tag{5.2}$$

From

$$|\mathcal{A}| = (1-q)(n_1 + \cdots + n_k) + r \quad \text{with } |r| \le k \tag{5.3}$$

we see that the largest possible value with respect to our construction occurs if the inequalities (5.2) turn to equations

$$n_1 = n, \quad n_j = \alpha n_{j-1} + \beta n_k \qquad (j = 2, \ldots, k), \tag{5.4}$$

where $\alpha := \frac{aq}{c}$ and $\beta := \frac{bq}{c}$. In contrast to Section 2 the resulting recursion may not be simplified by omitting the second term on the right hand side.

By multiplying with α^{-j} in (5.4), summing over $j = 2, \ldots, j' \le k$ and replacing j' by j we get

$$n_j = \alpha^{j-1} n + \frac{1-\alpha^{j-1}}{1-\alpha} \beta n_k \qquad (j = 1, \ldots, k).$$

For $j = k$ it follows that

$$n_k = \frac{(1-\alpha)\alpha^{k-1}}{1-(\alpha+\beta)+\alpha^{k-1}\beta} n$$

and therefore

$$n_j = \frac{1-(\alpha+\beta)+\alpha^{k-j}\beta}{1-(\alpha+\beta)+\alpha^{k-1}\beta}\, \alpha^{j-1} n \qquad (j = 1, \ldots, k). \tag{5.5}$$

Summation leads to

$$n_1 + \cdots + n_k = \left(\frac{1-(\alpha+\beta)}{1-(\alpha+\beta)+\alpha^{k-1}\beta} \frac{1-\alpha^k}{1-\alpha} + \frac{k\alpha^{k-1}\beta}{1-(\alpha+\beta)+\alpha^{k-1}\beta} \right) n$$

or, equivalently,

$$n_1 + \cdots + n_k = \frac{1+t_k}{1-\alpha} n$$

with

$$t_k = \frac{(1-\alpha)\big((k-1)\,\beta - \alpha\big)}{1-(\alpha+\beta)+\alpha^{k-1}\beta} \alpha^{k-1}. \tag{5.6}$$

We insert this into (5.3) to obtain

$$|\mathcal{A}| = \frac{1-q}{1-\alpha}\, (1+t_k)\, n + r \quad \text{with } |r| \le k. \tag{5.7}$$

In particular, it follows that

$$t_1 = -\alpha, \quad t_2 = \frac{\alpha(\beta-\alpha)}{1-\beta}, \quad t_3 = \frac{\alpha^2(2\beta-\alpha)}{1-\beta-\alpha\beta}. \tag{5.8}$$

Part (a) of the following lemma is already verified.

Lemma 5. *Let σ be given by (1.6) with $a, b, c \in \mathbb{N}$, $a \le b$, $\gcd(a,b) = 1$ and $c > ab(a+b)$. For $k \in \mathbb{N}$ fixed and $n \in \mathbb{N}$ sufficiently large, let the numbers $n_1, \ldots, n_k \ge 1$ and the segments $\mathcal{S}_1, \ldots, \mathcal{S}_k$ be defined according to (5.1) and (5.5), where $q = \frac{a+b}{c}$, $\alpha = \frac{aq}{c}$, $\beta = \frac{bq}{c}$. Then the following assertions hold:*

(a) The set $\mathcal{A} = \mathcal{S}_1 \cup \ldots \cup \mathcal{S}_k \subseteq \{1, \ldots, n\}$ is σ-free with

$$|\mathcal{A}| = \frac{1-q}{1-\alpha}(1+t_k)n + r \qquad (|r| \le k)$$

elements, where t_k is given by (5.6).

(b) If $a = b$ then the maximal number $|\mathcal{A}|$ of elements is taken for $k = 3$, in which case $\mathcal{A} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$ with $n_1 = n$, $n_2 = \frac{(1-\beta)\alpha}{1-\beta-\alpha\beta}n$, $n_3 = \frac{\alpha^2}{1-\beta-\alpha\beta}n$ and

$$|\mathcal{A}| = (1-q)\frac{1-\alpha^2}{1-\alpha-\alpha^2}n + r \qquad (|r| \le 3).$$

(c) If $a < b$ then the maximal number $|\mathcal{A}|$ of elements is taken for $k = 2$, in which case $\mathcal{A} = \mathcal{S}_1 \cup \mathcal{S}_2$ with $n_1 = n$, $n_2 = \frac{\alpha}{1-\beta}n$ and

$$|\mathcal{A}| = (1-q)\frac{1-\beta+\alpha}{1-\beta}n + r \qquad (|r| \le 2).$$

Proof. For the proof of (b) and (c) it suffices to verify that the sequence of numbers t_k, from the first positive term on, strictly decreases. From (5.8) we see that this is the term t_3 for $a = b$ and the term t_2 for $a < b$. With $k^* = 3$ or $k^* = 2$ we thus estimate for $k \ge k^*$:

$$\begin{aligned}
\frac{t_k}{t_{k+1}} &= \frac{1}{\alpha}\,\frac{(k-1)\beta-\alpha}{k\beta-\alpha}\,\frac{1-(\alpha+\beta)+\alpha^k\beta}{1-(\alpha+\beta)+\alpha^{k-1}\beta} > \\
&> \frac{1}{\alpha}\,\frac{(k^*-1)\beta-\alpha}{k^*\beta-\alpha}\,\frac{1-(\alpha+\beta)}{(1-\alpha)(1-\beta)} \ge \\
&\ge \frac{(k^*-1)\beta-\alpha}{k^*\beta-\alpha}\,\frac{1-(\alpha+\beta)}{\alpha}.
\end{aligned}$$

Since
$$\frac{(k^*-1)\beta-\alpha}{k^*\beta-\alpha}=\frac{(k^*-1)b-a}{k^*b-a}\geq\frac{1}{2b},$$
we obtain for $c\geq ab(a+b)+1$ that
$$\frac{t_k}{t_{k+1}}\geq\frac{1-(\alpha+\beta)}{2b\alpha}=\frac{c^2-(a+b)^2}{2ab(a+b)}>1.$$
Finally insertion of the value of t_{k^*} for $k^*=3$ and $k^*=2$ into (5.7) yields the asserted formulas for $|\mathcal{A}|$ and at the same time proves Theorem 2.

The description of the fine structure of extremal finite σ-free sets seems to be more complicated than in the infinite case. Baltz and Schoen [1] consider the special case $a=b=1$ and explicitly construct an injective mapping from any σ-free set $\mathcal{A}\subseteq\{1,\ldots,n\}$ into a σ-free union $\mathcal{A}'$ of segments $\mathcal{S}_j$ of the type (5.1) so that $|\mathcal{A}|\leq|\mathcal{A}'|$. Further they use a result of Chung and Goldwasser [2] which originally states that the maximum measure of a union of intervals $M\subseteq(0,1]$ containing no real solution to $x+y=cz$ is attained for the union M of three specific subintervals of $(0,1]$ (compare our Lemma 5 (b)). In the general case of systems σ defined by an equation $ax+by=cz$ with $a,b,c\in\mathbb{N}$, $a<b$, and c sufficiently large, we conjecture the optimality of the lower bound for $\overline{D}(\sigma)$ in Theorem 2.

We remark that in this case the method of Section 3 can be used to show that extremal finite σ-free sets necessarily have large gaps, too. Let $\varepsilon>0$. Due to the definition of $\overline{D}(\sigma)$ there exists a number $n_0=n_0(\varepsilon,\sigma)$ such that *for all* $n\geq n_0$ and *all* σ-free sets $\mathcal{A}\subseteq\{1,\ldots,n\}$

$$\rho(\mathcal{A},n)\leq\overline{D}(\sigma)+\varepsilon, \tag{5.9}$$

and there *exist infinitely many* $n\in\mathbb{N}$ and σ-free sets $\mathcal{A}\subseteq\{1,\ldots,n\}$ satisfying

$$\rho(\mathcal{A},n)\geq\overline{D}(\sigma)-\varepsilon. \tag{5.10}$$

Now the proof of the following lemma follows precisely that of Lemma 2 with $\overline{D}(\sigma)$ instead of $\overline{d}(\mathcal{A})$.

Lemma 6. *Let σ be given by (1.6) with $a,b,c\in\mathbb{N}$, $a\leq b$, $\gcd(a,b)=1$ and $c>a(a+b)$. Then, for every $\varepsilon>0$, there are infinitely many numbers $n\in\mathbb{N}$ and σ-free sets $\mathcal{A}\subseteq\{1,\ldots,n\}$ such that both inequalities (5.9) and (5.10) are valid. Moreover, there exists a constant $K>0$ depending only on σ such that for all numbers n and σ-free sets $\mathcal{A}\subseteq\{1,\ldots,n\}$ of this kind the following statements hold:*

(a) *If* $\overline{D}(\sigma) > 1 - \frac{1}{a+b}$ *then* $\left|\mathcal{A} \cap \left(\frac{b}{c}n,\ \frac{a+b}{c}n\right]\right| \le K\varepsilon n.$

(b) *If* $\overline{D}(\sigma) > 1 - \frac{1}{2b}$ *then* $\mathcal{A} \cap \left(\frac{a}{c}n,\ \frac{b}{c}n\right] = \emptyset.$

(c) *If* $\overline{D}(\sigma) > 1 - \frac{1}{2ab}$ *then* $\mathcal{A} \cap \left(\frac{1}{c}n,\ \frac{a}{c}n\right] = \emptyset.$

A recursive application of Lemma 6 would suffice to obtain equality in Theorem 2, at least for $a = 1$, if the open problem whether $\overline{D}(\sigma) = \underline{D}(\sigma)$ could be decided positively.

References

[1] **Baltz A. and Schoen T.,** *The structure of maximum subsets of* $\{1, \dots, n\}$ *with no solutions to* $a + b = kc$, preprint, 2001.

[2] **Chung R.K. and Goldwasser J.L.,** Integer sets containing no solutions to $x + y = 3z$, *The mathematics of Paul Erdős,* eds. R.L. Graham and J. Nešetřil, Springer, 1997, 218-227.

[3] **Klotz W.,** Generalization of some theorems on sets of multiples and primitive sequences, *Acta Arith.,* **32** (1977), 15-26.

[4] **Lucht L.G.,** Dichteschranken für die Lösbarkeit gewisser linearer Gleichungen, *J. Reine Angew. Math.,* **285** (1976), 209-217.

[5] **Ostmann H.,** *Additive Zahlentheorie I.,* Springer, Heidelberg, 1956.

[6] **Ruzsa I.Z.,** Solving a linear equation in a set of integers I., *Acta Arith.,* **65** (1993), 259-282.

[7] **Ruzsa I.Z.,** Solving a linear equation in a set of integers II., *Acta Arith.,* **72** (1995), 385-397.

[8] **Schoen T.,** On sets of natural numbers without solution to a noninvariant linear equation, *Acta Arith.,* **93** (2000), 149-155.

[9] **Szemerédi E.,** On sets of integers containing no four elements in arithmetic progression, *Acta Arith.,* **20** (1969), 89-104.

[10] **Szemerédi E.,** On sets of integers containing no k elements in arithmetic progression, *Acta Arith.,* **27** (1975), 199-245.

L.G. Lucht
Institut für Mathematik
Technische Universität Clausthal
Erzstraße 1
D-38678 Clausthal-Zellerfeld, Deutschland
lucht@math.tu-clausthal.de