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LACUNARY POWER SERIES WITH VARIOUS UNIVERSAL PROPERTIES

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Dedicated to our colleague and friend
Karl-Heinz Indlekofer
on the occasion of his sixtieth birthday

Abstract. We construct holomorphic functions on $\mathbb{D}$ and $\mathbb{C}$ which simultaneously are universal with respect as well to translates as to derivatives. The corresponding universal functions are defined by lacunary power series with gaps of positive lower Poisson density.

1. Introduction

For a compact set K in the complex plane $\mathbb{C}$ we denote by $A(K)$ the set of all complex valued functions, which are continuous on K and holomorphic in its interior K^0. Introducing the uniform norm $A(K)$ becomes a Banach space. By $\mathcal{M}$ we denote the family of all compact sets which have connected complement.

The problems of "universal approximation" of functions by so-called "universal elements" are classical and there exists an extended literature on the theory of functions which are universal in different respects. The first example (which we cite in a slightly different form) was given by Fekete in 1914 (see Pál [19]), who proved the existence of a universal real power series

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$\sum_{\nu=0}^{\infty} a_\nu x^\nu$ with the property that for every interval $[a,b]$ with $0 \notin [a,b]$ and every continuous function f on $[a,b]$ there exists a sequence $\{n_k\}$ such that $\left\{\sum_{\nu=0}^{n_k} a_\nu x^\nu\right\}$ converges uniformly to $f(x)$ on $[a,b]$. Obviously this power series has radius of convergence $r=0$.

Universal power series (with respect to overconvergence) in the complex plane with positive radii of convergence were constructed by the second author in 1970 [7] and independently by Chui and Parnes in 1971 [3].

Perhaps the best known example of a universal function was obtained by Birkhoff in 1929 [1], who proved the existence of an entire function ϕ with universal translates, i.e. for every entire function f there exists a subsequence $\{n_k\}$ of $\mathbb{N}$ such that $\{\phi(z+n_k)\}$ converges to $f(z)$ compactly on $\mathbb{C}$.

Since then many papers have dealt with this theory of universal families; the approximation theorems of Runge and Mergelian are in general the basic tools for the construction of elements which are universal in a certain specified sense. For details we refer to the excellent article of Grosse-Erdmann [6], where a survey of the various universalities as well as the full bibliography of relevant contributions updated till 1998 can be found.

We mention some further results which are of interest for our investigations.

In 1953 MacLane [17] has constructed an entire function ϕ with universal derivatives, that is for every entire function f there exists a subsequence $\{n_k\}$ of $\mathbb{N}$ such that $\{\phi^{(n_k)}(z)\}$ converges to $f(z)$ compactly on $\mathbb{C}$.

Motivated by Birkhoff's result the second author has in a series of papers [8]-[13] holomorphic functions in more general open sets investigated which behave universal under certain translations. For instance in [10] the existence of a holomorphic function in the unit disk $\mathbb{D} := \{z : |z| < 1\}$ was shown which has the property that for every $\zeta \in \partial\mathbb{D}$, every $K \in \mathcal{M}$ and every $f \in A(K)$ there exist sequences $\{a_n\}, \{b_n\}$ such that $a_n \to 0$, $b_n \to \zeta$ and $\{\phi(a_n z + b_n)\}$ converges to $f(z)$ uniformly on K.

In very recent papers authors fixed their attentions to functions which together with universalities satisfy additional properties. We mention the contributions of Schneider [22], Tenthoff [23] and in several articles, Martirosian, Müller and the second author [14], [15], [16] have dealt with functions which have lacunary power series expansions and are universal with respect to translates.

In this note we construct holomorphic functions which share several different universalities. It is our object to prove the existence of holomorphic functions on $\mathbb{D}$ and $\mathbb{C}$ respectively with lacunary power series which at the same time are universal with respect to translates as well as to derivatives. In addition it is shown that the "paths of approximation" can be prescribed in an arbitrary way. Our basic tool is a very modern result (Lemma 1, stated below) on the approximation by lacunary polynomials with gaps of positive lower Poisson density.

2. Remarks on density properties

For a subsequence $Q = \{q_\nu\}_{\nu \in \mathbb{N}_0}$ of $\mathbb{N}_0$ various notions of densities have been introduced. We denote by $n(t)$ the number of elements of Q in the interval $[0, t]$.

The upper density $\overline{\Delta}(Q)$ and the lower density $\underline{\Delta}(Q)$ of Q are given by

$$\overline{\Delta}(Q) := \overline{\lim_{t\to\infty}} \frac{n(t)}{t}, \qquad \underline{\Delta}(Q) := \underline{\lim_{t\to\infty}} \frac{n(t)}{t}.$$

If $\overline{\Delta}(Q) = \underline{\Delta}(Q)$ then the common value is called the density of Q. Moreover the maximal and minimal density of Q are defined by

$$\Delta_{\max}(Q) := \lim_{r\to 1-} \left(\overline{\lim_{t\to\infty}} \frac{n(t) - n(tr)}{(1-r)t} \right),$$

$$\Delta_{\min}(Q) := \lim_{r\to 1-} \left(\underline{\lim_{t\to\infty}} \frac{n(t) - n(tr)}{(1-r)t} \right),$$

respectively. These notions were essentially utilized by Pólya [20].

In this article we deal with two density conceptions which were introduced by Poisson. The expressions

$$\overline{\Delta}_p(Q) := \frac{2}{\pi} \overline{\lim_{s\to\infty}} \int_0^\infty \frac{n(t)}{t} \cdot \frac{s}{t^2 + s^2} dt,$$

$$\underline{\Delta}_p(Q) := \frac{2}{\pi} \underline{\lim_{s\to\infty}} \int_0^\infty \frac{n(t)}{t} \cdot \frac{s}{t^2 + s^2} dt$$

are called upper Poisson density and lower Poisson density of Q respectively. The following inequalities are well known:

$$\Delta_{\min}(Q) \le \underline{\Delta}(Q) \le \underline{\Delta}_p(Q) \le \overline{\Delta}_p(Q) \le \overline{\Delta}(Q) \le \Delta_{\max}(Q).$$

For further properties of the various notions of densities and their interdependences we refer to [21], see also [2].

3. Auxiliary results

For the proofs of our main results two Lemmas are needed which are already known.

We consider a subsequence $Q = \{q_\nu\}_{\nu\in\mathbb{N}_0}$ of $\mathbb{N}_0$ and denote by $\mathcal{P}_Q$ the set of all polynomials of the form

$$p(z) = \sum_{\nu=0}^{m} p_\nu z^{q_\nu}.$$

The first Lemma describes the possibilities of approximating functions by the polynomials in $\mathcal{P}_Q$ when Q has positive lower Poisson density.

Lemma 1. *Consider for $r > 0$ and $s > 0$ the disks*

$$D_r := \{z : |z| < r\}, \qquad D_s := \{z : |z - a| < s\};$$

assume that $r + s < |a|$ and $\overline{D_r} \cap \overline{D_s(a)} = \phi$ and define $K := \overline{D_r} \cup \overline{D_s(a)}$.

Let $Q = \{q_\nu\}_{\nu\in\mathbb{N}_0}$ be a subsequence of $\mathbb{N}_0$ with $\underline{\Delta}_p(Q) > \dfrac{1}{\pi}\arcsin\dfrac{s}{|a|}$ and suppose that $f \in A(K)$ is a function which in a neighborhood of the origin has a representation of the form

$$f(z) = \sum_{\nu=0}^{\infty} f_\nu z^{q_\nu}.$$

Then for every $\varepsilon > 0$ there exists a polynomial $p \in \mathcal{P}_Q$ with

$$\max_K |f(z) - p(z)| < \varepsilon.$$

For a proof see [5].

The following Lemma is a generalization of MacLane's result which was mentioned in the introduction.

Lemma 2. *Let be prescribed a subsequence $\lambda = \{\lambda_n\}_{n\in\mathbb{N}}$ of $\mathbb{N}_0$. Then there exists an entire function $\varphi = \varphi_\lambda$ with the following property: for any* *entire function f there exists a subsequence $\{m_k\}_{k\in\mathbb{N}}$ of $\mathbb{N}$, such that the corresponding sequence of derivatives $\{\varphi^{(\lambda_{m_k})}(z)\}$ converges to $f(z)$ compactly on $\mathbb{C}$.*

For an elementary proof see [12].

4. Universal entire functions

Let $Q=\{q_\nu\}_{\nu\in\mathbb{N}_0}$ be a subsequence of $\mathbb{N}_0$. We denote by $\mathcal{E}_Q$ the set of all entire functions with a power series representation

$$f(z)=\sum_{\nu=0}^{\infty} f_\nu z^{q_\nu}.$$

We first prove the existence of an entire function which is universal with respect as well to translates as to derivatives.

Theorem 1. *Let be prescribed*

- *a subsequence Q of $\mathbb{N}_0$ with $\underline{\Delta}_p(Q)>0$,*
- *an unbounded sequence $\{z_n\}$ in $\mathbb{C}$,*
- *a subsequence $\{\lambda_n\}$ of $\mathbb{N}_0$.*

Then there exists an entire function ϕ with the following properties.

Let be given any set $K\in\mathcal{M}$ and any function $f\in A(K)$. Then there exist subsequences $\{p_k\}$ and $\{q_k\}$ of $\mathbb{N}$ such that $\{\phi(z+z_{p_k})\}$ and $\{\phi^{(\lambda_{q_k})}(z)\}$ converge to $f(z)$ uniformly on K.

The function ϕ has the form $\phi=\Psi+\varphi$, where $\Psi\in\mathcal{E}_Q$ and φ is an entire function.

Proof. 1. Let $d\geq 1$ be a number with

$$\underline{\Delta}_p(Q)>\frac{1}{\pi}\arcsin\frac{1}{2d}\,.$$

We define

$$d_0:=1\,;\qquad d_n:=\frac{|z_n|-|z_{n-1}|}{2d}\qquad (n\in\mathbb{N})$$

and without loss of generality we may assume that $\{d_n\}$ is strictly increasing with $\lim\limits_{n\to\infty} d_n=\infty$ (otherwise we can choose a suitable subsequence of $\{z_n\}$ which has the desired property).

By $\{\Omega_n\}_{n\in\mathbb{N}}$ we denote an enumeration of all polynomials whose coefficients have rational real and imaginary parts. Let φ be the entire function from Lemma 2 with respect to $\{\lambda_n\}$.

2. We construct a sequence of polynomials $P_n \in \mathcal{P}_Q$ and a sequence of numbers $m_n \in \mathbb{N}_0$ by induction. Suppose that $P_0(z) \equiv 0$, $m_0 = 0$ and that for an $n \in \mathbb{N}$

$$P_0, P_1, ..., P_{n-1}\,; \qquad m_0, m_1, ..., m_{n-1}$$

have already been determined.

According to Lemma 2 we can choose the natural number $m_n := \lambda_{j_n}$ so great that

$$m_n > m_{n-1} + n\,, \qquad m_n > \deg(P_{n-1})$$

and

$$\max_{|z|\le|z_{n-1}|}\left|\varphi^{(m_n)}(z) - \Omega_n(z)\right| < \frac{1}{n}. \tag{1}$$

By Lemma 1 we find a polynomial $P_n \in \mathcal{P}_Q$ with

$$\max_{|w|\le|z_{n-1}|+d_{n-1}}\left|P_n(w) - P_{n-1}(w)\right| < \varepsilon_n := \frac{1}{(n+1)^2\cdot m_n!\cdot(|z_{n-1}|+d_{n-1})}, \tag{2}$$

$$\max_{|w-z_n|\le d_n}\left|P_n(w) - \Omega_n(w-z_n) + \varphi(w)\right| < \frac{1}{n}. \tag{3}$$

By induction we get $\{P_n(w)\}$ and $\{m_n\}$.

It follows from (2) that

$$\Psi(w) := \sum_{\nu=1}^{\infty}\{P_\nu(w) - P_{\nu-1}(w)\}$$

is an entire function which belongs to $\mathcal{E}_Q$. We will show that

$$\phi(w) := \Psi(w) + \varphi(w)$$

has the desired universal properties.

3. We obtain for all $\nu \ge n$

$$\max_{|z|\le|z_{\nu-1}|}\left|P_\nu^{(m_n)}(z)-P_{\nu-1}^{(m_n)}(z)\right| =$$

$$= \max_{|z|\le|z_{\nu-1}|}\left|\frac{m_n!}{2\pi i}\int\limits_{|w|=|z_{\nu-1}|+d_{\nu-1}}\frac{P_\nu(w)-P_{\nu-1}(w)}{(w-z)^{m_n+1}}\,dw\right| \le$$

$$\le m_n!\cdot(|z_{\nu-1}|+d_{\nu-1})\cdot\varepsilon_\nu\cdot\frac{1}{(d_{\nu-1})^{m_n+1}} \le$$

$$\le m_\nu!\cdot(|z_{\nu-1}|+d_{\nu-1})\cdot\varepsilon_\nu < \frac{1}{(\nu+1)^2}\,.$$

It follows

$$\max_{|z|\le|z_{n-1}|}\left|\Psi^{(m_n)}(z)\right| \le \sum_{\nu=n}^{\infty}\max_{|z|\le z_{\nu-1}|}\left|P_\nu^{(m_n)}(z)-P_{\nu-1}^{(m_n)}(z)\right| \le$$

$$\le \sum_{\nu=n}^{\infty}\frac{1}{(\nu+1)^2} < \frac{1}{n}\,.$$

We therefore obtain together with (1)

$$\max_{|z|\le|z_{n-1}|}\left|\phi^{(m_n)}(z)-\Omega_n(z)\right| < \frac{2}{n}\,. \tag{4}$$

4. From (2) and (3) we get

$$\max_{|w-z_n|\le d_n}|\Phi(w)-\Omega_n(w-z_n)| \le$$

$$\le \max_{|w-z_n|\le d_n}|\Psi(w)-P_n(w)| + \max_{|w-z_n|\le d_n}|P_n(w)-\Omega_n(w-z_n)+\varphi(w)| \le$$

$$\le \sum_{\nu=n+1}^{\infty}\max_{|w|\le|z_{\nu-1}|+d_{\nu-1}}|P_\nu(w)-P_{\nu-1}(w)| + \frac{1}{n} \le$$

$$\le \sum_{\nu=n+1}^{\infty}\frac{1}{(\nu+1)^2}+\frac{1}{n} < \frac{2}{n}$$

or equivalently

$$\max_{|z|\le d_n}|\phi(z+z_n)-\Omega_n(z)| < \frac{2}{n}\,. \tag{5}$$

5. Let now be given a set $K \in \mathcal{M}$ and a function $f \in A(K)$. Then by Mergelian's theorem [18], see also [4], there exists a subsequence $\{n_k\}$ of $\mathbb{N}$ with $\Omega_{n_k}(z) \underset{K}{\Longrightarrow} f(z)$.

a) There exists a k_0 such that $K \subset \{z : |z| \le |z_{n_k-1}|\}$ for all $k > k_0$ and it follows from (4) that

$$\phi^{(m_{n_k})}(z) \underset{K}{\Longrightarrow} f(z)\,.$$

b) On the other hand there exists a $\tilde{k}_0$ such that $K \subset \{z : |z| \le d_{n_k}\}$ for all $k \ge \tilde{k}_0$ and it follows from (5) that also

$$\phi(z + z_{n_k}) \underset{K}{\Longrightarrow} f(z)\,.$$

This finishes the proof of Theorem 1.

Remark 1. The proof shows that the function $\phi = \Psi + \varphi$ was constructed in such a way that

- the function $\Psi \in \mathcal{E}_Q$ has universal translates with respect to the prescribed sequence $\{z_n\}$,
- the entire function φ has universal derivatives with respect to the prescribed sequence $\{\lambda_n\}$

and that ϕ has both of these universalities simultaneously.

5. Universal holomorphic functions in the unit disk

Let again $Q = \{q_\nu\}_{\nu \in \mathbb{N}_0}$ be a subsequence of $\mathbb{N}_0$. We now denote by $\mathcal{U}_Q$ the set of all functions f which are holomorphic in the unit disk $\mathbb{D}$ with a power series expansion

$$f(z) = \sum_{\nu=0}^{\infty} f_\nu z^{q_\nu}\,.$$

For a sequence $\{w_n\}$ of complex numbers we denote by $V(\{w_n\})$ the set of all its accumulation points.

Theorem 2. *Let be prescribed:*

- *a subsequence Q of $\mathbb{N}_0$ with $\underline{\Delta}_p(Q) > 0$,*
- *a sequence $\{a_n\}$ in $\mathbb{C}\backslash\{0\}$ with $0 \in V(\{a_n\})$,*
- *a sequence $\{b_n\}$ in $\mathbb{D}$ with $V(\{b_n\}) = \partial\mathbb{D}$,*

- *a subsequence* $\{\lambda_n\}$ *of* $\mathbb{N}_0$.

Then there exists a holomorphic function ϕ *in* $\mathbb{D}$ *with the following properties.*

Let be given any set $K \in \mathcal{M}$, *any function* $f \in A(K)$ *and any* $\zeta \in \partial\mathbb{D}$. *Then there are subsequences* $\{p_k\}, \{q_k\}$ *of* $\mathbb{N}$ *such that*

$$\lim_{k\to\infty} a_{p_k} = 0, \qquad \lim_{k\to\infty} b_{q_k} = \zeta,$$

$$a_{p_k} z + b_{q_k} \in \mathbb{D} \qquad \textit{for all} \quad z \in K,$$

$$\{\phi(a_{p_k} z + b_{q_k})\} \qquad \textit{converges to} \quad f(z) \quad \textit{uniformly on } K.$$

If $K \subset \mathbb{D}$ *then there exists in addition a subsequence* $\{r_k\}$ *of* $\mathbb{N}$ *such that*

$$\{\phi^{(\lambda_{r_k})}(z)\} \quad \textit{converges to } f(z) \textit{ uniformly on } K.$$

The function ϕ *has the form* $\phi = \Psi + \varphi$, *where* $\Psi \in \mathcal{U}_Q$ *and* φ *is an entire function.*

Proof. 1. We consider any sequence $\{\zeta^{(k)}\}_{k\in\mathbb{N}_0}$ of points $\zeta^{(k)} \in \partial\mathbb{D}$ with $V(\{\zeta^{(k)}\}) = \partial\mathbb{D}$. For each $k \in \mathbb{N}_0$ we choose a subsequence $\{z_\nu^{(k)}\}_{\nu\in\mathbb{N}}$ of $\{b_n\}$ with the properties

$$\lim_{\nu\to\infty} z_\nu^{(k)} = \zeta^{(k)},$$

$$|z_\nu^{(0)}| < |z_\nu^{(1)}| < \ldots < |z_\nu^{(\nu)}| \qquad \text{for each } \nu \in \mathbb{N}$$

and that there exists a sequence $\{G_\nu\}_{\nu\in\mathbb{N}}$ of disks $G_\nu := \{z : |z| < \varrho_\nu\}$ with $\varrho_1 < \varrho_2 < \ldots < \varrho_\nu \to 1$ and

$$z_\nu^{(k)} \in G_{\nu+1} \backslash \overline{G_\nu} \qquad \text{for} \quad k = 0, 1, ..., \nu.$$

Next we choose ℓ_ν so great that $s_\nu := \sqrt{|a_{\ell\nu}|} \to 0$ for $\nu \to \infty$ and the following properties hold:

$$|z_\nu^{(k)}| - |z_\nu^{(k-1)}| > 2s_\nu \qquad \text{for} \quad k = 0, ..., \nu,$$

$$\{z : |z - z_\nu^{(k)}| < s_\nu\} \subset G_{\nu+1} \backslash \overline{G_\nu} \qquad \text{for} \quad k = 1, ..., \nu,$$

$$\underline{\Delta}_p(Q) > \frac{1}{\pi} \arcsin \frac{s_\nu}{|z_\nu^{(k)}|} \qquad \text{for} \quad k = 1, ..., \nu; \ \nu = 1, 2, ...$$

Let $\{\Omega_n\}_{n\in\mathbb{N}}$ be again an enumeration of all polynomials with coefficients whose real and imaginary parts are rational and let φ be the entire function from Lemma 2 with respect to $\{\lambda_n\}$.

2. We construct polynomials $P_{n\mu} \in \mathcal{P}_Q$ ($\mu = 0, ..., n$; $n = 1, 2, ...$) and integers m_n by induction, where in the n-th step the polynomials P_{n1}, $P_{n2}, ..., P_{nn}$ and an integer m_n are defined by an approximation process, using Lemma 1. To begin with we choose $P_{10}(z) \equiv P_{11}(z) \equiv 0$ and $m_1 := 0$.

Suppose that $n \geq 2$ and that the groups of polynomials

$$P_{\nu 1}, P_{\nu 2}, ..., P_{\nu\nu}$$

and the numbers $m_\nu \in \mathbb{N}$ have already been determined for all $\nu = 1, ..., n-1$, where we define in any case $P_{\nu+1,0}(z) := P_{\nu\nu}(z)$. Hence, when starting our construction the polynomial P_{n0} is already known. According to Lemma 2 we can choose $m_n := \lambda_{j_n}$ so great that

$$m_n > m_{n-1} + n\,, \qquad m_n > \deg(P_{n0})$$

and

$$\max_{\overline{G_n}} |\varphi^{(m_n)}(z) - \Omega_n(z)| < \frac{1}{n}\,. \tag{6}$$

If now for a μ with $1 \leq \mu \leq n$ the polynomial $P_{n,\mu-1}$ has already been chosen, then we find according to Lemma 1 a polynomial $P_{n\mu} \in \mathcal{P}_Q$ which satisfies

$$\max_{|w| \leq |z_n^{(\mu-1)}| + s_n} \Big| P_{n\mu}(w) - P_{n,\mu-1}(w) \Big| < \varepsilon_n := \frac{(s_n)^{m_n+1}}{(n+1)^3 \cdot m_n!}, \tag{7}$$

$$\max_{|w - z_n^{(\mu)}| \leq s_n} \left| P_{n\mu}(w) - \Omega_n\left(\frac{w - z_n^{(\mu)}}{a_{\ell_n}}\right) + \varphi(w) \right| < \frac{1}{n}. \tag{8}$$

By induction we get all polynomials $P_{n\mu}$ and all numbers m_n. It follows easily from (7) that the function

$$\Psi(w) := \sum_{\nu=1}^{\infty} \sum_{\mu=1}^{\nu} \{P_{\nu\mu}(w) - P_{\nu,\mu-1}(w)\}$$

belongs to $\mathcal{U}_Q$. We will show that

$$\phi(w) := \Psi(w) + \varphi(w)$$

has the asserted universal properties.

3. We obtain for all $\nu \geq n$ and $\mu = 1, ..., \nu$ the estimates

$$\max_{|z|\leq|z_\nu^{(\mu-1)}|}\left|P_{\nu\mu}^{(m_n)}(z) - P_{\nu,\mu-1}^{(m_n)}(z)\right| =$$

$$= \max_{|z|\leq|z_\nu^{(\mu-1)}|}\left|\frac{m_n!}{2\pi i}\int_{|w|=|z_\nu^{(\mu-1)}|+s_\nu}\frac{P_{\nu\mu}(w) - P_{\nu,\mu-1}(w)}{(w-z)^{m_n+1}}dw\right| \leq$$

$$\leq m_n! \cdot \left(|z_\nu^{(\mu-1)}| + s_\nu\right) \cdot \varepsilon_\nu \cdot \frac{1}{(s_\nu)^{m_n+1}} \leq$$

$$\leq m_\nu! \cdot \varepsilon_\nu \cdot \frac{1}{(s_\nu)^{m_\nu+1}} = \frac{1}{(\nu+1)^3}.$$

It follows

$$\max_{G_n}|\Psi^{(m_n)}(z)| \leq \sum_{\nu=n}^{\infty}\sum_{\mu=1}^{\nu}\max_{G_n}\left|P_{\nu\mu}^{(m_n)}(z) - P_{\nu,\mu-1}^{(m_n)}(z)\right| \leq$$

$$\leq \sum_{\nu=n}^{\infty}\sum_{\mu=1}^{\nu}\max_{|z|\leq|z_\nu^{(\mu-1)}|}\left|P_{\nu\mu}^{(m_n)}(z) - P_{\nu,\mu-1}^{(m_n)}(z)\right| \leq$$

$$\leq \sum_{\nu=n}^{\infty}\sum_{\mu=1}^{\nu}\frac{1}{(\nu+1)^3} < \frac{1}{n}.$$

Together with (6) we therefore have

$$\max_{G_n}|\phi^{(m_n)}(z) - \Omega_n(z)| < \frac{2}{n}. \tag{9}$$

4. It follows from (7) for $n \in \mathbb{N}$ and $\mu = 1, ..., n$

$$\max_{|w-z_n^{(\mu)}|\leq s_n}|\Psi(w) - P_{n\mu}(w)| \leq$$

$$\leq \sum_{\nu=\mu+1}^{n}\max_{|w|\leq|z_n^{(\nu-1)}|+s_n}|P_{n\nu}(w) - P_{n,\nu-1}(w)| +$$

$$+ \sum_{m=n+1}^{\infty}\sum_{\nu=1}^{m}\max_{|w|\leq|z_m^{(\nu-1)}|+s_m}|P_{m\nu}(w) - P_{m,\nu-1}(w)| \leq$$

$$\leq \sum_{\nu=\mu+1}^{n}\frac{1}{(n+1)^3} + \sum_{m=n+1}^{\infty}\sum_{\nu=1}^{m}\frac{1}{(m+1)^3} < \frac{2}{n}.$$

We have

$$\phi(w) - \Omega_n\left(\frac{w - z_n^{(\mu)}}{a_{\ell_n}}\right) =$$

$$= \{\Psi(w) - P_{n\mu}(w)\} + \left\{P_{n\mu}(w) - \Omega\left(\frac{w - z_n^{(\mu)}}{a_{\ell_n}}\right) + \varphi(w)\right\}$$

and therefore we obtain by (8) for $n \in \mathbb{N}$ and $\mu = 1, ..., n$

$$\max_{|w - z_n^{(\mu)}| \leq s_n} \left|\phi(w) - \Omega_n\left(\frac{w - z_n^{(\mu)}}{a_{\ell_n}}\right)\right| < \frac{3}{n}$$

or equivalently

$$\max_{|z| \leq \frac{1}{s_n}} |\phi(a_{\ell_n} z + z_n^{(\mu)}) - \Omega_n(z)| < \frac{3}{n}. \tag{10}$$

5. Let now be given a set $K \in \mathcal{M}$ and a function $f \in A(K)$. Then by Mergelian's theorem [18], see also [4], there exists a subsequence $\{n_k\}$ of $\mathbb{N}$ with $\Omega_{n_k}(z) \underset{K}{\Longrightarrow} f(z)$.

a) If $K \subset \mathbb{D}$ then there exists a k_0 such that $K \subset G_{n_k}$ for all $k > k_0$ and (9) implies that $\{\phi^{(m_{n_k})}(z)\}$ converges to $f(z)$ uniformly on K.

b) Suppose that $K \in \mathcal{M}$ is arbitrary; then there exists a $\tilde{k}_0$ such that $K \subset \left\{z : |z| \leq \dfrac{1}{s_{n_k}}\right\}$ for all $k > \tilde{k}_0$. If any $\zeta \in \partial\mathbb{D}$ is given, then ζ is an accumulation point of the set

$$\{z_{n_k}^{(\mu)} : \mu = 1, ..., n_k\,;\ k > \tilde{k}_0\}$$

and there exists $\mu_k, 1 \leq \mu_k \leq n_k$ such that $z_{n_k}^{(\mu_k)} \to \zeta$ for $k \to \infty$. Hence it follows from [10] that $\{\phi(a_{\ell_{n_k}} z + z_{n_k}^{(\mu_k)})\}$ converges to $f(z)$ uniformly on K.

This proves Theorem 2.

Remark 2. The proof shows that the function $\phi = \Psi + \varphi$ was constructed in such a way that

- the function $\Psi \in \mathcal{U}_Q$ has universal translates with respect to the prescribed pair of sequences $\{a_n\}, \{b_n\}$,
- the entire function φ has universal derivatives with respect to the prescribed sequence $\{\lambda_n\}$

and that ϕ has both of these universalities simultaneously. In addition, the power series expansion of ϕ around the origin

$$\phi(z) := \sum_{\nu=0}^{\infty} \phi_\nu z^\nu$$

has "quasi-gaps" according to the sequence Q (with positive lower Poisson density) in that sense that we have

$$\lim_{\substack{\nu\to\infty \\ \nu\in Q}} |\phi_\nu|^{1/\nu} = 0\,.$$

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