

ON THE DISTRIBUTION OF EXPONENTIAL DIVISORS

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*Dedicated to Professor Karl-Heinz Indlekofer
on his sixtieth birthday*

1. Introduction

1.1. Let \mathcal{P} be the whole set of the primes. p, p_i, p_j always denote prime numbers. For some integer n with prime decomposition $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ the divisor $d = p_1^{\beta_1} \dots p_r^{\beta_r}$ is said to be an exponential divisor of n , if β_j divides α_j for every $j = 1, \dots, r$. It is obvious, that the number of the exponential divisors of n (it is denoted as $\tau^{(e)}(n)$) is $\tau(\alpha_1) \dots \tau(\alpha_r)$, where $\tau(m)$ is the number of the divisors of m . The notion of exponential divisors was introduced by Subbarao [4]. In [13] Fabrykowski and Subbarao proved that

$$\sum \tau^{(e)}(n) = A_1 x + O\left(x^{1/2} \log x\right).$$

Recently Wu [2] observed that the generating Dirichlet series

$$F(s) := \sum_{n=1}^{\infty} \frac{\tau^{(e)}(n)}{n^s}$$

can be written as $F(s) = \zeta(s)\zeta(2s)U(s)$, where $U(s)$ can be written as an absolute convergent Dirichlet series in the halfplane $\sigma > 1/5$, whence, by using the estimate

$$\sum_{d^2 \leq x} \left(\left\{ \frac{x}{d^2} \right\} - 1/2 \right) \ll x^{2/9} \log x$$

The research was supported by the NSERC grant of the second named author It was done during the visit of the first named author to Edmonton.

(see [6]) he was able to deduce that

$$\sum \tau^{(e)}(n) = A_1 x + A_2 \sqrt{x} + O\left(x^{2/9} \log x\right).$$

Smati and Wu [3] recently proved that

$$(1.1) \quad \sum_{p \leq x} \tau^{(e)}(p-1) = c \operatorname{li} x + O_A\left(\frac{x}{(\log x)^A}\right)$$

holds for every fixed A .

1.2. Let \mathcal{B} be the set of square full numbers. For some integer n , let $E(n)$ be the square full, and $F(n)$ be the square free part of n . Then $n = E(n)F(n)$, $(E(n), F(n)) = 1$ and $E(n)$ is the largest divisor of n which belongs to \mathcal{B} .

For some $b \in \mathcal{B}$ let \mathcal{R}_b be the set of those integers n for which $E(n) = b$.
Let

$$(1.2) \quad \begin{aligned} \nu_x(b) &:= \frac{1}{x} \# \{n < x, n \in \mathcal{R}_b\}, \\ \nu(b) &:= \lim_{x \rightarrow \infty} \nu_x(b). \end{aligned}$$

By elementary sieve one can deduce that

$$(1.3) \quad \nu(b) = \frac{1}{\zeta(2)b} \prod_{p|b} \frac{1}{1+1/p}.$$

Let $m \in \mathbb{N}$, $m = p_1^{\alpha_1} \dots p_r^{\alpha_r}$. Let

$$D(m) := \{p_1^{\gamma_1} \dots p_r^{\gamma_r} \mid \gamma_1, \dots, \gamma_r \in \mathbb{N}_0\},$$

where \mathbb{N}_0 is the set of nonnegative integers.

Let

$$(1.4) \quad M(x) = \sum_{n \leq x} |\mu(n)|; \quad M(x|b) = \sum_{\substack{n \leq x \\ (n,b)=1}} |\mu(n)|.$$

It is known that

$$M(x) - \frac{x}{\zeta(2)} \ll \sqrt{x} \exp\left(-c(\log x)^{3/5}(\log \log x)^{-1/5}\right)$$

(Walfisz [7]). We shall use the somewhat weaker inequality

$$(1.5) \quad M(x) - \frac{x}{\zeta(2)} = O(\sqrt{x}).$$

Theorem 1. *We have, for $b \in \mathcal{B}$*

$$(1.6) \quad \nu_x(b) = \nu(b) + O\left(\frac{1}{b\sqrt{x}} \prod_{p|b} \left(1 + \frac{1}{\sqrt{p}}\right)\right).$$

Remark. A. Ivić [10] proved: if f is a multiplicative function such that $f(p^\alpha) = g(\alpha) \in \mathbb{N}$, $g(1) = 1$, then

$$\frac{1}{x} \sum_{\substack{n \leq x \\ f(n)=k}} = d_k + O\left(\frac{1}{\sqrt{x}} \exp\left(-c(\log x)^{3/5}(\log \log x)^{-1/5}\right)\right).$$

From our Theorem 1 one can deduce a similar theorem which is weaker than his, if k is small, and stronger than his for large $|k|$.

Theorem 2. *Let f be a multiplicative function for which $f(p^\alpha) = g(\alpha)$, $g(1) = 1$, $g(2) > 0$. Assume furthermore that*

$$\frac{|g(2)|}{2^2} + \frac{|g(3)|}{2^3} + \dots$$

is finite.

Then

$$(1.7) \quad \frac{1}{x} \sum_{n \leq x} f(n) = \sum_{b \in \mathcal{B}} f(b)\nu(b) + O\left(\sqrt{x}(\log x)^{g(2)-1}\right),$$

$$(1.8) \quad \sum_{b \in \mathcal{B}} f(b)\nu(b) = \frac{1}{\zeta(2)} \prod_{p \in \mathcal{P}} \left(1 + \sum_{j=2}^{\infty} \frac{g(j)}{(1+1/p)p^j}\right).$$

Corollary of Theorem 1. We have

$$(1.9) \quad \frac{1}{x} \sum_{n \leq x} \Omega\left(\tau^{(e)}(n)\right) = A + O\left(\frac{\log \log x}{\sqrt{x}}\right),$$

where

$$(1.10) \quad A = \sum_{b \in \mathcal{B}} \Omega\left(\tau^{(e)}(b)\right) \nu(b),$$

and

$$(1.11) \quad \frac{1}{x} \sum_{n \leq x} \omega\left(\tau^{(e)}(n)\right) = B + O\left(\frac{\log \log x}{\sqrt{x}}\right),$$

where

$$(1.12) \quad B = \sum_{b \in \mathcal{B}} \omega\left(\tau^{(e)}(b)\right) \nu(b).$$

Here $\omega(n)$ is the number of the prime divisors, and $\Omega(n)$ is the number of prime-power divisors of n .

Remark. (1.11) is somewhat stronger than Corollary 1 in [3].

We shall prove

Theorem 3. *We have*

$$\begin{aligned} \#\{n \in [X, X+H], \quad n \in \mathcal{R}_b\} &= \\ &= H\nu(b) + O\left(X^{\Theta+\varepsilon} \cdot 2^{\omega(b)}\right) + O\left(H^{1/2} X^\varepsilon \prod_{p|b} (1+1/\sqrt{p})\right) \end{aligned}$$

uniformly as $0 < H < x$. Here $\Theta = 0,2204$ and ε is an arbitrary positive constant. The implied constants in the order terms may depend on ε .

1.3 We have

Theorem 4. *Let $\Theta = 7/12$, A and B be arbitrary positive constants. Assume that $x^{\Theta+\varepsilon} \leq y \leq x$. Let $b \in \mathcal{B}$ and $b < (\log x)^A$. Then*

$$(1.13) \quad \#\{p < x \mid p-1 \in \mathcal{R}_b\} = \rho(b) \operatorname{li} x + O\left(\frac{x}{(\log x)^{Bb}}\right)$$

$$(1.14) \quad \begin{aligned} \#\{p \in [x, x+y] \mid p-1 \in \mathcal{R}_b\} &= \\ &= \rho(b) (\operatorname{li}(x+y) - \operatorname{li} x) + O\left(\frac{y}{(\log x)^{Bb}}\right), \end{aligned}$$

where

$$(1.15) \quad \rho(b) := \frac{c}{b} \prod_{\substack{\pi|b \\ \pi \in \mathcal{P}}} \frac{\pi(\pi-1)}{\pi^2 - \pi - 1}, \quad C = \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p(p-1)}\right).$$

Corollary of Theorem 4. Let $y \in [x^{\Theta+\varepsilon}, x]$, $\Theta = 7/12$, $r \in \mathbb{N}$. Then

$$(1.16) \quad \sum_{p \in [x, x+y]} \tau^{(e)r}(p-1) = D_r(\text{li}(x+y) - \text{li } x) + O\left(\frac{y}{(\log x)^{B_1}}\right),$$

where B_1 is an arbitrary constant.

Here

$$(1.17) \quad D_r = \sum_{b \in \mathcal{B}} \tau^{(e)}(b)^r \cdot \rho(b).$$

1.4. Let a_1, a_2, \dots, a_k be distinct positive integers, $G := \prod_{i < j} (a_i - a_j)$, $G = q_1^{\gamma_1} \dots q_r^{\gamma_r}$, q_1, \dots, q_r be primes.

Let $T = G^{[A \log \log x]}$, and for $l \in [1, T-1]$, $(l, T) = 1$, $1 \leq l \leq T-1$ let $t_1(l), t_2(l), \dots, t_k(l)$ be defined as

$$t_j(l) = \prod_{p^\alpha \parallel l + a_j} p^\alpha \quad (l = 1, \dots, k).$$

Let furthermore $e_j^{(l)} := E(t_j(l))$, the square full part of $t_j(l)$. Let $c_1, \dots, c_k \in \mathcal{B}$, such that $(c_i, c_j) = 1$ ($i \neq j$), and $(c_i, G) = 1$. Assume that $\max c_j < (\log x)^A$.

Theorem 5. Let $x^{\Theta+\varepsilon} \leq y \leq x$, $\Theta = 7/12$. Then

$$(1.18) \quad \begin{aligned} & \# \{p \in [x, x+y] \mid p \equiv l \pmod{T}, \quad p + a_j \in \mathcal{R}_{e_j c_j} \quad (j = 1, \dots, k)\} = \\ & = \frac{(\text{li}(x+y) - \text{li}(x))}{\varphi(T)} E(c_1, \dots, c_k) + O_B\left(\frac{y}{(\log x)^B}\right), \end{aligned}$$

where

$$(1.19) \quad E(c_1, \dots, c_k) = \frac{1}{c_1, \dots, c_k} \prod_{\substack{\pi | G c_1 \dots c_k \\ \pi \in \mathcal{P}}} \left(1 - \frac{k}{\pi(\pi-1)}\right).$$

As a consequence, for $h(p) := \tau^{(e)r_1}(p + a_1) \dots \tau^{(e)r_k}(p + a_k)$ we have

$$(1.20) \quad \sum_{p \in [x, x+y]} h(p) = K_{r_1, \dots, r_k}(a_1, \dots, a_k)(\text{li}(x+y) - \text{li } x) + O_B \left(\frac{y}{(\log x)^B} \right),$$

B is an arbitrary positive constant.

1.5. We would be able to prove the following theorems.

Theorem A. *Let $f_1, f_2, \dots, f_k \in \mathbb{Z}[x]$ be such that every f_j is a product of distinct irreducible polynomials of degree not higher than three. Then*

$$\sum_{n \leq x} \tau^{(e)}(f_1(n)) \tau^{(e)}(f_2(n)) \dots \tau^{(e)}(f_k(n)) = Cx + o(x)$$

with some positive constant C .

Theorem B. *If $f_1, \dots, f_k \in \mathbb{Z}[x]$, and every f_j is a product of distinct irreducible polynomials of degree not higher than two, then*

$$\sum_{p \leq x} \tau^{(e)}(f_1(p)) \tau^{(e)}(f_2(p)) \dots \tau^{(e)}(f_k(p)) = C^* \text{li } x + o(\text{li } x)$$

with some positive constant C^* .

Theorem A can be proved on a routine way by using the following theorem of C. Hooley: if $f \in \mathbb{Z}[x]$ is irreducible, $\deg f \leq 3$, then the number of the integers $n \leq x$ for which there is a prime $p > \log x$ such that $p^2 | f(n)$ is at most $O(x(\log x)^{-1/3})$. See C. Hooley [5], Chapter 4, Theorem 3, or [12] for a better estimate.

Let $g \in \mathbb{Z}[x]$ be an irreducible polynomial of degree 2. Let $\varepsilon > 0$ and $y = x^{1/2+\varepsilon}$. One can prove that the number of the integers $n \in [x, x+y]$ for which there is a prime $q > (\log x)^2$ such that $q^2 | g(n)$ is at most $O(Y/(\log x)^2)$.

Hence we can deduce Theorem B, or even a short interval version of it.

Let $\rho(m)$ be the number of solution of $n^2 + 1 \equiv 0 \pmod{m}$. We shall prove

Theorem 6. *Let $Y = x^{2/3+\varepsilon}$, $\varepsilon > 0$ be a small constant. Then, for every fixed $A > 0$,*

$$\sum_{n \in [X, X+Y]} \tau^{(e)}(n^2 + 1) = CY + O_A(Y/(\log x)^A),$$

where

$$C = \sum_{b \in \mathcal{B}} \frac{\tau^{(e)}(b)\rho(b)\varphi(b)}{b^2} \prod_{\substack{\pi|b \\ \pi \in \mathcal{P}}} \left(1 - \frac{\rho(\pi^2)}{\pi^2}\right).$$

Theorem 7. *Let $Y = x^{2/3+\varepsilon}$, A be an arbitrary positive constant. Then*

$$\sum_{p \in [X, X+Y]} \tau^{(e)}(p^2 + 1) = C_1(\text{li}(X + Y) - \text{li} X) + O_A(Y/(\log x)^A),$$

where

$$C_1 = \sum_{b \in \mathcal{B}} \frac{\tau^{(e)}(b)\rho(b)}{b} \prod_{\substack{\pi|b \\ \pi \in \mathcal{P}}} \left(1 - \frac{\rho(\pi)}{\pi(\pi - 1)}\right).$$

2. Proof of Theorems 1, 2

Since

$$\sum_{(n,b)=1} \frac{|\mu(n)|}{n^s} = \prod_{p|b} \left(1 + \frac{1}{p^s}\right)^{-1} \cdot \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s}$$

holds for $\text{Re } s > 1$, therefore

$$M(x|b) = \sum_{v \in D(b)} \lambda(v)M\left(\frac{x}{v}\right),$$

whence

$$M(x|b) = \frac{x}{\zeta(2)} \sum_{v \in D(b)} \frac{\lambda(v)}{v} + O\left(x^{1/2} \sum_{v \in D(b)} \frac{1}{v^{1/2}}\right) + O\left(x \sum_{\substack{v \geq x \\ v \in D(b)}} 1/v\right).$$

Since $v \geq \sqrt{x} \cdot v^{1/2}$ in the last sum, therefore

$$M(x|b) = \frac{x}{\zeta(2)} \prod_{p|b} \frac{1}{1 + 1/p} + O\left(x^{1/2} \prod_{p|b} \left(1 + \frac{1}{\sqrt{p}}\right)\right).$$

Observing that $\nu_x(b) = x^{-1}M\left(\frac{x}{b} \mid b\right)$, Theorem 1 immediately follows.

To prove Theorem 2, we start from the equation

$$\frac{1}{x} \sum_{n \leq x} f(n) = \sum_{\substack{b \leq x \\ b \in \mathcal{B}}} f(b) \nu_x(b) = \sum_1 + \sum_2,$$

where

$$\begin{aligned} \sum_1 &= \frac{1}{\zeta(2)} \sum_{\substack{b \in \mathcal{B} \\ b < x}} \frac{f(b)}{b \prod_{p|b} (1 + 1/p)}, \\ \sum_2 &= O\left(x^{-1/2} \sum_{b \in \mathcal{B}} \frac{f(b)}{b} \prod_{p|b} \left(1 + \frac{1}{\sqrt{p}}\right)\right). \end{aligned}$$

Let $t(y) = (1 + \sqrt{y}) (|g(2)| \cdot y^2 + |g(3)| \cdot y^3 + \dots)$. Then

$$\sum_{b \in \mathcal{B}} \frac{|f(b)|}{b} \prod_{p|b} \left(1 + \frac{1}{\sqrt{p}}\right) \leq \exp\left(2 \sum_p t(1/p)\right).$$

Since

$$\sum_{p \in \mathcal{P}} \frac{1}{p^s} \leq \frac{1}{2^{s-2}} \sum \frac{1}{p^2}, \quad s = 3, 4, \dots,$$

therefore

$$\sum_p t(1/p) \leq \left(\sum \frac{1}{p^2}\right) \left(|g(2)| + \frac{|g(3)|}{2} + \frac{|g(4)|}{2^2} + \dots\right)$$

and the right hand side is finite. Thus $\sum_2 = O(X^{-1/2})$.

We shall prove that

$$\sum_0 := \sum_{\substack{b \in \mathcal{B} \\ b \geq x}} \frac{f(b)}{b \prod_{p|b} (1 + 1/p)} \ll \frac{1}{\sqrt{x}} (\log x)^{g(2)-1},$$

and this completes the proof of Theorem 2.

We can write each b as v^2u , where v is square free and u is three full, i.e. $p|u$ implies that $p^3|u$.

Thus

$$\Delta(Y) := \sum_{uv^2 \leq Y} |f(uv^2)| \leq \sum_{u \leq Y} |f(u)| \cdot \sum_{v < \sqrt{Y/u}} |f(v^2)|.$$

Since

$$\sum_{v < \kappa} |f(v^2)| \ll \kappa(\log \kappa)^{g(2)-1},$$

therefore

$$\Delta(Y) \ll (\log Y)^{g(2)-1} \sqrt{Y} \sum_{u \leq Y} \frac{|f(u)|}{u^{1/2}}.$$

Furthermore

$$(2.1) \quad \sum \frac{|f(u)|}{u^{1/2}} \leq \prod_{p \leq Y} \left(1 + \frac{|g(3)|}{p^{3/2}} + \frac{|g(4)|}{p^2} + \dots \right).$$

Arguing as earlier, we can deduce that

$$\sum_{p \in \mathcal{P}} \left(\frac{|g(3)|}{p^{3/2}} + \frac{|g(4)|}{p^2} + \dots \right)$$

is convergent, thus the right hand side of (2.1) is bounded.

Thus $\Delta(Y) \ll (\log Y)^{g(2)-1} \sqrt{Y}$, and so

$$\sum_0 \ll \sum_{j=0}^{\infty} \frac{1}{2^j X} \Delta(2^j X) \ll \frac{(\log X)^{g(2)-1}}{\sqrt{X}}.$$

3. Proof of the Corollary of Theorem 1

We shall prove (1.9) only. The proof of (1.11) is almost the same.

From (1.6) we obtain that

$$\frac{1}{x} \sum_{n \leq x} \Omega(\tau^{(e)}(n)) = A + O\left(\sum_1\right) + O\left(\frac{1}{\sqrt{x}} \sum_2\right),$$

where

$$\begin{aligned}\sum_1 &= \sum_{\substack{b > x \\ b \in \mathcal{B}}} \frac{\Omega(\tau^{(e)}(b))}{b}, \\ \sum_2 &= \sum_{\substack{b < x \\ b \in \mathcal{B}}} \frac{\Omega(\tau^{(e)}(b))}{b} \prod_{p|b} \left(1 + \frac{1}{\sqrt{p}}\right).\end{aligned}$$

Let $h(n) := \Omega(\tau^{(e)}(n))$. h is completely additive, therefore

$$\sum_2 \leq 2 \sum_{p^\nu} \frac{h(p^\nu)}{p^\nu} \left(\sum_{c \in \mathcal{B}} \frac{1}{c} \prod_{p|c} (1 + 1/\sqrt{p}) \right).$$

The inner sum is convergent,

$$= \prod_p \left(1 + \left(1 + \frac{1}{\sqrt{p}} \right) \left(\frac{1}{p^2} + \frac{1}{p^3} + \dots \right) \right) \leq C \prod_p \left(1 + \frac{2}{p^2} \right).$$

Furthermore

$$\sum_{\substack{p \in \mathcal{P} \\ \nu \geq 2}} \frac{h(p^\nu)}{p^\nu} = O(1),$$

thus $\sum_2 = O(1)$.

Since

$$\begin{aligned}E(Y) &:= \sum_{\substack{b < Y \\ b \in \mathcal{B}}} h(b) = \sum_{\substack{p^\nu \leq Y \\ \nu \geq 2}} h(p^\nu) \sum_{\substack{c \leq Y/p^\nu \\ c \in \mathcal{B}}} 1 \leq \\ &\leq \sqrt{Y} \sum_{\substack{p \leq \sqrt{Y} \\ p \in \mathcal{P}}} \frac{h(p^2)}{p} + \sqrt{Y} \sum_{\substack{p \in \mathcal{P} \\ \nu \geq 3}} \frac{h(p^\nu)}{p^{\nu/2}},\end{aligned}$$

thus

$$E(Y) \ll \sqrt{Y} \log \log Y,$$

and so

$$\sum_1 \ll \sum_{j=0}^{\infty} \frac{1}{2^j x} E(2^{j+1}x) \ll \frac{\log \log x}{\sqrt{x}}.$$

The proof is completed.

4. Proof of Theorem 3

We shall use the next Lemma 1 due to P. Varbanec [8].

Lemma 1. *Let $\phi(d)$ be a multiplicative function, such that $\phi(d) = O(d^\varepsilon)$ for $\varepsilon > 0$. Let*

$$f(n) = \sum_{d^2|n} \phi(d).$$

Then

$$\sum_{x \leq n \leq x+h} f(n) = h \sum_{d=1}^{\infty} \frac{\phi(d)}{d^2} + O\left(h^{1/2}x^\varepsilon\right) + O\left(x^{\Theta+\varepsilon}\right),$$

uniformly in h , $h < x$, where ε is an arbitrary positive constant and $\Theta = 0, 2204$.

The exponent $\Theta < 2/9$, thus Lemma 1 is somewhat stronger than that of Graham and Kolesnik in [9].

Let $b \in \mathcal{B}$. Then

$$\begin{aligned} & \#\{n \in [X, X+H] \mid n \in \mathcal{R}_b\} = \\ & = \#\left\{m \in \left[\frac{X}{b}, \frac{X+H}{b}\right] \mid (m, b) = 1, |\mu(m)| = 1\right\} = \\ & = \sum_{\delta|b} \mu(\delta) \#\left\{\nu \in \left[\frac{X}{b\delta}, \frac{X+H}{b\delta}\right] \mid p^2 \nmid \nu \text{ if } p \nmid b\right\}. \end{aligned}$$

Let us apply Lemma 1 with $x = \frac{X}{b\delta}$, $h = \frac{H}{b\delta}$,

$$\phi(p) = \begin{cases} 0 & \text{if } p|b, \\ 1 & \text{if } (p, b) = 1. \end{cases}$$

We have

$$\begin{aligned} & \#\{n \in [X, X+H], n \in \mathcal{R}_b\} = \\ & = H\nu(b) + O\left(X^{\Theta+\varepsilon} \cdot 2^{\omega(b)}\right) + O\left(H^{1/2}X^\varepsilon \prod_{p|b} \left(1 + \frac{1}{\sqrt{p}}\right)\right), \end{aligned}$$

which proves the theorem.

5. Proof of Theorem 4 and that of the Corollary

Let $\prod(x|b) = \#\{p < x \mid p - 1 \in \mathcal{R}_b\}$. We shall start from the identity

$$(5.1) \quad \prod(x|b) = \sum_{\substack{\delta|b \\ (\kappa, b)=1}} \mu(\delta)\mu(\kappa)\pi(x, \delta\kappa^2b, 1),$$

which can be proved similarly as we argued at the proof of Theorem 1.

Let A and B be arbitrary constants. Assume that $b \leq (\log x)^A$. Let

$$(5.2) \quad \prod(x|b) = \prod_1(x|b) + \prod_2(x|b),$$

where in \prod_1 we sum over $\kappa \leq (\log x)^B$ ($=: L$), and in \prod_2 over $\kappa > L$.

By the Siegel-Walfisz theorem we obtain that

$$\prod_1(x|b) = (\text{li } x) \sum_{\substack{\delta|b \\ (\kappa, b)=1 \\ \kappa < L}} \frac{\mu(\delta)\mu(\kappa)}{\varphi(b\delta)\varphi(\kappa^2)} + O\left((\text{li } x) \log x)^{-B} \sum_{\substack{\delta|b \\ (\kappa, b)=1 \\ \kappa \geq L}} \frac{1}{\varphi(b\delta\kappa^2)}\right).$$

Since

$$\sum_{\kappa \geq L} \frac{1}{\varphi(\kappa^2)} \ll \frac{1}{L},$$

we have

$$\prod_1(x|b) = \nu(b) \text{li } x + O\left(\frac{x}{(\log x)^{Bb}}\right),$$

where

$$(5.3) \quad \rho(b) = \frac{c}{b} \prod_{\substack{\pi|b \\ \pi \in \mathcal{P}}} \frac{\pi(\pi-1)}{\pi^2 - \pi - 1},$$

$$(5.4) \quad C = \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p(p-1)}\right).$$

To estimate $\prod_2(x|b)$, we observe that $\pi(x, D, 1) \leq \frac{c \operatorname{li} x}{\varphi(D)}$ if $D \leq x^{3/4}$, and $\pi(x, D, 1) \leq \frac{x}{D}$ if $D \geq x^{1/2}$. Thus

$$\prod_2(x|b) \ll (\operatorname{li} x) \sum_{\substack{\delta|b \\ L < \kappa \leq \sqrt{x}}} \frac{1}{\varphi(b\delta\kappa^2)} + x \sum_{\substack{\delta|b \\ \kappa > \sqrt{x}}} \frac{1}{\varphi(b\delta\kappa^2)},$$

and the right hand side is less than

$$\ll \frac{\operatorname{li} x}{(\log x)^B} \frac{1}{\varphi(b)} \prod_{p|b} (1 + 1/p) \ll \frac{1}{b} \frac{x}{(\log x)^B}.$$

Thus

$$(5.5) \quad \prod(x|b) = \rho(b) \operatorname{li} x + O\left(\frac{x}{b(\log x)^B}\right)$$

if $b \leq (\log x)^A$.

We can prove (1.14) similarly. We have to use the short interval version of the Siegel-Walfisz theorem (i.e. the theorem of Hoheisel and Tatzuza, see K. Prachar [11], Theorem 3.2 in Chapter IX) and that

$$\pi(x + y, D, 1) - \pi(x, D, 1) \ll \frac{y}{\varphi(D) \log x} \quad \text{for } D < y^{1-\varepsilon},$$

and $\pi(x + y, D, 1) - \pi(x, D, 1) \ll (y/D + 1)$ if $y < D$. We omit the details.

Now we prove the Corollary.

From sieve theorems we know that

$$\begin{aligned} \prod(x + y|b) - \prod(x|b) &\ll \frac{y}{\varphi(b) \log x} && \text{if } b \leq \sqrt{x}, \\ &\ll \frac{y}{b} && \text{if } \sqrt{x} < b. \end{aligned}$$

Thus

$$\begin{aligned} \sum_0 &:= \sum_{p \in [x, x+y]} \left(\tau^{(e)}(p-1)\right)^r = \sum_{\substack{b < x+y \\ b \in \mathcal{B}}} \tau^{(e)r}(b) \left(\prod(x + y|b) - \prod(x|b)\right) = \\ &= \sum_1 + \sum_2, \end{aligned}$$

where in \sum_1 we sum over $b < (\log x)^A$ and in \sum_2 over the others.

One can prove simply that

$$\sum_{\substack{b \in \mathcal{B} \\ b < z}} \tau^{(e)r}(b) \ll z,$$

whence one gets that

$$\begin{aligned} \sum_2 &\ll \frac{y}{\log x} \left(\sum_{\substack{b > (\log x)^A \\ b \in \mathcal{B}}} \frac{\tau^{(e)r}(b)}{b} + (\log x) \sum_{b > \sqrt{x}} \frac{\tau^{(e)r}(b)}{b} \right) \ll \\ &\ll \frac{y}{(\log x)^{A/2+1}}. \end{aligned}$$

Furthermore,

$$\begin{aligned} \sum_1 &= (\text{li}(x+y) - \text{li } x) \sum_{b \leq (\log x)^A} \tau^{(e)r}(b) \rho(b) + O\left(\sum_3\right), \\ \sum_3 &\ll \frac{y}{(\log x)^B} \sum_{b \in \mathcal{B}} \frac{\tau^{(e)r}(b)}{b} \ll \frac{y}{(\log x)^B}. \end{aligned}$$

Finally, we observe that

$$\sum_{\substack{b > (\log x)^A \\ b \in \mathcal{B}}} \tau^{(e)r}(b) \rho(b) \ll \frac{1}{(\log x)^{A/2}},$$

whence

$$\sum_0 = (\text{li}(x+y) - \text{li } x) D_r + O\left(\frac{y}{(\log x)^{B_1}}\right),$$

D_r is defined in (1.17).

Since $A = 2B_1$ can be chosen, the Corollary is true.

6. Proof of Theorem 5

Let $p \equiv l \pmod{T}$. Then $p + a_j \in \mathcal{R}_{e_j c_j}$ holds, if $\left(\frac{p + a_j}{c_j}, c_j\right) = 1$, and $\kappa^2 \nmid \frac{p + a_j}{c_j}$ if $(\kappa, Gc_j) = 1$.

Thus $p + a_j \in \mathcal{R}_{e_j c_j}$, if

$$\sum_{\substack{\delta | c_j \\ \kappa^2 | p + a_j \\ (\kappa, Gc_j) = 1}} \mu(\delta)\mu(\kappa) = 1$$

and the above sum is zero if $p + a_j \notin \mathcal{R}_{e_j c_j}$.

Thus the left hand side of (1.18) is

$$(6.1) \quad \sum \mu(\delta_1) \dots \mu(\delta_k) \mu(\kappa_1) \dots \mu(\kappa_k) (\pi((x + y), \Delta, r) - \pi(x, \Delta, r)),$$

where Δ is the least common multiple of T and $\delta_j c_j \kappa_j^2$ ($j = 1, \dots, k$), i.e.

$$\Delta = [T, \delta_1 c_1 \kappa_1^2, \dots, \delta_k c_k \kappa_k^2],$$

and $r \pmod{\Delta}$ is such a residue, for which $r \equiv l \pmod{T}$ and $r + a_j \equiv 0 \pmod{\delta_j c_j \kappa_j^2}$ ($j = 1, \dots, k$) hold true simultaneously.

The sum is extended over all $\delta_1, \dots, \delta_k, \kappa_1, \dots, \kappa_k$ such that

$$\delta_j | c_j, (\kappa_j, Gc_j) = 1.$$

Since $(\delta_{i_1} c_{i_1} \kappa_{i_1}^2, \delta_{i_2} c_{i_2} \kappa_{i_2}^2) | a_{i_2} - a_{i_1}$ if $i_1 \neq i_2$, and r satisfies the above equations, therefore $(\delta_{i_1} c_{i_1} \kappa_{i_1}^2, \delta_{i_2} c_{i_2} \kappa_{i_2}^2) = 1$ for every couple $i_1 \neq i_2$. Thus

$$\begin{aligned} \varphi(\Delta) &= \varphi(T) \cdot \varphi(\delta_1 c_1 \kappa_1^2) \dots \varphi(\delta_k c_k \kappa_k^2) = \\ &= \varphi(T) \delta_1 \dots \delta_k \varphi(c_1) \dots \varphi(c_k) \varphi((\kappa_1 \dots \kappa_k)^2). \end{aligned}$$

Furthermore,

$$\begin{aligned} \sum \frac{\mu(\delta_1) \dots \mu(\delta_k) \mu(\kappa_1) \dots \mu(\kappa_k)}{\varphi(\Delta)} &= \frac{1}{\varphi(T) \varphi(c_1) \dots \varphi(c_k)} \times \\ &\times \prod_{p | c_1} \left(1 - \frac{1}{p}\right) \dots \prod_{p | c_k} \left(1 - \frac{1}{p}\right) \prod_{\pi \nmid Gc_1 \dots c_k} \left(1 - \frac{k}{\pi(\pi - 1)}\right) = \\ &= \frac{1}{\varphi(T) c_1 \dots c_k} \prod_{\pi | G(c_1 \dots c_k)} \left(1 - \frac{k}{\pi(\pi - 1)}\right) = \frac{1}{\varphi(T)} E(c_1, \dots, c_k). \end{aligned}$$

By the prime number theorem for short intervals we obtain that

$$\varepsilon(\Delta) := \pi(x + y, \Delta, r) - \pi(x, \Delta, r) = \frac{(\text{li}(x + y) - \text{li } x)}{\varphi(\Delta)} + O\left(\frac{y}{(\log x)^A}\right)$$

whenever $\kappa_j \ll (\log x)^{A_1}$, A_1 is large. For larger values of κ_j we can use the upper bounds

$$\begin{aligned} \varepsilon(\Delta) &\ll \frac{y}{\varphi(\Delta) \log x}, & \Delta < y^{1-\varepsilon}, \quad \text{and} \\ \varepsilon(\Delta) &\ll \frac{y}{\varphi(\Delta)} & \text{for } \Delta > y^{1-\varepsilon}. \end{aligned}$$

Substituting this estimates into (6.1), we obtain (1.19) easily. The relation (1.20) is a simple consequence of (1.18).

7. Proof of Theorems 6, 7

We shall use the following

Lemma 2. *For every fixed $A > 0$ the number of solutions of $n^2 - Am^2 = -1$, $0 < n < x$ is at most $O(\log x)$, where the implied constant is absolute.*

Proof. If (n, m) is a solution, then $\left| \frac{n}{m} - \sqrt{A} \right| \leq \frac{1}{2m^2}$, thus $\frac{n}{m}$ is an approximant of the continous fraction of \sqrt{A} , therefore the assertion is true.

Lemma 3. *Let $y = x^{2/3+\varepsilon}$, and $E(x, y)$ be the number of those integers $n \in [x, x+y]$ for which there is a prime q such that $q^2 \geq y$ and $q^2 | n^2 + 1$. Then*

$$E(x, x+y) \ll x^{2/3}(\log x) = y \frac{\log x}{x^\varepsilon}.$$

Proof. If n_1, n_2 are such integers for which $n_j^2 + 1 \equiv 0 \pmod{q^2}$, ($j = 1, 2$), then $n_2^2 - n_1^2 \equiv 0 \pmod{q^2}$. Since $n_1 - n_2 \equiv 0 \pmod{q}$, $n_1 + n_2 \equiv 0 \pmod{q}$ cannot hold, therefore either $n_1 + n_2 \equiv 0 \pmod{q^2}$, or $n_1 - n_2 \equiv 0 \pmod{q^2}$. It implies that for every q , $q^2 \geq y$ no more than two n exist in $[x, x+y]$ for which $q^2 | n^2 + 1$.

If $q^2 | n^2 + 1$, $n \in [X, X+Y]$, and $q > X^\lambda$, then $n^2 + 1 = Aq^2$ and $A < 2X^{2-2\lambda}$. From Lemma 1 we obtain that for fixed A no more than $\log x$ such n exists. Thus the whole contribution of these q is less than $O(X^{2-2\lambda} \log X)$.

Thus no more than $O(X^\lambda / \log x) + O(X^{2-2\lambda} \log x)$ integers $n \in [X, X+Y]$ exists for which $q^2 | n^2 + 1$ for some $q \geq \sqrt{Y}$. By $\lambda = \frac{2}{3}$ the inequality follows.

Let $\rho(m)$ be the number of solutions of the congruence $n^2 + 1 \equiv 0 \pmod{m}$. As it is known, $\rho(m)$ is multiplicative, $\rho(2) = 1$, $\rho(2^\alpha) = 0$ ($\alpha \geq 2$), $\rho(p^\alpha) = \rho(p) = 2$ or 0 according to $p \equiv 1$ or $p \equiv -1 \pmod{4}$.

Lemma 4. *Let A be an arbitrary constant, $B = 2A$. Then*

$$\sum_{b > (\log x)^B} \sum_{\substack{n \in [x, x+y] \\ n^2 + 1 \in \mathcal{R}_b}} \tau^{(e)}(n^2 + 1) \ll \frac{y}{(\log x)^A}.$$

Proof. Let $\varepsilon_1 > 0$ be a small constant. Let us consider first those integers $n \in [x, x + y]$ for which $q^2 | n^2 + 1$, $q > x^{\varepsilon_1}$. The sum of $\tau^{(e)}(n^2 + 1)$ for those n for which $q^2 | n^2 + 1$, $q > \sqrt{y}$ is less than $y \cdot \frac{(\log x)}{x^{\varepsilon/2}}$. (See Lemma 3, and that $\tau^{(e)}(n^2 + 1) \ll x^{\varepsilon/2}$).

It is obvious that

$$(7.1) \quad \sum_{(\log x)^B \leq b < y} \sum_{\substack{n \in [x, x+y] \\ n^2 + 1 \in \mathcal{R}_b}} \tau^{(e)}(n^2 + 1) \ll y \sum_{(\log x)^B \leq b < y} \frac{\tau^{(e)}(b)\rho(b)}{b}.$$

Since

$$\sum \frac{\tau^{(e)}(b)\rho(b)}{b^s} = \prod_{p \equiv 1 \pmod{4}} \left(1 + \frac{2 \cdot 2}{p^{2s}} + \frac{2 \cdot 2}{p^{3s}} + \dots \right),$$

we can get that

$$(7.2) \quad \sum_{\substack{b \in \mathcal{B} \\ b < z}} \tau^{(e)}(b)\rho(b) \ll \sqrt{z} \quad (z \rightarrow \infty),$$

and so the right hand side of (5.1) is less than $y/(\log x)^{B/2}$.

Finally we consider those $b \in \mathcal{B}$ for which $b > y$ and for each prime divisor q of b , $q < x^{\varepsilon_1}$. If $n^2 + 1 \in \mathcal{R}_b$, then there exists some $b_1 | b$, such that $b_1 \in \mathcal{B}$, and $y \cdot x^{-3\varepsilon_1} < b_1 < x$. For such an n , $\tau^{(e)}(n^2 + 1) \ll x^\varepsilon$, and the remaining part of the left hand side of (7.1) is less than

$$x^\varepsilon \sum_{\substack{y \cdot x^{-3\varepsilon_1} < b_1 < y \\ b_1 \in \mathcal{B}}} \sum_{\substack{n^2 + 1 \equiv 0 \pmod{b_1} \\ n \in [x, x+y]}} \tau^{(e)}(n^2 + 1) \ll yx^\varepsilon \sum \frac{\rho(b_1)}{b_1} \ll y^{3/4},$$

say. The proof is completed.

Let $H = (\log x)^B$, $B = 2A$. For some $b \in \mathcal{B}$ let $S(b)$ be the number of those $n \in [x, x + y]$ for which $\frac{n^2 + 1}{b}$ is squarefree, and $S^*(b)$ the number of those n , for which $\frac{n^2 + 1}{b}$ does not have prime square divisor κ^2 , if $\kappa < H$. From Lemma 4 we obtain that

$$\sum_{b \in \mathcal{B}} \tau^{(\epsilon)}(b) (S^*(b) - S(b)) \ll y/(\log x)^A.$$

We have

$$S^*(b) = \sum_{\delta, \kappa} \mu(\delta)\mu(\kappa) \#\{n \in [x, x + y] \mid n^2 + 1 \equiv 0 \pmod{b\delta\kappa^2}\},$$

where $\delta|b$, $(\kappa, b) = 1$, and the largest prime factor of κ is less than H . Let $b < H$. Thus

$$\begin{aligned} S^*(b) &= Y \sum_{\delta, \kappa} \frac{\mu(\delta)\mu(\kappa)\rho(b\delta\kappa^2)}{b\delta\kappa^2} + \\ &+ O\left(\sum_{b\delta\kappa^2 < 4x^2} \rho(b\delta\kappa^2)\right) + O\left(Y \sum_{b\delta\kappa^2 > y} \frac{\rho(b\delta\kappa^2)}{b\delta\kappa^2}\right). \end{aligned}$$

The error terms are clearly less than x^ϵ . Thus the first sum

$$\begin{aligned} &= \frac{\rho(b)}{b} \prod_{p|b} \left(1 - \frac{1}{p}\right) \prod_{\substack{\pi|b \\ \pi < H}} \left(1 - \frac{\rho(\pi^2)}{\pi^2}\right) = \\ &= \frac{\rho(b)\varphi(b)}{b^2} \prod_{\pi|b} \left(1 - \frac{\rho(\pi^2)}{\pi^2}\right) \left(1 + O\left(\frac{1}{H}\right)\right). \end{aligned}$$

We can continue on a routine way, and deduce Theorem 6.

The proof of Theorem 7 is similar. Doing the same as earlier, we reduce the proof to estimate $\#\{p \in [x, x + y] \mid p^2 + 1 \equiv 0 \pmod{b\delta\kappa^2}\}$ for $b < H$, $\delta|b$, $\kappa^2 < H$, for which we can use the short interval version of Siegel-Walfisz theorem, according to it equals

$$\frac{\rho(b\delta\kappa^2)}{\varphi(b\delta\kappa^2)} (\text{li}(x + y) - \text{li } x) + O(y/(\log x)^A).$$

Hence one can deduce that the number of primes $p \in [x, x + y]$, for which $\left(\frac{p^2 + 1}{b}, b\right) = 1$, and $\pi^2 \nmid \frac{p^2 + 1}{b}$ for the primes $\pi < H$, equals

$$\begin{aligned} & (\text{li}(x + y) - \text{li } x) \frac{\rho(b)}{\varphi(b)} \prod_{p|b} \left(1 - \frac{1}{p}\right) \cdot \prod_{\substack{\pi|b \\ \pi < H}} \left(1 - \frac{\rho(\pi)}{\pi(\pi - 1)}\right) + \\ & + O\left(\frac{y}{(\log x)^A} \frac{1}{\varphi(b)}\right), \end{aligned}$$

whence one can deduce Theorem 7 on a routine way. We have

$$C_1 = \sum_{b \in \mathcal{B}} \frac{\tau^{(e)}(b) \rho(b)}{b} \prod_{\pi|b} \left(1 - \frac{\rho(\pi)}{\pi(\pi - 1)}\right).$$

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