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ONCE MORE ABOUT WIRSING'S THEOREM ON MULTIPLICATIVE FUNCTIONS: A SIMPLE PROBABILISTIC PROOF

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To Professor Karl-Heinz Indlekofer on his 60th birthday

Abstract. Let $g(m)$ be a multiplicative arithmetical function taking the values +1 or -1 only. We give a purely probabilistic proof for Wirsing's theorem stating that the asymptotic mean of $g(m)$ always exists and it is zero if, and only if, either $g(2^k) = -1$ for all $k \geq 1$, or $\sum_{g(p)=-1} 1/p = +\infty$. Since for $g(m)$ above the existence of the asymptotic mean value is equivalent to the existence of the asymptotic distribution of $g(m)$, we compare the distribution of $g(m)$ with the distribution of products of random variables, and then apply a recent result of Simonelli (2001) from probability theory.

1. Introduction

An arithmetical function $g(m)$ is called multiplicative if for coprime m and n

$$g(mn) = g(m)g(n).$$

A remarkable theorem of Wirsing (1967) states that if $g(m)$ is a multiplicative function with values in $[-1, 1]$, the mean value

$$M(g) = \lim_{N\to+\infty} \frac{1}{N} \sum_{m\leq N} g(m)$$

always exists and equals zero if, and only if, either $g(2^k) = -1$ for $k \geq 1$, or

$$\sum_p \frac{1 - g(p)}{p} = +\infty.$$

The importance and difficulty of this problem was shown by a theorem of Landau (1909), that says that the prime number theorem is equivalent to $M(g) = 0$ for the special case $g(m)$ is the Möbius function. In this paper we limit our investigation to multiplicative functions with values in $\{-1, +1\}$. However our method of proof can be modified to multiplicative functions which are zero at all prime powers larger than one (thus covering the Möbius function).

Consider the probability space $(\Omega_N, \mathcal{A}_N, P_N)$, where Ω_N denotes the first N positive integers, $\mathcal{A}_N$ the collection of all subsets of Ω_N, and P_N the probability measure that assign mass $1/N$ to each element in Ω_N. Then every multiplicative function $g(m)$ can be viewed as a random variable in $(\Omega_N, \mathcal{A}_N, P_N)$. Moreover $g(m)$ can be expressed as

$$g(m) = \prod_{p \leq N} g\left(p^{s_p(m)}\right),$$

where the product is over all primes less than or equal to N, $s_p(m)$ is the integer in the prime factorization of m, and for arbitrary primes $p_1, \cdots, p_t$,

$$P_N[s_{p_i}(m) \geq k_i,\ i = 1, \cdots, t] = \frac{1}{N}\left[\frac{N}{p_1^{k_1} \cdots p_t^{k_t}}\right],$$

where $[x]$ denotes the integer part of x.

The multiplicative function $g(m)$ is said to have a limit distribution if there is a distribution function $F(x)$ such that

$$\lim_{N \to +\infty} P_N\left[\prod_{p \leq N} g(p^{s_p(m)}) \leq x\right] = F(x)$$

for all continuity points of $F(x)$. Hence in the case $g(m) \in \{-1, +1\}$, $M(g)$ is zero if, and only if, $g(m)$ has a symmetric limit distribution $F(x)$.

In this paper we give a simple probabilistic proof of Wirsing's theorem for the case $g(m) = -1, 1$. For the general case, besides the original proof by Wirsing, simple proofs were also given by Hildebrand (1985), Daboussi and Indlekofer (1992), and Indlekofer (1993).

2. Tools from probability theory

In this section we collect the main probability tools used in our proof.

We start with a result of Simonelli (2001).

Theorem 1. *Let $X_1, X_2, \cdots$ be a sequence of independent random variables, $X_i = -1, 1$, $i = 1, 2, \cdots$. Then*

$$\lim_{n\to+\infty} P\left(\prod_{i=1}^{n} X_i = -1\right) = P$$

exists if, and only if, either $\lim_{n\to+\infty} P(X_n = -1) = 0$, or $P(X_i = 1) = 1/2$ for some i, or

$$\sum_{i=1}^{+\infty} \min\{P(X_i = -1),\, P(X_i = 1)\} = +\infty. \tag{1}$$

Moreover $P = 1/2$ if, and only if, either $P(X_i = -1) = 1/2$ for some i, or (1) holds.

Let $A_1, A_2, \cdots$ be arbitrary events in some given probability space, and for $k \geq 1$ put

$$S_k = S_{k,n} = E\left[\binom{m_n}{k}\right] = \sum P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}),$$

where m_n is the number of those A_j which occur and $\sum$ is summation over all integers $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. Galambos and Simonelli (2003) developed Bonferroni-type identities and inequalities for an odd number of occurrences in a sequence of events, which can be applied to products of random variables with values -1, 1.

Theorem 2. *Let $X_1, X_2, \cdots$ be a sequence of random variables, each with values -1, 1. Let $A_i = \{X_i = -1\}$, $i = 1, 2, \cdots$. Then for arbitrary integers r, d, $2 \leq 2r$, $1 \leq 2d + 1$,*

$$\sum_{k=1}^{2r} (-2)^{k-1} S_k \leq P\left(\prod_{i=1}^{n} X_i = -1\right) \leq \sum_{k=1}^{2d+1} (-2)^{k-1} S_k.$$

When $2r \geq n$ and $2d + 1 \geq n$, the above inequalities become identities.

In the case the X_i are independent, Theorem 2 gives

$$P\left(\prod_{i=1}^{n} X_i = -1\right) = \frac{1}{2} - \frac{1}{2}\prod_{i=1}^{n}(1 - 2P(A_i)), \tag{2}$$

from which one can easily obtain necessary and sufficient conditions for $P\left(\prod_{i=1}^{n} X_i = -1\right)$ to increase (decrease) either as a function of n or as a function of $P(A_i)$, for any given i. For example, given $P(A_i) < 1/2$ for $i = 1, 2, \cdots, n$, from (2) it is easy to see that $P\left(\prod_{i=1}^{n} X_i = -1\right)$ increases if any $P(A_j)$ is replaced by $P(B) > P(A_j)$. This fact will be used in the next section.

From elementary probability theory we will use several times the total probability rule with the simple form of

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c), \qquad 0 < P(B) < 1,$$

and the simple formula

$$P(A \cap B^c) = P(A) - P(A \cap B). \tag{3}$$

2. Probabilistic proof of Wirsing's theorem

The aim of this section is to give a simple probabilistic proof of the following theorem of Wirsing.

Theorem 3. *Let $g(m)$ be a multiplicative function with values in $\{-1, 1\}$. Then $g(m)$ has always a limit distribution $F(x)$. Moreover $F(x)$ is symmetric if, and only if, either $g(2^k) = -1$ for $k \geq 1$, or*

$$\sum_{p:g(p)<0} \frac{1}{p} = +\infty. \tag{4}$$

Proof. In some abstract probability space let e_{p_i} be independent random variables such that $e_{p_i} = 0, 1, \cdots$, and

$$P[e_{p_i} \geq k_i,\ i = 1, \cdots, t] = \frac{1}{p_1^{k_1} \cdots p_t^{k_t}}.$$

Then, for any T,

$$\lim_{N\to+\infty} P_N\left[\prod_{p\le T} g(p^{s_p(m)})<0\right] = P\left[\prod_{p\le T} g(p^{e_p})<0\right].$$

Moreover, by using (3), one obtains

$$\begin{aligned}
&P_N\left[\prod_{p\le N} g(p^{s_p(m)})<0\right] = \\
=&P_N\left[\prod_{p\le T} g(p^{s_p(m)})<0\right] + P_N\left[\prod_{p\le T} g(p^{s_p(m)})>0,\ \prod_{T<p\le N} g(p^{s_p(m)})<0\right] - \\
&- P_N\left[\prod_{p\le T} g(p^{s_p(m)})<0,\ \prod_{T<p\le N} g(p^{s_p(m)})<0\right],
\end{aligned}$$

which implies that, for any T,

$$\left|P_N\left[\prod_{p\le N} g(p^{s_p(m)})<0\right] - P_N\left[\prod_{p\le T} g(p^{s_p(m)})<0\right]\right| \le 2\sum_{\substack{p\ge T\\ g(p)=-1}} \frac{1}{p}.$$

Hence if

$$\sum_{p:g(p)<0} \frac{1}{p} < +\infty,$$

the above calculation, Theorem 2 and (2) imply

$$\lim_{N\to+\infty} P_{N,\alpha}\left[\prod_{p\le N} g(p^{s_p(m)})<0\right] = \lim_{T\to+\infty} P\left[\prod_{p\le T} g(p^{e_p})<0\right] = b,$$

with $b=1/2$ if, and only if, $g(2^k)=-1$ for $k\ge 1$, and $b<1/2$ otherwise.

Next let us assume the validity of (4), and let us assume that the conclusion of the theorem does not hold. Then there exist an $\epsilon>0$, a sequence of positive integers N_k, $N_k\to+\infty$ with k, and a positive integer M, $M=M(\epsilon)$, such that either

$$P_{N_k}\left[\prod_{p\le N_k} g(p^{s_p(m)})>0\right] - P_{N_k}\left[\prod_{p\le N_k} g(p^{s_p(m)})<0\right] > 2\epsilon \quad \text{for all } N_k\ge M$$

or

$$P_{N_k}\left[\prod_{p\le N_k} g(p^{s_p(m)})>0\right]-P_{N_k}\left[\prod_{p\le N_k} g(p^{s_p(m)})<0\right]<-2\epsilon \quad \text{for all } N_k\ge M.$$

Let us assume the first inequality holds, and let q_o be such that

$$\frac{1}{q_o}<\frac{\epsilon}{2}.$$

For $1\le\alpha<2$, define

$$P_{N,\alpha}[s_{p_i}(m)\ge k_i,\ i=1,\cdots,t]=\frac{1}{N}\left[\frac{N}{p_1^{\delta_1k_1}\cdots p_t^{\delta_tk_t}}\right], \tag{5}$$

where $\delta_ik_i=\alpha$ if $k_i=1$, $p_i\ge q_o$, and $g(p_i)=-1$, and $\delta_ik_i=k_i$ otherwise. Then $P_{N,\alpha}$ defines a probability measure on $\Omega_N=\{1,2,\cdots,N\}$. Similarly we define

$$P_\alpha[e_{p_i}\ge k_i,\ i=1,\cdots,t]=\frac{1}{p_1^{\delta_1k_1}\cdots p_t^{\delta_tk_t}},$$

where δ_ik_i is as in (5). Then for arbitrary T,

$$\lim_{N\to+\infty}P_{N,\alpha}\left[\prod_{p\le T}g(p^{s_p(m)})<0\right]=P_\alpha\left[\prod_{p\le T}g(p^{e_p})<0\right],$$

and from Theorem 2,

$$\lim_{T\to+\infty}P_\alpha\left[\prod_{p\le T}g(p^{e_p})<0\right]=b_\alpha.$$

By proceeding as in the previous case we immediately have that for any T

$$\left|P_{N,\alpha}\left[\prod_{p\le N}g(p^{s_p(m)})<0\right]-P_{N,\alpha}\left[\prod_{p\le T}g(p^{s_p(m)})<0\right]\right|\le 2\sum_{\substack{p\ge T\\ g(p)=-1}}\frac{1}{p^\alpha},$$

which implies that for $\alpha>1$,

$$\lim_{N\to+\infty}P_{N,\alpha}\left[\prod_{p\le N}g(p^{s_p(m)})<0\right]=\lim_{T\to+\infty}P_\alpha\left[\prod_{p\le T}g(p^{e_p})<0\right]=b_\alpha.$$

By (2) one further obtains that either $b_\alpha = 1/2$ for $1 \le \alpha < 2$ (in the case $g(2^k) = -1$, for $k \ge 1$) or $b_\alpha < 1/2$, if $\alpha > 1$, and $b_\alpha \nearrow 1/2$ as $\alpha \searrow 1$. In either case, there exist α_1 and a positive integer L, $L = L(\epsilon)$, such that for $N \ge L$,

$$P_{N,\alpha_1}\left[\prod_{p\le N} g(p^{s_p(m)}) > 0\right] - P_{N,\alpha_1}\left[\prod_{p\le N} g(p^{s_p(m)}) < 0\right] < \epsilon. \tag{6}$$

Choose N in the sequence N_k such that $N \ge \max\{M, L\}$. Since P_N only depends on a finite number of primes, our initial assumption implies the existence of α_o such that whenever $1 \le \alpha \le \alpha_o$,

$$P_{N,\alpha}\left[\prod_{p\le N} g(p^{s_p(m)}) > 0\right] - P_{N,\alpha}\left[\prod_{p\le N} g(p^{s_p(m)}) < 0\right] > \epsilon. \tag{7}$$

If $\alpha_1 \le \alpha_o$, we immediately get a contradiction ((6) and (7) cannot hold at the same time). So let us assume that $\alpha_o < \alpha_1$. Let $q_o \le q \le N$, $g(q) < 0$, and denote by $D(q, \alpha_o)$ the left hand side of (7), with $\alpha = \alpha_o$. We claim that if we change the distribution of P_{N,α_o} by changing q^{α_o} to q^{α_1} in the marginal distributions of P_{N,α_o} (given by (5)), then the left hand side of (7) computed with respect to this new distribution function, which we denote by $D(q, \alpha_1)$, satisfies

$$D(q, \alpha_1) - D(q, \alpha_o) \ge 0. \tag{8}$$

In fact our computation will show that if we view $D(q, \alpha)$ as a function of q^α (q fixed, but α is allowed to take values on $[\alpha_o, \alpha_1]$), then $D(q, \alpha)$ is a non-decreasing function of q^α. We start by rewriting $D(q, \alpha_o)$ as

$$\begin{aligned}
&P_{N,\alpha_o}[s_q(m) = 1]\, P_{N,\alpha_o}\left[\prod_{p\le N} g(p^{s_p(m)}) < 0 \,\middle|\, s_q(m) = 1\right] + \\
&+ (1 - P_{N,\alpha_o}[s_q(m) = 1])\, P_{N,\alpha_o}\left[\prod_{p\le N} g(p^{s_p(m)}) > 0 \,\middle|\, s_q(m) \ne 1\right] - \\
&- P_{N,\alpha_o}[s_q(m) = 1]\, P_{N,\alpha_o}\left[\prod_{p\le N} g(p^{s_p(m)}) > 0 \,\middle|\, s_q(m) = 1\right] - \\
&- (1 - P_{N,\alpha_o}[s_q(m) = 1])\, P_{N,\alpha_o}\left[\prod_{p\le N} g(p^{s_p(m)}) < 0 \,\middle|\, s_q(m) \ne 1\right] =
\end{aligned}$$

$$(9)\ = P_{N,\alpha_o}[s_q(m)=1]\left(P_{N,\alpha_o}\left[\prod_{p\le N} g(p^{s_p(m)})<0\Big|\, s_q(m)=1\right]-\right.$$
$$\left.-P_{N,\alpha_o}\left[\prod_{p\le N} g(p^{s_p(m)})>0\Big|\, s_q(m)=1\right]\right)+$$
$$+\,(1-P_{N,\alpha_o}[s_q(m)=1])\left(P_{N,\alpha_o}\left[\prod_{p\le N} g(p^{s_p(m)})>0\Big|\, s_q(m)\neq 1\right]-\right.$$
$$\left.-\ P_{N,\alpha_o}\left[\prod_{p\le N} g(p^{s_p(m)})<0\Big|\, s_q(m)\neq 1\right]\right)=$$
$$(10) = P_{N,\alpha_o}[\prod_{\substack{p\le N\\ p\neq q}} g(p^{s_p(m)})<0]-P_{N,\alpha_o}[\prod_{\substack{p\le N\\ p\neq q}} g(p^{s_p(m)})>0]+$$
$$+\,2\,(1-P_{N,\alpha_o}[s_q(m)=1])\left(P_{N,\alpha_o}\left[\prod_{p\le N} g(p^{s_p(m)})>0\Big|\, s_q(m)\neq 1\right]-\right.$$
$$\left.-\ P_{N,\alpha_o}\left[\prod_{p\le N} g(p^{s_p(m)})<0\Big|\, s_q(m)\neq 1\right]\right).$$

The last two representations of $D(q,\alpha_o)$ contain the proof of our claim. Since $D(q,\alpha_o)>0$ and

$$P_{N,\alpha_o}[s_q(m)=1]=\frac{1}{N}\left[\frac{N}{q^{\alpha_o}}\right]-\frac{1}{N}\left[\frac{N}{q^2}\right]\le\frac{1}{q}<\frac{\epsilon}{2}$$

then the last difference in (9), and consequently the last difference in (10), must be nonnegative. Moreover the only term in (10) which depends on q^α is

$$(1-P_{N,\alpha_o}[s_q(m)=1])=1-\frac{1}{N}\left[\frac{N}{q^{\alpha_o}}\right]+\frac{1}{N}\left[\frac{N}{q^2}\right],$$

which clearly non-decreases as α increases from α_o to α_1. Hence (8) holds. So let us change the distribution P_{N,α_o} by changing q^{α_o} to q^{α_1} in its marginal distributions. This procedure can now be repeated for every prime $p\neq q$, $q_o\le p\le N$, $g(p)<0$, and at the end of this process the left hand side of (7) will be changed into the left hand side of (6). Since at each iteration we are non-decreasing the difference between the probabilities of the events

$$\left\{\prod_{p\le N} g(p^{s_p(m)})>0\right\}\quad\text{and}\quad\left\{\prod_{p\le N} g(p^{s_p(m)})<0\right\},$$

we immediately get a contradiction (if (7) holds, (6) cannot hold).

If the second inequality at the beginning of our proof holds, instead of the first one, then α_1, α_o, and N can be chosen such that

$$P_{N,\alpha_1}\left[\prod_{p\le N} g(p^{s_p(m)})>0\right]-P_{N,\alpha_1}\left[\prod_{p\le N} g(p^{s_p(m)})<0\right]>-\epsilon, \tag{11}$$

and

$$P_{N,\alpha}\left[\prod_{p\le N} g(p^{s_p(m)})>0\right]-P_{N,\alpha}\left[\prod_{p\le N} g(p^{s_p(m)})<0\right]<-\epsilon, \tag{12}$$

for $1\le\alpha\le\alpha_o$. We can again assume $\alpha_o<\alpha_1$. Let $q_o\le q\le N$, $g(q)<0$, $D(q,\alpha_o)$ and $D(q,\alpha_1)$ be as before. In this case we claim

$$D(q,\alpha_1)-D(q,\alpha_o)\le 0. \tag{13}$$

Since $D(q,\alpha_o)<0$, then the last difference in (9), and consequently the last difference in (10), is less than or equal to zero, and considerations similar to the ones made in the previous case immediately imply the validity of (13). So let us consider $D(q,\alpha_1)$. As before by repeating the above procedure for every $p\ne q$, $q_o\le p\le N$, $g(p)<0$, we will eventually change the left hand side of (12) into the left hand side of (11). Since at each iteration we are non-increasing the difference between the probabilities of the events

$$\left\{\prod_{p\le N} g(p^{s_p(m)})>0\right\} \quad\text{and}\quad \left\{\prod_{p\le N} g(p^{s_p(m)})<0\right\},$$

we immediately get a contradiction (if (12) holds, (11) cannot hold).

The proof of the theorem is now complete.

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