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THE JOINT UNIVERSALITY FOR GENERAL DIRICHLET SERIES

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In honour of Professor Karl-Heinz Indlekofer on the occasion of his 60th birthday

1. Introduction

In 1975 S.M. Voronin [14] discovered one more remarkable property of the Riemann zeta-function $\zeta(s)$, $s=\sigma+it$. Roughly speaking he proved that any analytic function can be approximated uniformly on some sets by translations $\zeta(s+i\tau)$. The latter property of $\zeta(s)$ is called the universality. To state the last version of Voronin's theorem we need some notations. Let *meas*$\{A\}$ denote the Lebesgue measure of the set A, and let, for $T>0$,

$$\nu_T(\dots)=\frac{1}{T}\text{meas}\{\tau\in[0,T]:\ \dots\},$$

where in place of the dots a condition satisfied by τ is to be written. As usual, $\mathbb{C}$ stands for the complex plane.

Voronin's theorem ([5]). *Let K be a compact subset of the strip $\{s\in\in\mathbb{C}:\frac{1}{2}<\sigma<1\}$ with connected complement, and let $g(s)$ be a non-vanishing continuous function on K which is analytic in the interior K. Then, for every $\varepsilon>0$,*

$$\liminf_{T\to\infty}\nu_T\left(\sup_{s\in K}\left|\zeta(s+i\tau)-g(s)\right|<\varepsilon\right)>0.$$

Later the universality of zeta-functions was studied by many mathematicians, among them by S.M. Gonek, A. Reich, B. Bagchi, K. Matsumoto, H. Mishou, W. Schwarz, J. Steuding, by the author, and by others.

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The paper [11] is devoted to the universality of general Dirichlet series

$$\sum_{m=1}^{\infty} a_m e^{-\lambda_m s}, \tag{1}$$

where $a_m \in \mathbb{C}$ and $\{\lambda_m\}$ is an increasing sequence, $\lambda_m \to +\infty$. Denote by σ_a the abscissa of absolute convergence of series (1), and let $f(s)$, for $\sigma > \sigma_a$, be its sum. The universality for the function $f(s)$ requires several additional conditions.

Let the system of exponents $\{\lambda_m\}$ of series (1) be linearly independent over the field of rational numbers. We suppose that $f(s)$ cannot be represented by an Euler product over primes in the half-plane $\sigma > \sigma_a$. Moreover, we suppose that $f(s)$ is meromorphically continuable to the half-plane $\sigma > \sigma_1$ with some $\sigma_1 < \sigma_a$, and that it is analytic in the strip

$$D = \{s \in \mathbb{C} : \ \sigma_1 < \sigma < \sigma_a\}.$$

For conditions of the continuation see, for example, [13]. Furthermore we assume that, for $\sigma > \sigma_1$, the estimates

$$f(s) = B|t|^{\alpha}, \qquad |t| \geq t_0, \quad \alpha > 0,$$

and

$$\int_{-T}^{T} \left|f(\sigma + it)\right|^2 dt = BT, \qquad T \to \infty,$$

are satisfied. Here and what follows B denotes a quantity bounded by some constant. Moreover, we require some conditions on the sequences $\{a_m\}$ and $\{\lambda_m\}$. Let, for $x > 0$,

$$r(x) = \sum_{\lambda_m \leq x} 1,$$

and $c_m = a_m e^{-\lambda_m \sigma_a}$. Suppose that there exists a real $\theta > 0$ with the property

$$\sum_{\lambda_m \leq x} |c_m|^2 = \theta r(x)(1 + o(1))$$

as $x \to \infty$, and that

$$|c_m| \leq d$$

for some $d > 0$. Finally, we suppose that

$$r(x) = C_1 x^{\kappa} + B \tag{2}$$

with $\kappa \geq 1$ and positive C_1, and $|B| \leq C_2$. Then in [11] the following statement was obtained.

Theorem A. *Suppose that the function $f(s)$ satisfies all the conditions stated. Let K be a compact subset of the strip $\{s \in \mathbb{C} : \sigma_1 < \sigma < \sigma_a\}$ with connected complement, and let $g(s)$ be a continuous function on K which is analytic in the interior of K. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \nu_T\left(\sup_{s\in K}\left|f(s+i\tau)-g(s)\right| < \varepsilon\right) > 0.$$

The aim of this note is to obtain a joint universality theorem for general Dirichlet series, i.e. to prove that a collection of analytic functions can be simultaneously approximated by translations of general Dirichlet series. We recall that the joint universality for Dirichlet L-functions $L(s,\chi)$ was proved independently by S.M. Voronin [15-16], S. M. Gonek [4] and B. Bagchi [1-2]. For example, in [15] we find the following statement.

Theorem B. *Suppose that $0 < r < 1/4$. Let $\chi_1, \dots, \chi_n$ be pairwise nonequivalent Dirichlet characters, and let $f_1(s), \dots, f_n(s)$ be continuous and non-vanishing on $|s| \leq r$ functions which are analytic on $|s| < r$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty}\left(\max_{1\leq j\leq n}\max_{|s|\leq r}\left|L\left(s+\frac{3}{4}+i\tau,\chi_j\right)-f_j(s)\right| < \varepsilon\right) > 0.$$

The paper [6] contains a joint conditional universality theorem for Matsumoto zeta-functions, while in [7] the joint universality for Lerch zeta-functions is obtained. The paper [8] is devoted to the joint universality for zeta-functions attached to certain cusp forms. In [9] the joint universality of twisted automorphic L-functions is proved.

During the conference "Theory of the Riemann zeta and allied functions" at Oberwolfach in 2001 Professor E. Bombieri in discussion with the author noted that joint universality theorems for zeta-functions are an interesting and important problem of analytic number theory. It seems to be that the joint universality for general Dirichlet series is rather complicated problem. Therefore, we limit ourselves by the investigation of a collection of general Dirichlet series with the same sequence of exponents $\{\lambda_m\}$.

Let, for $\sigma > \sigma_{aj}$, the series

$$f_j(s) = \sum_{m=1}^{\infty} a_{mj} e^{-\lambda_m s}$$

converges absolutely, $j = 1, \dots, n$. Suppose that $f_j(s)$ is meromorphically continuable to the half-plane $\sigma > \sigma_{1j}$ with some $\sigma_{1j} < \sigma_{aj}$, all poles being included in a compact set, that it is analytic in the strip $D_j = \{s \in \mathbb{C} : \sigma_{1j} < < \sigma < \sigma_{aj}\}$, and that $f_j(s)$ cannot be represented by an Euler product over primes in the region $\sigma > \sigma_{aj}$, $j = 1, \dots, n$. We also require that, for $\sigma > \sigma_{1j}$, the estimates

$$f_j(\sigma + it) = B|t|^{\alpha_j}, \qquad |t| \geq t_0, \quad \alpha_j > 0, \tag{3}$$

and

$$\int_{-T}^{T} \left| f_j(\sigma + it) \right|^2 dt = BT, \qquad T \to \infty, \tag{4}$$

hold, $j = 1, \dots, n$. Moreover, we need some conditions on the sequences $\{a_{mj}\}$ and $\{\lambda_m\}$. We suppose that the system $\{\log 2\} \cup \bigcup_{m=1}^{\infty} \{\lambda_m\}$ is linearly independent over the field of rational numbers. Let

$$c_{mj} = a_{mj} e^{-\lambda_m \sigma_{aj}}, \qquad j = 1, \dots, n.$$

Suppose that there exist $r \geq n$ sets $\mathbb{N}_k$, $\mathbb{N}_{k_1} \cup \mathbb{N}_{k_2} = \emptyset$ for $k_1 \neq k_2$, $\mathbb{N} = \bigcup_{k=1}^{r} \mathbb{N}_k$, such that $c_{mj} = b_{kj}$ for $m \in \mathbb{N}_k$, $k = 1, \dots, r$, $j = 1, \dots, n$. We set

$$B = \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \dots & \dots & \dots \\ b_{r1} & \dots & b_{rn} \end{pmatrix}.$$

We also suppose that the sequence of exponents $\{\lambda_m\}$ satisfies the relation (2), and that

$$\sum_{\substack{\lambda_m \leq x \\ m \in \mathbb{N}_k}} 1 = \kappa_k r(x)(1 + o(1)), \qquad x \to \infty, \tag{5}$$

with positive κ_k, $k = 1, \dots, r$.

Theorem. *Suppose that conditions (2)–(4) are satisfied, and that* $\operatorname{rank}(B) = n$. *Let K_j be a compact set of the strip D_j with connected complement, and let $g_j(s)$ be a continuous function on K_j which is analytic in the interior of K_j, $j = 1, \dots, n$. Then, for any $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \nu_T \left(\sup_{1 \leq j \leq n} \sup_{s \in K_j} \left| f_j(s + i\tau) - g_j(s) \right| < \varepsilon \right) > 0.$$

2. A limit theorem for the functions $f_j(s)$

For any region G on the complex plain, by $H(G)$ denote the space of analytic on G functions equipped with the topology of uniform convergence on compacta. Let $N > 0$,

$$D_{j,N} = \{s \in \mathbb{C} : \ \sigma_{1j} < \sigma < \sigma_{aj}, \quad |t| < N\}, \qquad j = 1, \dots, n,$$

and let

$$H_{n,N} = H_n(D_{1,N}, \dots, D_{n,N}) = H(D_{1,N}) \times \dots \times H(D_{n,N}).$$

Denote by $\mathcal{B}(S)$ the class of Borel sets of the space S, and define the probability measure

$$P_T(A) = \nu_T\Big(\big(f_1(s_1 + i\tau), \dots, f_n(s_n + i\tau)\big) \in A\Big), \qquad A \in \mathcal{B}(H_{n,N}).$$

To prove the theorem we need a limit theorem for the measure P_T as $T \to \infty$ in the sense of the weak convergence of probability measures. Moreover, the limit measure in such theorem must be explicitly given. For this, define the following topological structure. Let γ denote the unit circle on $\mathbb{C}$, and

$$\Omega = \prod_{m=1}^{\infty} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m \in \mathbb{N}$. The infinitedimensional torus Ω is a compact topological Abelian group, therefore on $(\Omega, \mathcal{B}(\Omega))$ the probability Haar measure m_H exists, and we obtain a probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Let $\omega(m)$ stand for the projection of $\omega \in \Omega$ onto the coordinate space γ_m.

Now on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ we define the $H_{n,N}$-valued random element $f(s_1, \dots, s_n; \omega)$ by the formula

$$f(s_1, \dots, s_n; \omega) = \big(f_1(s_1, \omega), \dots, f_n(s_n, \omega)\big),$$

where

$$f_j(s_j, \omega) = \sum_{m=1}^{\infty} a_{mj}\omega(m)e^{-\lambda_m s_j}, \qquad s_j \in D_{j,N}, \quad j = 1, \dots, n,$$

and let P_f be the distribution of the random element $f(s_1, \ldots, s_n; \omega)$, i.e.

$$P_f(A) = m_H\big(\omega \in \Omega : \ f(s_1, \ldots, s_n; \omega) \in A\big), \qquad A \in \mathcal{B}(H_{n,N}).$$

Lemma 1. *The probability measure P_T converges weakly to the measure P_f as $T \to \infty$.*

Proof. Let $\widehat{D}_j = \{s \in \mathbb{C} : \sigma > \sigma_{1j}\}$,

$$\widehat{M}_n = M(\widehat{D}_1) \times \ldots \times M(\widehat{D}_n),$$

and

$$\widehat{P}_T(A) = \nu_T\Big(\big(f_1(s_1 + i\tau), \ldots, f_n(s_n + i\tau)\big) \in A\Big), \qquad A \in \mathcal{B}(M_n),$$

where $M(G)$ denotes the space of meromorphic on G functions equipped with the topology of uniform convergence on compacta. We put

$$\widehat{H}_n = \widehat{H}_n(\widehat{D}_1, \ldots, \widehat{D}_n) = H(\widehat{D}_1) \times \ldots \times H(\widehat{D}_n),$$

and define on $(\Omega, \mathcal{B}(\Omega), m_H)$ an $\widehat{H}_n$-valued random element $\widehat{f}(s_1, \ldots, s_n; \omega)$ by the formula

$$\widehat{f}(s_1, \ldots, s_n; \omega) = \big(\widehat{f}(s_1, \omega), \ldots, \widehat{f}(s_n, \omega)\big),$$

where

$$\widehat{f}_j(s_j, \omega) = \sum_{m=1}^{\infty} a_{mj}\omega(m)e^{-\lambda_m s_j}, \qquad s_j \in \widehat{D}_j, \quad j = 1, \ldots, n.$$

Then in [12] it was proved that the probability measure P_T converges weakly to the distribution of the random element $\widehat{f}(s_1, \ldots, s_n; \omega)$ as $T \to \infty$. Note that in [12] the condition

$$\lambda_{mj} \geq c_j(\log m)^{\theta_j},$$

with positive c_j and θ_j, $j = 1, \ldots, n$, is covered by (2)–(4). Let

$$M_{n,N} = M(D_{1,N}) \times \ldots \times M(D_{n,N}).$$

Since the function $h : \widehat{M}_n \to M_{n,N}$, defined by the coordinatewise restriction, is continuous, hence in view of Theorem 5.1 from [3], we obtain the assertion of the lemma.

3. Functions of exponential type

We recall that an entire function $g(s)$ is of exponential type if

$$\limsup_{r\to\infty} \frac{\log|g(re^{i\theta})|}{r} < \infty$$

uniformly in θ, $|\theta| \le \pi$.

Now we state some lemmas on the functions of exponential type.

Lemma 2. *Let $g(s)$ be an entire function of exponential type, and let $\{\xi_m\}$ be a sequence of complex numbers. Moreover, let α_1, α_2 and α_3 be real positive numbers satisfying*

1^0 $\displaystyle\limsup_{x\to\infty} \frac{\log|g(\pm ix)|}{x} \le \alpha_1$;

2^0 $|\xi_m - \xi_n| \ge \alpha_2 |m-n|$;

3^0 $\displaystyle\lim_{m\to\infty} \frac{\xi_m}{m} = \alpha_3$;

4^0 $\alpha_1\alpha_2 < \pi$.

Then

$$\limsup_{m\to\infty} \frac{\log|g(\xi_m)|}{|\xi_m|} = \limsup_{r\to\infty} \frac{\log|g(r)|}{r}.$$

The lemma is a version of Bernstein's theorem, for the proof see Theorem 6.4.12 of [5].

Lemma 3. *Let μ be a complex-valued measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ having compact support contained in the half-plane $\sigma > \sigma_0$. Define*

$$g(s) = \int\limits_{\mathbb{C}} e^{sz}\, d\mu(z)$$

and suppose that $g(s) \not\equiv 0$. Then

$$\limsup_{r\to\infty} \frac{\log|g(r)|}{r} > \sigma_0.$$

Proof of the lemma can be found in [5], Theorem 6.4.14.

4. The support of the random element f

This section is devoted to the support of the measure P_f. We recall that the minimal closed set $S_{P_f} \subseteq H_{n,N}$ such that $P_f(S_{P_f}) = 1$ is called the support of P_f. The set S_{P_f} consists of all $\underline{g} = (g_1(s_1), \dots, g_n(s_n)) \in H_{n,N}$ such that for every neighbourhood $\mathcal{G}$ of $\underline{g}$ the inequality $P_f(\mathcal{G}) > 0$ is satisfied.

The support of distribution of the random element X is called the support of X and is denoted by S_X.

We begin with some auxiliary lemmas.

Lemma 4. *Let $\{X_m\}$ be a sequence of independent $H_{n,N}$-valued random elements such that the series*

$$\sum_{m=1}^{\infty} X_m$$

converges almost surely. Then the support of the sum of the latter series is the closure of the set of all $\underline{g} \in H_{n,N}$ which may be written as convergent series

$$\underline{g} = \sum_{m=1}^{\infty} \underline{g}_m, \qquad \underline{g}_m \in S_{X_m}.$$

Proof of the lemma is given in [8].

Lemma 5. *Let $\{\underline{g}_m\} = \{(g_{1m}, \dots, g_{nm})\}$ be a sequence in $H_{n,N}$ which satisfies:*

1^0 *If $\mu_1, \dots, \mu_n$ are complex measures on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ with compact supports contained in $D_{1,N}, \dots, D_{n,N}$, respectively, such that*

$$\sum_{m=1}^{\infty} \left| \sum_{j=1}^{n} \int_{\mathbb{C}} g_{jm}\, d\mu_j \right| < \infty,$$

then

$$\int_{\mathbb{C}} s^r \, d\mu_j(s) = 0$$

for $j = 1, \dots, n$, $r = 0, 1, 2, \dots$;

2^0 *The series*

$$\sum_{m=1}^{\infty} \underline{g}_m$$

converges in $H_{n,N}$*;*

3^0 *For any compacts* $K_1 \subseteq D_{1,N}, \dots, K_n \subseteq D_{n,N}$*,*

$$\sum_{m=1}^{\infty} \sum_{j=1}^{n} \sup_{s \in K_j} \left| g_{jm}(s) \right|^2 < \infty.$$

Then the set of all convergent series

$$\sum_{m=1}^{\infty} a_m \underline{g}_m$$

with $a_m \in \gamma$ *is dense in* $H_{n,N}$*.*

Proof of the lemma can be found in [8].

Let ν be a complex measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ with compact support contained in

$$\left\{ s \in \mathbb{C} : \min_{1 \le j \le n} (\sigma_{1j} - \sigma_{aj}) < \sigma < 0, |t| < N \right\}.$$

Define

$$w(z) = \int_{\mathbb{C}} e^{-sz} \, d\nu(s), \qquad z \in \mathbb{C}.$$

Lemma 6. *Suppose that, for some* k*,*

$$\sum_{\substack{m=1 \\ m \in \mathbb{N}_k}}^{\infty} \left| w(\lambda_m) \right| < \infty.$$

Then

$$\int_{\mathbb{C}} s^l \, d\nu(s) = 0, \qquad l = 0, 1, 2, \dots.$$

Proof. Let $g(s) = w(s)$ in Lemma 3. Since, for $x > 0$,

$$|w(\pm x)| \le e^{Nx} \int_{\mathbb{C}} |d\nu(s)|,$$

condition 1^0 of Lemma 2 is satisfied with $\alpha_1 = N$. Now we fix an α_3 satisfying

$$0 < \alpha_3 < \frac{\pi}{N},$$

and a real number ξ with

$$C_1\alpha_3\xi > C_2, \tag{6}$$

where the constants C_1 and C_2 are given by formula (2). Define

$$A = \Big\{ l \in \mathbb{N}:\ \exists r \in \big((l-\xi)\alpha_3, (l+\xi)\alpha_3\big] \quad \text{with} \quad |w(r)| \le \frac{1}{r^\kappa} \Big\}.$$

We have

$$\sum_{\substack{m=1\\ m\in\mathbb{N}_k}}^{\infty} |w(\lambda_m)| \ge \sum_{l\notin A}{\sum_m}' |w(\lambda_m)| \ge \sum_{l\notin A}{\sum_m}' \frac{1}{\lambda_m^\kappa},$$

where ${\sum}'_m$ denotes the sum over all $m \in \mathbb{N}_k$ satisfying the inequalities

$$(l-\xi)\alpha_3 < \lambda_m \le (l+\xi)\alpha_3.$$

This and the hypothesis of the lemma yield

$$\sum_{l\notin A} \sum_{\substack{m\in\mathbb{N}_k\\ a<\lambda_m\le b}} \frac{1}{\lambda_m^\kappa} < \infty \tag{7}$$

with $a = (l-\xi)\alpha_3$, $b = (l+\xi)\alpha_3$. Summing by parts and applying conditions (2) and (5), we find

$$\sum_{\substack{m\in\mathbb{N}_k\\ a<\lambda_m\le b}} \frac{1}{\lambda_m^\kappa} =$$

$$= \frac{1}{b^\kappa} \sum_{\substack{m\in\mathbb{N}_k\\ a<\lambda_m\le b}} 1 + \kappa\int_a^b \Big(\sum_{\substack{m\in\mathbb{N}_k\\ a<\lambda_m\le u}} 1 \Big) \frac{du}{u^{\kappa+1}} \ge \big(r(b)-r(a)\big)\frac{\kappa_k}{b^\kappa}(1+o(1)) \ge$$
$$\ge \frac{C_1\kappa_k}{b^\kappa}(1+o(1))\Big(\big((l+\xi)\alpha_3\big)^\kappa - \big((l-\xi)\alpha_3\big)^\kappa\Big) - \frac{2\kappa_k}{b^\kappa}C_2(1+o(1)) =$$
$$= \frac{C_1\kappa_k\alpha_3^\kappa l^\kappa}{b^\kappa}(1+o(1))\bigg(\Big(1+\frac{\xi}{l}\Big)^\kappa - \Big(1-\frac{\xi}{l}\Big)^\kappa\bigg) - \frac{2\kappa_k C_2}{b^\kappa}(1+o(1)) =$$
$$= \frac{2C_1\kappa_k\alpha_3^\kappa l^\kappa\kappa\xi}{b^\kappa l}(1+o(1)) + \frac{B}{l^2} - \frac{2\kappa_k C_2}{b^\kappa}(1+o(1))$$

as $l\to\infty$. Therefore, (6) and (7) yield

$$\sum_{l\notin A} \frac{1}{l} < \infty. \tag{8}$$

Let

$$A = \{a_l : \ l \in \mathbb{N}\}, \qquad a_1 < a_2 < \dots .$$

Then from (8) we find

$$\lim_{l\to\infty} \frac{a_l}{l} = 1. \tag{9}$$

By the definition of the set A there exists a sequence ξ_l such that

$$(a_l - \xi)\alpha_3 < \xi_l \le (a_l + \xi)\alpha_3$$

and

$$|w(\xi_l)| \le \frac{1}{\xi_l^{\kappa}} .$$

This and (9) show that

$$\lim_{l\to\infty} \frac{\xi_l}{l} = \alpha_3,$$

and

$$\limsup_{l\to\infty} \frac{\log|w(\xi_l)|}{\xi_l} \le 0.$$

Moreover, in view of (9)

$$|\xi_m - \xi_n| > |a_m - a_n|\alpha_3 \ge \alpha_2|m-n|$$

with some positive constant α_2. Therefore, applying Lemma 2, we find that

$$\limsup_{r\to\infty} \frac{\log|w(r)|}{r} \le 0. \tag{10}$$

On the other hand, by Lemma 3, if $w(s) \not\equiv 0$, then

$$\limsup_{r\to\infty} \frac{\log|w(r)|}{r} > 0,$$

and this contradicts (10). Consequently, $w(s) \equiv 0$, and the lemma follows by differentiation.

Lemma 7. *The support of the random element $f(s_1,\dots,s_n;\omega)$ is the whole of $H_{n,N}$.*

Proof. Let, for $m = 1, 2, \dots$,

$$\underline{f}_m(s_1,\dots,s_n;\omega(m)) =$$

$$= \big(f_m(s_1,\omega),\dots,f_m(s_n,\omega)\big) = \big(a_{m1}\omega(m)e^{-\lambda_m s_1},\dots,a_{mn}\omega(n)e^{-\lambda_m s_n}\big).$$

It follows from the definition of Ω that $\{\omega(m)\}$ is a sequence of independent random variables with respect to the measure m_H. Hence $\{\underline{f}_m(s_1,\dots,s_n;\omega(m)\}$ is a sequence of independent $H_{n,N}$-valued random elements defined on the probability space $(\Omega,\mathcal{B}(\Omega),m_H)$. The support of each random variable $\omega(m)$ is the unit circle γ. Therefore, the set $\{\underline{f}_m(s_1,\dots,s_n;a) : a\in\gamma\}$ is the support of the random element $\underline{f}_m(s_1,\dots,s_n;\omega(m))$. Hence in virtue of Lemma 4 the closure of the set of all convergent series

$$\sum_{m=1}^{\infty}\underline{f}_m(s_1,\dots,s_n;a_m),\qquad a_m\in\gamma,$$

is the support of the random element $f(s_1,\dots,s_n;\omega)$. To prove the lemma it remains to check that the latter set is dense in $H_{n,N}$. For this we will apply Lemma 5.

Let $\mu_1,\dots,\mu_n$ be complex measures on $(\mathbb{C},\mathcal{B}(\mathbb{C}))$ with compact supports contained in $D_{1,N},\dots,D_{n,N}$, respectively, such that

$$\sum_{m=1}^{\infty}\left|\sum_{j=1}^{n}\int\limits_{\mathbb{C}} a_{mj}e^{-\lambda_m s}\,d\mu_j(s)\right| < \infty. \tag{11}$$

Now let

$$h_j(s) = s-\sigma_{aj},\qquad j=1,\dots,n.$$

Then we have that

$$\mu_j h_j^{-1}(A) = \mu_j(h_j^{-1}A),\qquad A\in\mathcal{B}(\mathbb{C}),$$

is a complex measure of $(\mathbb{C},\mathcal{B}(\mathbb{C}))$, with compact support contained in $\widehat{D}_{j,N} = \{s\in\mathbb{C} : \sigma_{1j}-\sigma_{aj}<\sigma<0, |t|<N\}$, $j=1,\dots,n$. Now (11) can be rewritten in the form

$$\sum_{m=1}^{\infty}\left|\sum_{j=1}^{n}c_{mj}\int\limits_{\mathbb{C}} e^{-\lambda_m s}\,d\mu_j h_j^{-1}(s)\right| < \infty.$$

This together with hypotheses on c_{mj} leads to

$$\sum_{\substack{m=1\\ m\in\mathbb{N}_k}}^{\infty}\left|\sum_{j=1}^{n}b_{kj}\int\limits_{\mathbb{C}} e^{-\lambda_m s}\,d\mu_j h_j^{-1}(s)\right| < \infty,\qquad k=1,\dots,r. \tag{12}$$

Taking

$$\widehat{\mu}_k(A) = \sum_{j=1}^{n} b_{kj}\mu_j h_j^{-1}(A), \qquad A \in \mathcal{B}(\mathbb{C}),$$

$$v_k(z) = \int\limits_{\mathbb{C}} e^{-sz}\, d\widehat{\mu}_k(s), \qquad z \in \mathbb{C}, \quad k = 1, \dots, r,$$

we write (12) in the form

$$\sum_{\substack{m=1 \\ m \in \mathbb{N}_k}}^{\infty} |v_k(\lambda_m)| < \infty, \qquad k = 1, \dots, r.$$

Clearly, $\widehat{\mu}_k$, $k = 1, \dots, r$, is a complex measure on $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ with compact support contained in

$$\Big\{s \in \mathbb{C} : \min_{1 \le j \le n} (\sigma_{1j} - \sigma_{aj}) < \sigma < 0, \quad |t| < N\Big\}.$$

Now Lemma 3 shows that $v_k(z) \equiv 0$, and thus

$$\int\limits_{\mathbb{C}} s^l\, d\widehat{\mu}_k(s) = 0, \qquad l = 0, 1, 2, \dots, \quad k = 1, \dots, r.$$

Hence, using the definition of $\widehat{\mu}_k$ and the properties of the matrix B, we obtain that

$$\int\limits_{\mathbb{C}} s^l\, d\mu_j h_j^{-1}(s) = 0, \qquad l = 0, 1, 2, \dots, \quad j = 1, \dots, n,$$

and this together with definition of the function h_j implies the relations

$$\int\limits_{\mathbb{C}} s^l\, d\mu_j(s) = 0, \qquad l = 0, 1, 2, \dots, \quad j = 1, \dots, n. \tag{13}$$

In the proof that

$$f_j(s_j, \omega) = \sum_{m=1}^{\infty} a_{mj}\omega(m) e^{-\lambda_m s_j}$$

is an $H(D_j)$-valued random element, $D_j = \{s \in \mathbb{C} : \sigma > \sigma_{1j}\}$, it is proved in [10] that, for almost all $\omega \in \Omega$, the series for $f_j(s_j, \omega)$ converges uniformly on compact subsets of D_j, $j = 1, \dots, n$. Therefore, there exists a sequence $\{b_m : b_m \in \gamma\}$ such that

$$\sum_{m=1}^{\infty} \underline{f}_m(s_1, \dots, s_n; b_m)$$

converges in $H_{n,N}$. Moreover, in [10] it was obtained that, for $\sigma > \sigma_{1j}$,

$$\sum_{m=1}^{\infty} |a_{mj}|^2 e^{-2\lambda_m \sigma} < \infty, \qquad j = 1, \dots, n.$$

Hence, by the well-known property of Dirichlet series, see Corollary 2.1.3 of [5], we have that for any compacts $K_j \subseteq D_{j,N}$, $j = 1, \dots, n$,

$$\sum_{m=1}^{\infty} \sum_{j=1}^{n} \sup_{s \in K_j} \left| f_{mj}(s, b_m) \right|^2 < \infty.$$

Since $|b_m| = 1$, condition (13) is valid also for $\underline{f}(s_1, \dots, s_n; b_m)$. Thus we have that all conditions of Lemma 3 for $\underline{f}_m(s_1, \dots, s_n; b_m)$ are satisfied, and therefore the set of all convergent series

$$\sum_{m=1}^{\infty} a_m^* \underline{f}_m(s_1, \dots, s_n; b_m)$$

with $a_m^* \in \gamma$ is dense in $H_{n,N}$. Hence the set of all convergent series

$$\sum_{m=1}^{\infty} \underline{f}_m(s_1, \dots, s_n; a_m), \qquad a_m \in \gamma,$$

is dense in $H_{n,N}$, and the closure of this set is the whole of $H_{n,N}$. The lemma is proved.

5. Proof of the Theorem

The proof is similar to that given in [8]. First we suppose that the functions $g_j(s)$ can be continued analytically to the whole of $D_{j,N}$, respectively, $j = 1, \dots, n$. Denote by G the set of all $(y_1, \dots, y_n) \in H_{n,N}$ such that

$$\sup_{1 \le j \le n} \sup_{s \in K_j} \left| y_j(s) - g_j(s) \right| < \frac{\varepsilon}{4}.$$

The set G is open. Therefore, Lemma 1, properties of the weak convergence and Lemma 5 show that

$$\liminf_{T\to\infty}\left(\sup_{1\le j\le n}\sup_{s\in K_j}\left|f_j(s+i\tau)-g_j(s)\right|<\frac{\varepsilon}{4}\right)\ge P_f(G)>0.$$

Now let the functions $g_j(s)$, $j=1,\dots,n$, be the same as in the statement of the theorem. By the Mergelyan theorem, see, for example, [17], there exist polynomials $p_j(s)$, $j=1,\dots,n$, such that

$$\sup_{1\le j\le n}\sup_{s\in K_j}\left|p_j(s)-g_j(s)\right|<\frac{\varepsilon}{2}. \tag{14}$$

By the first part of the proof we have

$$\liminf_{T\to\infty}\nu_T\left(\sup_{1\le j\le n}\sup_{s\in K_j}\left|f_j(s+i\tau)-p_j(s)\right|<\frac{\varepsilon}{2}\right)>0. \tag{15}$$

In view of (14)

$$\left\{\tau:\sup_{1\le j\le n}\sup_{s\in K_j}\left|f_j(s+i\tau)-p_j(s)\right|<\frac{\varepsilon}{2}\right\}\subseteq$$
$$\subseteq\left\{\tau:\sup_{1\le j\le n}\sup_{s\in K_j}\left|f_j(s+i\tau)-g_j(s)\right|<\varepsilon\right\}.$$

Therefore this and (15) yield that

$$\liminf_{T\to\infty}\nu_T\left(\sup_{1\le j\le n}\sup_{s\in K_j}\left|f_j(s+i\tau)-g_j(s)\right|<\varepsilon\right)>0,$$

and the theorem is proved.

Now we give an example. Let a_{mj} be a periodic sequence with period $r\ge n$, $j=1,\dots,n$, and $\lambda_m=(m+\alpha)^\beta$ with some transcendental $\alpha>0$ and $\beta\in(0,1)$. Then

$$f_j(s)=\sum_{m=1}^{\infty}a_{mj}e^{-(m+\alpha)^\beta s}$$

converges absolutely for $\sigma>\sigma_{aj}=0$, $j=1,\dots,n$, and $c_{mj}=a_{mj}$ is constant on the set $\mathbb{N}_k=\{m\in\mathbb{N}: m\equiv k \pmod r\}$, $j=1,\dots,n$. Moreover,

$$r(x)=\sum_{(m+\alpha)^\beta\le x}1=x^{1/\beta}+B.$$

Clearly, elements b_{kj} can be chosen so that rank $(B) = n$. Therefore, assuming that $f_j(s)$ is analytically continuable to a half-plane $\sigma > \sigma_{1j}$ with $\sigma_{1j} < \sigma_{aj}$, and the estimates (3) and (4) are satisfied, we obtain the joint universality of functions $f_1(s), \ldots, f_n(s)$.

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