

## DIGITAL EXPANSION IN $\mathbb{Q}(\sqrt{2})$

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*Dedicated to Prof. K.-H. Indlekofer on his 60th birthday*

**Abstract.** One objective of our research is to extend the concept of number system to various algebraic structures. A study of the quadratic fields automatically raises the question: for what algebraic integers  $\alpha$  can we find such a digit set  $E_\alpha$  where  $(\alpha, E_\alpha)$  is a number system.

### 1. Introduction

Let  $D$  be a square free (positive or negative) integer, or  $D = -1$ . The quadratic fields  $\mathbb{Q}(\sqrt{D})$  are classified: we say that it is an imaginary extension of  $\mathbb{Q}$ , if  $D < 0$ , while in the case  $D > 0$  we say that it is a real extension field. For some fixed  $D$ , let  $I$  denote the set of algebraic integers in  $\mathbb{Q}(\sqrt{D})$ .

**Definition.** Let  $\alpha \in I$  and  $E_\alpha (\subseteq I)$  be a complete residue system mod  $\alpha$  containing 0.

We say that  $(\alpha, E_\alpha)$  is a *coefficient system* in  $I$ , the elements of  $E_\alpha$  are digits and  $\alpha$  is the base (number) of  $(\alpha, E_\alpha)$ .

**Definition.**  $(\alpha, E_\alpha)$  is a *number system* in  $I$  if each  $\gamma \in I$  can be written in a finite sum

$$(1.1) \quad \gamma = e_0 + e_1\alpha + \dots + e_k\alpha^k,$$

where  $e_i \in E_\alpha$ ,  $i = 0, 1, \dots, k$ .

Then  $\alpha$  is the *base (number)* of the number system  $(\alpha, E_\alpha)$ .

Since  $E_\alpha$  is a complete residue system mod  $\alpha$ , therefore the uniqueness of the representation (1.1) is clear. Furthermore, it is obvious that for every  $\gamma \in I$  there exists a unique  $f \in E_\alpha$  for which

$$(1.2) \quad \gamma = \alpha \cdot \gamma_1 + f$$

holds with a suitable  $\gamma_1 \in I$ .

Let  $J : I \rightarrow I$  be the function defined by  $J(\gamma) = \gamma_1$ , if  $\gamma, \gamma_1$  are related by (1.2). Then we can define a directed graph over  $I$ , by drawing an edge from  $\gamma$  to  $\gamma_1$ , labeling it with  $f$ :

$$(1.3) \quad \gamma \xrightarrow{(f)} \gamma_1.$$

We say that  $\pi \in I$  is a periodic element, if  $J^k(\pi) = \pi$  holds for some positive integer  $k$ . Let  $\mathcal{P}$  be the set of periodic elements.

It is clear that  $(\alpha, E_\alpha)$  is a number system, if  $\mathcal{P} = \{0\}$ .

The question we are interested in is the following: what are those  $\alpha \in I$  in some  $\mathbb{Q}(\sqrt{D})$  for which  $(\alpha, E_\alpha)$  is a number system with an appropriate digit set  $E_\alpha$ .

The question was completely solved by G. Steidl for  $D = -1$  [5], for arbitrary imaginary quadratic fields by I. Kátai. Namely they proved that  $\alpha$  is a base of a number system, if and only if  $\alpha \neq 0$  and  $\alpha, 1 - \alpha$  are not units. The explicit construction of  $E_\alpha$  is given.

The case of real quadratic extension field seems to be harder due to the fact that the module of  $\alpha$  and its conjugate  $\bar{\alpha}$  are not the same in general.

Assume that  $D > 0$ . If  $\alpha$  is a base of a number system then clearly

$$(1.4) \quad \begin{cases} \alpha \neq 0, \\ \alpha, 1 - \alpha \text{ are not units,} \\ |\alpha| > 1, |\bar{\alpha}| > 1 \end{cases}$$

should be satisfied.

It is true that  $\alpha$  is a base of a number system if (1.4) holds?

The first named author proved it [4] if (1.4) holds in a stronger form:

$$|\alpha| \geq 2, \quad |\bar{\alpha}| \geq 2.$$

We shall improve this in the case  $D = 2$ .

## 2. The formulation of our results

### 2.1. A previous achievement

The starting point of our investigation is a theorem published by I. Kátai in [1]. He proved for imaginary quadratic fields that  $\alpha \in I$  is a base of a number system with an appropriate digit set  $E_\alpha$  if and only if

$$\begin{aligned} \alpha &\neq 0, \\ \alpha, 1 - \alpha &\text{ are not units and} \\ |\alpha|, |\bar{\alpha}| &> 1, \end{aligned}$$

where  $\bar{\alpha}$  is the algebraic conjugate of  $\alpha$ . In that paper the explicit construction of digit set  $E_\alpha$  was given. The digit sets constructed in this way are called *K-type digit sets*.

Our prospective purpose is to prove the same result in real quadratic fields. In this paper we concentrate on the quadratic field  $\mathbb{Q}(\sqrt{2})$ .

### 2.2. The construction of the K-type digit sets

Let  $\alpha = a + b\sqrt{2}$  be an arbitrary algebraic integer in  $\mathbb{Q}(\sqrt{2})$ , that is  $a, b \in \mathbb{Z}$ , and  $d = \alpha \cdot \bar{\alpha}$ , where  $\bar{\alpha} = a - b\sqrt{2}$  is the conjugate of  $\alpha$ . Then  $E_\alpha^{(\varepsilon, \delta)}$  are the sets of those  $f = k + l\sqrt{2}$  ( $k, l \in \mathbb{Z}$ ) for which

$$f \cdot \bar{\alpha} = \left(k + l\sqrt{2}\right) \left(a - b\sqrt{2}\right) = (ka - bl2) + (la - kb)\sqrt{2} = r + s\sqrt{2}$$

satisfy the following conditions:

- if  $(\varepsilon, \delta) = (1, 1)$ , then  $r, s \in \left(\frac{-|d|}{2}, \frac{|d|}{2}\right]$ ,
- if  $(\varepsilon, \delta) = (-1, -1)$ , then  $r, s \in \left[\frac{-|d|}{2}, \frac{|d|}{2}\right)$ ,
- if  $(\varepsilon, \delta) = (-1, 1)$ , then  $r \in \left[\frac{-|d|}{2}, \frac{|d|}{2}\right)$ ,  $s \in \left(\frac{-|d|}{2}, \frac{|d|}{2}\right]$ ,
- if  $(\varepsilon, \delta) = (1, -1)$ , then  $r \in \left(\frac{-|d|}{2}, \frac{|d|}{2}\right]$ ,  $s \in \left[\frac{-|d|}{2}, \frac{|d|}{2}\right)$ .

We call the above constructed digit sets  $K$ -type digit sets. If we do not specify the value of  $(\varepsilon, \delta)$ , then we denote with  $E_\alpha$  an arbitrary element of the set  $\{E_\alpha^{(1,1)}, E_\alpha^{(1,-1)}, E_\alpha^{(-1,1)}, E_\alpha^{(-1,-1)}\}$ .

### 2.3. The construction of the $F$ -type digit sets

For an arbitrary  $\alpha \in I (\subseteq \mathbb{Q}(\sqrt{2}))$  let  $E_\alpha$  be a  $K$ -type digit set. We proved in [2] that if  $P \setminus \{0\}$  is not empty, then either  $G(P \setminus \{0\})$  is a disjoint union of loops, or a disjoint union of circles of length 2, which take the form:  
 $\pi \xrightarrow{f} (-\pi) \xrightarrow{-f} \pi$ .

We shall construct the appropriate  $F$ -type digit set  $A_\alpha$  by modifying  $E_\alpha$  as follows.

1) If  $(\alpha, E_\alpha)$  is a number system, then  $A_\alpha = E_\alpha$ .

2) If  $f \in E_\alpha$  and either

a)  $\pi \xrightarrow{f} \pi$  holds for some  $\pi \in \mathcal{P}$

or

b)  $\pi \xrightarrow{f} (\pi) \xrightarrow{-f} \pi$  holds for some  $\pi \in \mathcal{P}$ ,

then let  $f^* \in \mathcal{A}_\alpha$ , where  $f^*$  is one of  $f + \alpha$ ,  $f - \alpha$ , so that

$$|\overline{f^*}| = ||\overline{\alpha}| - |\overline{f}||$$

holds.

Let the other elements of  $E_\alpha$  belong to  $A_\alpha$ .

### 2.4. The formulation of our Theorem

**Theorem.** *Let  $\alpha \in \mathbb{Q}(\sqrt{2})$  be an arbitrary algebraic integer and*

$$\alpha \neq 0,$$

$$\alpha, 1 - \alpha \text{ not units and}$$

$$|\alpha|, |\overline{\alpha}| > \sqrt{2}$$

*hold. Then  $\alpha$  is a base of a number system.*

### 3. The proof of our Theorem

**Remark 1.** We saw in [3] that we can assume, without injuring the generality that  $|\alpha| > |\bar{\alpha}|$ . In case  $|\alpha| < |\bar{\alpha}|$  we can prove the same assertion in the same way.

**Remark 2.** In [4] we proved that if  $|\alpha|, |\bar{\alpha}| \geq 2$  and the conditions of our Theorem hold then  $(\alpha, E_\alpha)$  is a number system, where  $E_\alpha$  is the appropriate  $K$ -type digit set.

**Remark 3.** If  $|\alpha|, |\bar{\alpha}| < 2$  then  $1 - \alpha$  is a unit. Thus we have to prove our Theorem when the inequalities

$$\min(|\alpha|, |\bar{\alpha}|) < 2 \quad \text{and} \quad \max(|\alpha|, |\bar{\alpha}|) > 2$$

are simultaneously valid.

**Remark 4.** Relying on the data included in the appendix of [3], the validity of our Theorem can be easily verified for case  $\max(|\alpha|, |\bar{\alpha}|) \leq 6 + 3\sqrt{2}$ .

In line with the Remarks we have made so far, in the following we assume that

$$|\alpha| > 6 + 3\sqrt{2} \quad \text{and} \\ \sqrt{2} < |\bar{\alpha}| < 2$$

inequalities hold true.

In [3], [4] we proved some useful assertions assuming that  $1 + \alpha$  is not a unit. In order to use these results in this paper we present the following lemma:

**Lemma 1.** *If  $1 + \alpha$  is a unit in  $I$  then*

$$|\bar{\alpha}| < \sqrt{2}.$$

**Proof.** It is a well-known fact in number theory that a number  $\beta$  is a unit in an arbitrary quadratic field if and only if either  $\beta \cdot \bar{\beta} = 1$  or  $\beta \cdot \bar{\beta} = -1$ . Let us consider the following product. Assume that  $1 + \alpha$  is a unit. Then  $M = (1 + \alpha)(1 + \bar{\alpha}) = \pm 1$ , that is  $M = 1 + 2a + d$ , where naturally  $\alpha = a + b\sqrt{2}$  and  $\alpha \cdot \bar{\alpha} = d$ .

There are four cases:

1. If  $\alpha, \bar{\alpha} > 0$  then  $M = 1$ .
2. If  $\alpha < 0, \bar{\alpha} > 0$  then  $M = -1$ , that is  $2a + d = -2$ .

It is clear that both cases yield a contradiction.

3. If  $\alpha > 0$ ,  $\bar{\alpha} < 0$  then  $M = -1$ , that is  $a = -\frac{d}{2} - 1$ .

Since  $|\bar{\alpha}| = |b|\sqrt{2} - |a|$ , therefore  $|\alpha| = |a| + |b|\sqrt{2} = 2|a| + |\bar{\alpha}|$ . Thus  $|\alpha| = |\alpha| - 2 + |\bar{\alpha}|$ . We get from  $|d| = |\alpha||\bar{\alpha}|$  that

$$|\alpha| + 2 - |\bar{\alpha}| = |\alpha||\bar{\alpha}|, \quad \text{that is}$$

$$|\bar{\alpha}| = 1 + \frac{2 - |\bar{\alpha}|}{|\alpha|} < \sqrt{2}.$$

4. If  $\alpha, \bar{\alpha} < 0$  then  $M = 1$ , that is  $a = \frac{-d}{2}$ . We get that  $|\alpha| = |d| - |\bar{\alpha}|$  and from this

$$|\alpha| + |\bar{\alpha}| = |\alpha||\bar{\alpha}|.$$

Then

$$|\bar{\alpha}| = 1 + \frac{|\bar{\alpha}|}{|\alpha|} < \sqrt{2}$$

is true completing the proof of the Lemma 1. Thus, we can use the results of our previous papers, without losing the generality, because  $1 + \alpha$  never is unit.

The following assertions are related to the estimation of the modulus of the digits and their conjugates.

**Lemma 2.** *If  $f^* \in A_\alpha$  and  $f^* \notin E_\alpha$  then*

$$|f^*| < (3 - \sqrt{2})|\alpha| + 1$$

and

$$|\bar{f}^*| < |\bar{\alpha}| - 1.$$

**Proof.** If  $f^* \notin E_\alpha$ , then there exists a digit  $f$  in  $E_\alpha$  for which either  $f^* = f + \alpha$  or  $f^* = f - \alpha$  holds, thus

$$(3.1) \quad |f^*| \leq |f| + |\alpha|.$$

$f^* \notin E_\alpha$  implies further that there exists a  $\pi = p + q\sqrt{2}$  in  $(\alpha, E_\alpha)$ , for which either

$$(3.2) \quad f = \pi(1 - \alpha) \quad \text{or} \quad f = \pi(1 + \alpha)$$

is true. From this follows that

$$(3.3) \quad |\pi| \leq \frac{|f|}{|\alpha| - 1} \quad \text{and} \quad |\bar{\pi}| \leq \frac{|\bar{f}|}{|\bar{\alpha}| - 1}.$$

Let

$$S := \frac{|f|}{|\alpha| - 1} + \frac{|\bar{f}|}{|\bar{\alpha}| - 1}.$$

(3.3) implies that

$$|\pi - \bar{\pi}| = 2|q|\sqrt{2} \leq |\pi| + |\bar{\pi}| \leq S.$$

The construction of  $E_\alpha$  leads to

$$|f \cdot \bar{\alpha}| = |r + s\sqrt{2}|, \quad |\bar{f} \cdot \alpha| = |r - s\sqrt{2}| \quad \text{and} \quad |r|, |s| \leq \frac{|d|}{2}.$$

Now we can compute easily that

$$\text{if } \text{sgn}(r) = \text{sgn}(s) \text{ then } S \leq \frac{1 + \sqrt{2}}{2} \frac{|\alpha|}{|\alpha| - 1} + \frac{\sqrt{2}}{2} \frac{|\bar{\alpha}|}{|\bar{\alpha}| - 1} \quad \text{and,}$$

$$\text{if } \text{sgn}(r) \neq \text{sgn}(s) \text{ then } S \leq \frac{\sqrt{2}}{2} \frac{|\alpha|}{|\alpha| - 1} + \frac{\sqrt{2} + 1}{2} \frac{|\bar{\alpha}|}{|\bar{\alpha}| - 1}.$$

Since  $|q| \leq \frac{S}{2\sqrt{2}}$  we get that

$$|q| < 2.$$

In [3] we saw that  $|\pi| = |p + q\sqrt{2}| < 2$  and  $\pm 1 \notin \mathcal{P}$ , thus either

$$|\pi| = \sqrt{2} - 1 \quad \text{or} \quad |\pi| = 2 - \sqrt{2}$$

can be valid. Then, it follows from (3.1) and (3.2) that

$$|f^*| \leq \left(2 - \sqrt{2}\right) (|\alpha| + 1) + |\alpha| = \left(3 - \sqrt{2}\right) |\alpha| + 2 - \sqrt{2},$$

that is

$$|f^*| < \left(3 - \sqrt{2}\right) |\alpha| + 1$$

holds.

What can we say about  $|\bar{f}^*|$ ? We have to consider two cases. If  $|\pi| = \sqrt{2} - 1$  then  $|\bar{\pi}| = \sqrt{2} + 1$  and

$$|\bar{f}| = \left(1 + \sqrt{2}\right) (|\bar{\alpha}| - 1) > \left(1 + \sqrt{2}\right) \left(\sqrt{2} - 1\right) = 1.$$

If  $|\pi| = 2 - \sqrt{2}$  then  $|\bar{\pi}| = 2 + \sqrt{2}$  and

$$|\bar{f}| = (2 + \sqrt{2})(|\bar{\alpha}| - 1) > (2 + \sqrt{2})(\sqrt{2} - 1) = \sqrt{2}.$$

Thus we get that if  $|\bar{f}| > |\bar{\alpha}|$  then

$$|\bar{f}^*| = |\bar{f}| - |\bar{\alpha}| \geq \frac{1 + \sqrt{2}}{2}|\bar{\alpha}| - |\bar{\alpha}|,$$

that is

$$|\bar{f}^*| < \sqrt{2} - 1,$$

and if  $|\bar{f}| < |\bar{\alpha}|$  then

$$|\bar{f}^*| = |\bar{\alpha}| - |\bar{f}| < |\bar{\alpha}| - 1.$$

We can see that in both cases  $|\bar{f}^*| < |\bar{\alpha}| - 1$ , therefore we proved Lemma 2.

**Lemma 3.** *In  $(\alpha, A_\alpha)$  for an arbitrary  $\pi (= p + q\sqrt{2}) \in \mathcal{P}$*

$$|p| \leq 2 \quad \text{and} \quad |q| \leq 1$$

*hold.*

**Proof.** Let  $\pi_1$  be a periodic element of maximal modulus. There exists a transition

$$\pi = \pi_1\alpha + f^*$$

with suitable  $\pi \in \mathcal{P}$  and  $p^* \in A_\alpha$ .

Consequently, we can carry out the following deduction:

$$\begin{aligned} \pi_1\alpha &= \pi - f^*, \\ |\pi_1| &\leq \frac{|\pi| + |f^*|}{|\alpha|} \leq \frac{|\pi_1| + |f^*|}{|\alpha|}, \\ |\pi_1| \left(1 - \frac{1}{|\alpha|}\right) &\leq \frac{|f^*|}{|\alpha|}, \\ |\pi_1| &\leq \frac{|f^*|}{|\alpha| - 1}. \end{aligned}$$

In the same way we get that

$$|\bar{\pi}_1| \leq \frac{|\bar{f}^*|}{|\bar{\alpha}| - 1}.$$



In order to estimate the value of  $|p|$  and  $|q|$ , we use the method described in the proof of the Lemma 2. Thus let

$$S_{\max} = \max_{f^* \in A_\alpha} \left( \frac{|f^*|}{|\alpha| - 1} + \frac{|\overline{f^*}|}{|\overline{\alpha}| - 1} \right).$$

We know that

$$|\pi + \overline{\pi}| = 2|p| \leq S_{\max}$$

and

$$|\pi - \overline{\pi}| = 2|q\sqrt{2}| \leq S_{\max}.$$

If  $f^* \in E_\alpha$ , then we have already proved that  $|p| \leq 2$ ,  $|q| \leq 1$ , since  $|\overline{\alpha}| > \sqrt{2}$  and  $|\alpha| > 10$ .

If  $f^* \notin E_\alpha$ , then we get from Lemma 2 that

$$S_{\max} < \frac{(3 - \sqrt{2})|\alpha| + 1}{|\alpha| - 1} + \frac{|\overline{\alpha}| - 1}{|\overline{\alpha}| - 1} < 3.$$

The proof is completed.

The next assertion helps to describe the structure of  $G(\mathcal{P} \setminus \{0\})$ .

**Lemma 4.** In  $(\alpha, A_\alpha)$   $G(\mathcal{P} \setminus \{0\})$  does not contain loops, and circles of the form  $\pi \xrightarrow{f^*} (-\pi) \xrightarrow{-f^*} \pi$ , or in case  $\overline{\alpha} > 0$  a transition with form  $-\pi \xrightarrow{f^*} \pi$ .

**Proof.** Assume first that for some  $\pi \in \mathcal{P}$  and  $f^* \in A_\alpha$

$$\pi = \pi \cdot \alpha + f^*$$

holds, i.e. there exists a loop in  $G(\mathcal{P} \setminus \{0\})$ . Thus we get

$$f^* = \pi(1 - \alpha).$$

If  $\overline{\alpha} > 0$  then  $1 - \alpha \mid f^*$ , which implies that either

$$1 - \alpha \mid f + \alpha \quad \text{or} \quad 1 - \alpha \mid f - \alpha$$

for the appropriate  $f \in E_\alpha$ . Let us observe that this is a contradiction because in this case  $1 - \alpha$  is a unit.

If  $\overline{\alpha} < 0$ , then the inequality

$$|\overline{f^*}| = |\overline{\pi}|(|\overline{\alpha}| + 1) \geq \sqrt{2}(|\overline{\alpha}| + 1)$$

is true, which yields a contradiction, since  $|\overline{f^*}| < |\overline{\alpha}| - 1$ . Thus we have proved that there do not exist a loop.

Now let us assume that the equation

$$-\pi = \pi \cdot \alpha + f^*$$

holds for some  $f^* \in A_\alpha$ , and also  $-f^* \in A_\alpha$ . It is obvious that

if  $\overline{\alpha} > 0$ , then  $|f^*| = |\overline{\pi}|(|\overline{\alpha}| + 1)$ , and

if  $\overline{\alpha} < 0$ , then  $1 + \alpha$  is a unit, which can be proved in the above mentioned way.

Although we have proved Lemma 4, in case  $\overline{\alpha} < 0$  there may exist transition with form  $-\pi \xrightarrow{f^*} \pi$ .

Summing up the results we have reached so far, and taking into account that  $\pm 1 \notin \mathcal{P}$ , we can include the possible elements of  $\mathcal{P}$  in this table:

(3.4)

$ p $	$ q $	$ \pi $	$ \overline{\pi} $
0	1	$\sqrt{2}$	$\sqrt{2}$
1	1	$\sqrt{2} - 1$	$\sqrt{2} + 1$
2	0	2	2
2	1	$2 - \sqrt{2}$	$2 + \sqrt{2}$

As a matter of fact our last assertion completes the proof of the Theorem, namely:

**Lemma 5.**  $\mathcal{P} = \{0\}$ .

**Proof.** Let us observe that  $\pm 2 \notin \mathcal{P}$ , otherwise the equation

$$|\pi_1| = 2|\alpha| - |f^*|$$

would be valid, but this means that

$$|\pi_1| > 2|\alpha| - (3 - \sqrt{2})|\alpha| - 1,$$

and we would get that

$$|\pi_1| > (\sqrt{2} - 1)|\alpha| - 1 > 2,$$

which is a contradiction.

Now let us assume that  $\pi_1 \in \mathcal{P}$  such that  $|\pi_1| = \sqrt{2}$ . According to Lemma 4 it must be a transition

$$\pi_1 = \pi \cdot \alpha + f^*,$$

where  $|\pi| \neq \sqrt{2}$ , that is  $|\pi| \geq \sqrt{2} + 1$ . Then we get that the inequalities

$$\frac{\sqrt{2} + 1}{2} |\bar{\alpha}| \geq |\bar{f}^*| \geq (\sqrt{2} + 1) |\bar{\alpha}| - \sqrt{2}$$

hold and these imply that

$$\sqrt{2} \geq \frac{\sqrt{2} + 1}{2} |\bar{\alpha}| > \frac{\sqrt{2} + 1}{2} \sqrt{2},$$

which is impossible, thus  $\pm\sqrt{2} \notin \mathcal{P}$ .

Now, we only have to concentrate on the 2nd and 4th rows of the table (3.4). In accordance with Lemma 4 there must exist a transition

$$\pi_1 = \pi \cdot \alpha + f^*,$$

where  $|\bar{\pi}_1| = 1 + \sqrt{2}$  and  $|\bar{\pi}| = 2 + \sqrt{2}$ , but then

$$\frac{\sqrt{2} + 1}{2} |\bar{\alpha}| \geq |\bar{f}^*| = (2 + \sqrt{2}) |\bar{\alpha}| - \sqrt{2} - 1$$

is true and we get from this that

$$1 + \sqrt{2} \geq \frac{3 + \sqrt{2}}{2} |\bar{\alpha}| > \frac{3 + \sqrt{2}}{2} \sqrt{2}.$$

Obviously, this is a contradiction, therefore  $\mathcal{P} = \{0\}$ .

This concludes the proof of Lemma 5 as well as that of our Theorem.

## References

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