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A MATKOWSKI–SUTÔ–TYPE PROBLEM FOR WEIGHTED QUASI–ARITHMETIC MEANS

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Dedicated to the 60th birthday of
Professor Karl-Heinz Indlekofer

Abstract. The aim of this paper is to solve the functional equation

$$\lambda\varphi^{-1}\left(\lambda\varphi(x)+(1-\lambda)\varphi(y)\right)+(1-\lambda)\psi^{-1}\left(\lambda\psi(x)+(1-\lambda)\psi(y)\right)= \\ =\lambda x+(1-\lambda)y,$$

where φ, ψ are sctrictly monotone continuous real functions defined on an open real interval I and $\lambda \in]0,1[$ is a fixed number. The case $\lambda = \frac{1}{2}$ has recently been completely solved by the authors in [6]. The main result of the paper offers a complete solution for the case $\lambda \neq \frac{1}{2}$ and it states that if $\lambda \neq \frac{1}{2}$ then φ, ψ are solutions of the above equation if and only if there exist constants a, b, c, d with $ac \neq 0$ such that $\varphi(x) = ax + b$ and $\psi(x) = cx + d$ for all $x \in I$.

1. Introduction

Let $I \subset \mathbb{R}$ be a nonvoid open interval. A function $M : I^2 \to I$ is called a *strict mean* on I if it is continuous and $\min\{x, y\} < M(x, y) < \max\{x, y\}$ for

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all $x, y \in I,\ x \neq y$. Let $M_i : I^2 \to I$ $(i = 1, 2)$ be strict means. For any fixed $x, y \in I$, we define the Gauss-iteration in the following way

$$x_1 := x, \qquad\qquad y_1 := y,$$
$$x_{n+1} := M_1(x_n, y_n), \quad y_{n+1} := M_2(x_n, y_n) \quad (n \in \mathbb{N}).$$

It is known ([1],[6]) that, for any $x, y \in I$, the limit $M_3(x, y) = \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$ exists, and $M_1 \otimes M_2 := M_3 : I^2 \to I$ is a strict mean on I called the *Gauss-composition* of M_1 and M_2.

Denote by $\mathcal{CM}(I)$ the class of continuous and strictly monotone functions defined on the interval I. A function $M : I^2 \to I$ is called a *weighted quasi-arithmetic mean* on I if there exist $0 < \lambda < 1$ and $\varphi \in \mathcal{CM}(I)$ such that

$$M(x, y) = \varphi^{-1}(\lambda\varphi(x) + (1 - \lambda)\varphi(y)) =: A_\varphi(x, y; \lambda) \tag{1.1}$$

for all $x, y \in I$ (see [8], [15], [6]). The number λ in (1.1) is called the *weight* and the function φ is said to be the *generating function*. Let $0 < \lambda < 1$ be a fixed number and M_i $(i = 1, 2, 3)$ be *weighted quasi-arithmetic means* on I with the same weight λ. Our main concern is to find conditions so that

$$M_3 = M_1 \otimes M_2 \tag{1.2}$$

be an identity on I^2. In the particular case $\lambda = \frac{1}{2}$ we have recently determined all the solutions in full generality in [6].

In order to solve the problem (1.2), we need to study the functional equation

$$\lambda A_\varphi(x, y : \lambda) + (1 - \lambda)A_\psi(x, y; \lambda) = \lambda x + (1 - \lambda)y \quad (x, y \in I), \tag{1.3}$$

where $\varphi, \psi \in \mathcal{CM}(I)$ are unknown functions. The case $\lambda = \frac{1}{2}$ is called the Matkowski-Sutô problem (cf. [16], [17], [11], [3], [4], [5]). When $\lambda \neq \frac{1}{2}$, the continuously differentiable solutions of (1.3) were determined in [7].

Our approach is analogous to that of followed in [6]. First we prove certain regularity properties of the functions satisfying (1.3). Based on this and also applying the extension theorem known from [2], we then obtain the complete solution of the problem described above.

2. The locally Lipschitz property of solutions

Let $0 < \lambda < 1$ and let $\varphi, \psi \in \mathcal{CM}(I)$ be solutions for (1.3). Our aim is to prove that $\varphi, \varphi^{-1}, \psi, \psi^{-1}$ are locally Lipschitz functions on their domains.

Definition 2.1. Let $J \subset \mathbb{R}$ be a nonvoid open interval and $f : J \to \mathbb{R}$. We say that the function f is *locally Lipschitz* in J if, for any $u_0 \in J$, there exist constants $\delta > 0$ and $L > 0$ such that $U :=]u_0 - \delta, u_0 + \delta[\subset J$ and, for all $u, v \in U$,

$$|f(u) - f(v)| \leq L|u - v|.$$

The following theorem plays an important role in our investigations.

Theorem 2.2. *Let $f : J \to \mathbb{R}$ ($J \subset \mathbb{R}$ is a nonvoid open interval) be a strictly monotone increasing and continuous function such that, for all $v \in J$, the map*

$$u \longmapsto f(u) - f(\lambda u + (1-\lambda)v) \quad (u \in J)$$

is strictly monotone increasing. Then f and its inverse f^{-1} are Lipschitz functions on their domains J and $f(J)$, respectively.

Proof. For the case $\lambda = \dfrac{1}{2}$, the proof can be found in [6]. In the more general case $0 < \lambda < 1$ (including also the case $\lambda = \dfrac{1}{2}$) the result follows from a more general result stated in [14, Theorem 3].

Theorem 2.3. *Let $0 < \lambda < 1$ and $\varphi, \psi \in \mathcal{CM}(I)$ be solutions for the functional equation (1.3). Then $\varphi, \varphi^{-1}, \psi, \psi^{-1}$ are locally Lipschitz functions on their domains.*

Proof. It is sufficient to prove the statement for the functions φ, φ^{-1} because the role of functions φ and ψ can be interchanged. Applying (1.3) with the substitutions $u = \varphi(x),\ v = \varphi(y)$ $(u, v \in J := \varphi(I))$ we deduce the equation

$$\begin{aligned}(2.1)\qquad (1-\lambda)\psi^{-1}(\lambda\psi \circ \varphi^{-1}(u) + (1-\lambda)\psi \circ \varphi^{-1}(v)) = \\ = \lambda\varphi^{-1}(u) + (1-\lambda)\varphi^{-1}(v) - \lambda\varphi^{-1}(\lambda u + (1-\lambda)v)\end{aligned}$$

for all $u, v \in J$. Without loss of generality, we can assume that φ and ψ are strictly *increasing* functions. Then, for each fixed v, the left hand side of (2.1) is strictly increasing function of u, which results that the right hand side of (2.1) should also be strictly increasing in u. Therefore, for each fixed $v \in J$,

$$u \longmapsto \varphi^{-1}(u) - \varphi^{-1}(\lambda u + (1-\lambda)v) \quad (u \in J)$$

is a strictly increasing function. Hence, in virtue of Theorem 2.2, φ^{-1} and φ are locally Lipschitz functions on J and on $I = \varphi^{-1}(J)$, respectively.

Corollary 2.4. *If $\varphi, \psi \in \mathcal{CM}(I)$ are solutions of (1.3) and φ (or ψ) is differentiable at a point $x_0 \in I$, then $\varphi'(x_0) \neq 0$ (or $\psi'(x_0) \neq 0$).*

Proof. See Corollary 4.4 of Theorem 4.3 in [6].

3. Differentiability of solutions

Suppose that $\varphi, \psi \in \mathcal{CM}(I)$ are *increasing* functions satisfying equation (1.3). Let

$$x := \varphi^{-1}(t + (1-\lambda)s), \quad y := \varphi^{-1}(t - \lambda s),$$

where $t \in J := \varphi(I)$ and

$$s \in \left(\frac{J-t}{1-\lambda}\right) \bigcap \left(\frac{t-J}{\lambda}\right) := J_{t,\lambda}$$

are arbitrary elements. Then equation (1.3) yields that, for any $t \in J := \varphi(I)$ and for any $s \in J_{t,\lambda}$,

$$\begin{aligned} \lambda\varphi^{-1}(t) = \lambda\varphi^{-1}(t + (1-\lambda)s) + (1-\lambda)\varphi^{-1}(t - \lambda s) - \\ - (1-\lambda)\psi^{-1}\left[\lambda h(t + (1-\lambda)s) + (1-\lambda)h(t-\lambda s)\right], \end{aligned} \tag{3.1}$$

where $h := \psi \circ \varphi^{-1}$.

Definition 3.1. Let $f : J \to \mathbb{R}$ be an arbitrary function and $0 < \lambda < 1$ be fixed. An element $t \in J$ is said to be a *point of λ-symmetry* for f, in notation $t \in \sigma_\lambda(f)$, if the identity

$$\lambda f(t + (1-\lambda)s) + (1-\lambda)f(t - \lambda s) = f(t) \tag{3.2}$$

holds true for all $s \in J_{t,\lambda}$.

Lemma 3.2. *If $F : J \to \mathbb{R}$ is a continuous function then $\sigma_\lambda(F)$ is closed in J.*

Proof. The proof of this obvious statement is analogous to that of [6, Lemma 4.6] concerning the case $\lambda = \dfrac{1}{2}$.

Lemma 3.3. *Let $0 < \lambda < 1$ and $\varphi, \psi \in \mathcal{CM}(I)$ be solutions of (1.3). Then $\sigma_\lambda(h) = \sigma_\lambda(\varphi^{-1})$, where $h := \psi \circ \varphi^{-1}$.*

Proof. We have that, for any $t \in J := \varphi(I)$ and $s \in J_{t,\lambda}$, (3.1) holds.

If $t \in \sigma_\lambda(h)$ then, by (3.1), we obtain

$$\lambda\varphi^{-1}(t) = \lambda\varphi^{-1}(t + (1-\lambda)s) + (1-\lambda)\varphi^{-1}(t - \lambda s) - (1-\lambda)\psi^{-1} \circ h(t),$$

and since $\psi^{-1} \circ h(t) = \varphi^{-1}(t)$ holds, it follows that $t \in \sigma_\lambda(\varphi^{-1})$.

Conversely, if $t \in \sigma_\lambda(\varphi^{-1})$, then, by (3.1),

$$\varphi^{-1}(t) = \psi^{-1}(\lambda h(t + (1 - \lambda s)) + (1-\lambda)h(t - \lambda s)),$$

whence we obtain that $t \in \sigma_\lambda(h)$.

Theorem 3.4. *If $0 < \lambda < 1$ and $\varphi, \psi \in \mathcal{CM}(I)$ are solutions of (1.3) then φ^{-1} is differentiable at any point $t_0 \in J \backslash \sigma_\lambda(\varphi^{-1})$.*

Proof. Without loss of generality we can assume that φ and ψ are *increasing* functions. If $J\backslash\sigma_\lambda(\varphi^{-1}) \neq \emptyset$ then let $t_0 \in J\backslash\sigma_\lambda(\varphi^{-1})$ be arbitrary. For an arbitrary function $g : J_{t_0,\lambda} \to \mathbb{R}$ denote by N_g the set of points $s \in J_{t_0,\lambda}$ at which g is not *differentiable.* Define the following functions

$$\begin{aligned}
g_1(s) &:= \varphi^{-1}(t_0 + (1-\lambda)s),\\
g_2(s) &:= \varphi^{-1}(t_0 - \lambda s),\\
g_3(s) &:= h(t_0 + (1-\lambda)s),\\
g_4(s) &:= h(t_0 - \lambda s)
\end{aligned}$$

for all values $s \in J_{t_0,\lambda}$. Since φ^{-1} and h are strictly monotone functions, therefore, by Lebesgue's theorem on the almost everywhere differentiability of monotone functions, each N_{g_i} $(i = 1, 2, 3, 4)$ is a null set, that is, the set

$$N := \bigcup_{i=1}^{4} N_{g_i} \subset J_{t_0,\lambda}$$

is of measure zero. Since $t_0 \notin \sigma_\lambda(\varphi^{-1})$ thus, by Lemma 3.3, $t_0 \notin \sigma_\lambda(h)$. Therefore, the function

$$h_{t_0}(s) := \lambda h(t_0 + (1-\lambda)s) + (1-\lambda)h(t_0 - \lambda s) \quad (s \in J_{t_0,\lambda})$$

is not constant, which yields that its image is a *proper interval* $H_0 := h_{t_0}(J_{t_0,\lambda})$.

Let the set C be defined in the following way:

$$C := \{u \in H_0 \mid \psi^{-1} \quad \text{is not differentiable at} \quad u\}.$$

Then, by Lebesgue's theorem, C is a null set. Therefore $H_0 \backslash C$ is a set of *positive* measure. Define D as follows:

$$D := h_{t_0}^{-1}(H_0 \backslash C) \subseteq J_{t_0,\lambda}.$$

Then $h_{t_0}(D) = H_0 \backslash C$. If D were a null set, then $h_{t_0}(D)$ would also be a null set since by Theorem 2.2, h_{t_0} is a locally Lipschitz function. Therefore D is a set of positive measure in $J_{t_0,\lambda}$. This implies that $D \backslash N$ is also a set of positive measure, hence $D \backslash N$ is not empty. Let $s_0 \in D \backslash N$ be arbitrarily fixed. Then g_i is differentiable at s_0 $(i = 1, 2, 3, 4)$ and ψ^{-1} is differentiable at $h_{t_0}(s_0)$. Then, by (3.1), the equation

$$\begin{aligned}\lambda\varphi^{-1}(t) = \lambda\varphi^{-1}(t + (1-\lambda)s_0) + (1-\lambda)\varphi^{-1}(t - \lambda s_0) -\\ - (1-\lambda)\psi^{-1}(\lambda h(t + (1-\lambda)s_0) + (1-\lambda)h(t - \lambda s_0))\end{aligned} \tag{3.3}$$

holds for all $t \in J$ such that $s_0 \in J_{t,\lambda}$ is also valid. This latter set of values of t is an open interval containing t_0. Thus, ψ^{-1} is differentiable at $(t_0 + (1-\lambda)s_0)$ and at $(t_0 - \lambda s_0)$; h is differentiable at $(t_0 + (1-\lambda)s_0)$ and at $(t_0 - \lambda s_0)$, and ψ^{-1} is differentiable at $h_{t_0}(s_0)$, therefore, by the chain rule, the expression on the right side of (3.2) is differentiable at t_0. Thus, we obtain that φ^{-1} is differentiable at t_0.

Applying the previous result, we obtain the following important regularity theorem for the solutions of (1.3).

Theorem 3.5. *If $0 < \lambda < 1$ and $\varphi, \psi \in \mathcal{CM}(I)$ are solutions of (1.3), then there exists a nonvoid open interval $K \subset I$ on which φ and ψ are differentiable and $\varphi'(x) \neq 0$, $\psi'(x) \neq 0$ for all $x \in K$.*

Proof. Consider the function $\varphi^{-1} : J \to I$, where $J := \varphi(I)$. Then there are two possible cases:

(i) either $\sigma_\lambda(\varphi^{-1}) = J$, that is, every $t \in J$ is a point of λ-symmetry for φ^{-1};

(ii) or $\sigma_\lambda(\varphi^{-1}) \neq J$, that is, φ^{-1} has a point of non-λ-symmetry in J.

In case (i), for all $t \in J$ and $s \in J_{t,\lambda}$,

$$\varphi^{-1}(t) = \lambda\varphi^{-1}(t + (1-\lambda)s) + (1-\lambda)\varphi^{-1}(t - \lambda s)$$

holds. Since φ^{-1} is continuous, we have $\varphi^{-1}(u) = Au + B$ (where $A \neq 0$ and B are constants) for $u \in J$. This implies that $A_\varphi(x, y; \lambda) = \lambda x + (1-\lambda)y$ for all $x, y \in I$, hence, by (1.3), $A_\psi(x, y; \lambda) = \lambda x + (1-\lambda)y$ holds for all $x, y \in I$. Thus, ψ is also an affine function, therefore φ and ψ are differentiable functions with non-vanishing derivatives.

In case (ii), there exists $t_0 \notin \sigma_\lambda(\varphi^{-1})$. With the notation $G := \{t \in J \mid t \notin \notin \sigma_\lambda(\varphi^{-1})\}$, due to Lemma 3.2, we have that G is a *nonvoid open set*. Thus, by Theorem 3.4, there exists a nonvoid open interval $\Delta \subset G \subset J$ such that φ^{-1} is differentiable on Δ. Hence φ is differentiable on some nonvoid open interval $K_0 \subset I$ and Corollary 2.4 implies that $\varphi'(x) \neq 0$ if $x \in K_0$. Now let us restrict equation (1.3) to the interval K_0. Then the role of the functions φ and ψ can be interchanged and, by a similar argument, we obtain that there exists a nonvoid open interval $K \subset K_0 \subset I$ on which ψ is differentiable and $\psi'(x) \neq 0$ if $x \in K$. This completes the proof of the existence of the desired subinterval.

4. Continuous differentiability of solutions

If $\varphi, \psi \in \mathcal{CM}(I)$ are differentiable solutions of (1.3) with non-vanishing derivatives then (since the functions φ^{-1} and ψ^{-1} have the Darboux's property) we can assume that $\varphi'(x) > 0$ and $\psi'(x) > 0$ for every $x \in I$ without loss of generality.

Lemma 4.1. *If $0 < \lambda < 1$ is a fixed number and $\varphi, \psi \in \mathcal{CM}(I)$ are solutions of (1.3), moreover, φ and ψ are differentiable functions on I and $\varphi'(x) > 0$, $\psi'(x) > 0$ if $x \in I$ then, with the notation*

$$J := \varphi(I), \quad f := \varphi' \circ \varphi^{-1}, \quad g := \psi' \circ \varphi^{-1}, \tag{4.1}$$

the functions $f, g : J \to \mathbb{R}_+$ satisfy the functional equation

$$f(\lambda u + (1-\lambda)v)(g(v) - g(u)) = \lambda(f(u)g(v) - f(v)g(u)) \tag{4.2}$$

for all $u, v \in J$.

Proof. Let us differentiate the functional equation (1.3) first with respect to x and then with respect to y. The conditions of the lemma ensure the differentiability, and we get the equations

$$\lambda \frac{\lambda\varphi'(x)}{\varphi'(A_\varphi(x,y;\lambda))} + (1-\lambda)\frac{\lambda\psi'(x)}{\psi'(A_\psi(x,y;\lambda))} = \lambda$$

and

$$\lambda \frac{(1-\lambda)\varphi'(y)}{\varphi'(A_\varphi(x,y;\lambda))} + (1-\lambda)\frac{(1-\lambda)\psi'(y)}{\psi'(A_\psi(x,y;\lambda))} = 1-\lambda$$

for all $x, y \in I$. Multiplying the first equation by $(1-\lambda)\psi'(y)$, the second by $\lambda\psi'(x)$, and subtracting the new equations from each other, we obtain

$$\frac{\lambda(\varphi'(x)\psi'(y) - \varphi'(y)\psi(x))}{\varphi'[A_\varphi(x,y;\lambda)]} = \psi'(y) - \psi'(x)$$

for all $x, y \in I$. Let $u = \varphi(x)$, $v = \varphi(y)$ $(u, v \in J := \varphi(I))$ be arbitrary then with the notations of (4.1), we obtain equation (4.2).

Definition 4.2. We say that $h : J \to \mathbb{R}_+$ is an element of the set $\mathcal{D}(J)$ if $h = d \circ c$, where $c \in \mathcal{CM}(J)$ and $d : I := c(J) \to \mathbb{R}_+$ is a derivative function, that is there exists a differentiable function $D : I \to \mathbb{R}_+$, such that $D'(x) = d(x)$ for all $x \in I$.

According to the previous definition, the functions f and g involved in the functional equation (4.2) are elements of the set $\mathcal{D}(J)$.

Theorem 4.3. *If the functions $f, g \in \mathcal{D}(J)$ satisfy the functional equation (4.2) for all $u, v \in J$ (where $0 < \lambda < 1$ is fixed), then there exists a nonvoid open interval $J_0 \subset J$ on which f is continuous.*

Proof. (i) If there exists a nonvoid open interval $J_0 \subset J$ on which f is continuous then the statement is true. If there exists a nonvoid open interval $J_0 \subset J$ on which g is constant then let $g(t) =: k$ for $t \in J_0$. Substituting arbitrary values $u, v \in J_0 (\subset J)$ into (4.2), we get that $f(u)k - f(v)k = 0$ for all $u, v \in J_0$. Hence, f must be constant on J_0 and consequently, f is continuous on J_0.

Therefore, we may assume that $f, g \in \mathcal{D}(J)$ and that f and g are not constants on *any* nonvoid open subinterval $J_0 \subset J$. Denote by $\mathcal{D}_0(J)$ the set of functions in $\mathcal{D}(J)$ which are not constant on any proper subinterval of J.

(ii) Suppose that $f, g \in \mathcal{D}_0(J)$ satisfy (4.2) for all $y, v \in J$. Define the set $C(g)$ by

$$C(g) := \{t \mid t \in J,\ g \text{ is continuous at } t\}.$$

Then $g = d \circ c$, where c is continuous and strictly monotone, d is a derivative function; therefore g is continuous at each point $t \in J$ for which d is continuous at the point $c(t)$. Since the derivative function d is of Baire class 0 or 1, thus, according to Baire's theorem ([12], [13], [10]), the set of all points at which d is continuous is a *dense set of type* G_δ in $c(J)$, whence, because c is continuous and strictly monotone, $C(g)$ is also a dense G_δ set in J.

Now we will show that there exist points $u_0, v_0 \in C(g)$ such that $g(u_0) \neq$ $\neq g(v_0)$. Contrary to our assumption, suppose that $g(t) = k$ for every $t \in C(g)$, where $k > 0$ is a constant. Then substituting the values $u, v \in C(g)$ into (4.2), we obtain

$$f(u)k - f(v)k = 0,$$

whence $f(t) = l$ follows for every $t \in C(g)$, where $l > 0$ is a constant.

Because of the property of the set $C(g)$, for all $u \in J$, there exists $v \in C(g)$ such that $\lambda u + (1-\lambda)v \in C(g)$. Thus, by (4.2),

$$l[k - g(u)] = \lambda(f(u)k - lg(u))$$

for every $u \in J$. This implies

$$f(u) = \frac{l(\lambda-1)g(u) + lk}{\lambda k} \quad \text{if } u \in J.$$

If we substitute this form of the function f back into equation (4.2), after some calculations, we get

$$(k - g(\lambda u + (1-\lambda)v))(g(v) - g(u)) = 0 \tag{4.3}$$

for all $u, v \in J$.

Now let $v_0 \in J$ be fixed such that $c := g(v_0) \neq k$ holds. (Such a v_0 exists since g is non-constant.)

On the other hand, for any $t \in J$ and for any $\varepsilon > 0$ satisfying $]t-\varepsilon, t+\varepsilon[\subset J$, there exists $u \in]t-\varepsilon, t+\varepsilon[\subset J$ such that

$$g(\lambda u + (1-\lambda)v_0) \neq k.$$

This last statement is valid because g is non-constant on any proper subinterval. Thus, by (4.3) it is obvious that $g(u) = g(v_0) = c$ holds. So in any neighborhood of *any* point $t \in J$ there exists u such that $g(u) = c$ and there exists s such that $g(s) = k \neq c$ which yields that g is not continuous anywhere and it is a contradiction.

(iii) We have proved in the previous part (ii) that there exist points $u_0, v_0 \in$ $\in C(g)$ such that

$$g(u_0) \neq g(v_0)$$

holds. Then there exist a neighborhood $U \subset J$ of u_0 and a neighborhood $V \subset J$ of v_0 such that for any $u \in U$ and $v \in V$ we have $g(u) \neq g(v)$. Hence, by (4.2),

$$f(\lambda u + (1-\lambda)v) = \lambda \frac{f(u)g(v) - f(v)g(u)}{g(v) - g(u)}$$

follows for all $u \in U$ and $v \in V$. This implies

$$f(t) = \lambda \frac{f\left(\frac{t-(1-\lambda)v}{\lambda}\right) g(v) - f(v) g\left(\frac{t-(1-\lambda)v}{\lambda}\right)}{g(v) - g\left(\frac{t-(1-\lambda)v)}{\lambda}\right)} \tag{4.4}$$

for every pair of values

$$(t,v) \in S := \{(t,v) \mid v \in V,\ t \in \lambda U + (1-\lambda)v\},$$

where $t_0 := \lambda u_0 + (1-\lambda)v_0 \in J$. By (4.4) and in view of Járai's theorem ([9, Theorem 3.3]), we obtain that f is *continuous* in a neighborhood of the point $t_0 \in J$, that is, there exists a nonvoid open interval $t_0 \in J_0 \subset J$ on which f is continuous.

Finally we can state the following regularity theorem.

Theorem 4.4. *Let $0 < \lambda < 1$ and $\varphi, \psi \in \mathcal{CM}(I)$ be solutions of the functional equation (1.3). Then there exists a nonvoid open interval $K \subset I$ such that φ, ψ are continuously differentiable on K and $\varphi'(x) \neq 0,\ \psi'(x) \neq 0$ if $x \in K$.*

Proof. In virtue of Theorem 3.5, there exists a nonvoid open interval $K_1 \subset I$ on which φ and ψ are differentiable with non-vanishing derivatives. We can assume that $\varphi'(x) > 0$ and $\psi'(x) > 0$ if $x \in K_1$. Then by Lemma 4.1, with the notation of (4.1), we obtain that (4.2) holds, where $f, g \in \mathcal{D}(K_1)$. Thus, by Theorem 4.3, we obtain that there exists a nonvoid open interval $J_0 \subset J$ on which f is *continuous*. It means that $f := \varphi' \circ \psi^{-1} : J \to \mathbb{R}_+$ is continuous in J_0. Consequently, $K_2 := := \varphi^{-1}(J_0) \subset K_1 \subset I$ is a nonvoid open interval and

$$\varphi'(x) = \varphi' \circ \varphi^{-1}(s) = f(s) = f \circ \varphi(x)$$

for all $x \in K_2$. Hene φ' is *continuous* on the nonvoid open interval $K_2 \subset I$.

It is obvious that $\varphi, \psi \in \mathcal{CM}(K_2)$ and φ, ψ satisfy the functional equation (1.3) in K_2, where φ is continuously differentiable on K_2 and $\varphi'(x) > 0$ if $x \in K_2$. Now apply our previous results for ψ. Then there exists a nonvoid open interval $K \subset K_2$ such that ψ is continuously differentiable on K and $\psi'(x) > 0$ if $x \in K$. Thus the statement of the theorem holds on the interval K.

5. Solution for the problem

The case $\lambda = \dfrac{1}{2}$ (which is the original Matkowski-Sutô problem) was treated and completely solved in [6]. Therefore, it remains to consider the case $0 < \lambda < 1$ and $\lambda \neq \dfrac{1}{2}$ only. Then the following statement holds.

Theorem 5.1. *Let* $0 < \lambda < 1$ *and* $\lambda \neq \frac{1}{2}$. *If* $\varphi, \psi \in \mathcal{CM}(I)$ *are solutions of the functional equation (1.3) then there exist constants* $a, b, c, d \in \mathbb{R}$, $ac \neq 0$ *such that*

$$\varphi(x) = ax + b \quad \text{and} \quad \psi(x) = cx + d \tag{5.1}$$

for all $x \in I$. *Then*

$$A_\varphi(x, y; \lambda) = A_\psi(x, y; \lambda) = \lambda x + (1 - \lambda)y \tag{5.2}$$

for all $x, y \in I$.

Proof. In virtue of the Theorem 4.4, there exists a nonvoid open interval $K \subset I$ such that φ and ψ are continuously differentiable on K and their derivatives do not vanish. Then, according to [7], (5.1) holds in K. Due to the extension theorem of the paper [2], (5.1) holds for all $x \in I$. This immediately yields that (5.2) is also true.

Now we are going to examine the solution of the general Matkowski-Sutô problem stated in the introduction for the class of weighted quasi-arithmetic means.

Theorem 5.2. *If* $M_i : I^2 \to I$ $(i = 1, 2, 3)$ *are weighted quasi-arithmetic means with some weight* λ $\left(0 < \lambda < 1; \lambda \neq \frac{1}{2}\right)$ *on* I, *then the identity* $M_3 = M_1 \otimes M_2$ *holds on* I^2 *if and only if there exists* $f \in \mathcal{CM}(I)$ *such that* $M_i(x, y) = A_f(x, y; \lambda)$ $(i = 1, 2, 3)$ *holds for all* $x, y \in I$.

Proof. There exist generating functions $f_1, f_2, f_3 \in \mathcal{CM}(I)$ such that the invariance equation (c.f. [6], [1])

$$A_{f_3}(x, y; \lambda) = A_{f_3}(Af_1(x, y; \lambda), A_{f_2}(x, y; \lambda); \lambda) \tag{5.3}$$

holds for all $x, y \in I$. Thus with the notations $u := f_3(x)$, $v = f_3(y)$, $(u, v \in f_3(I) =: J)$, $\varphi := f_1 \circ f_3^{-1}$, $\psi := f_2 \circ f_3^{-1}$, (5.3) holds if and only if $\varphi, \psi \in \mathcal{CM}(J)$ satisfy the functional equation (1.3) for all $u, v \in J$. Then, by Theorem 5.1, we get that

$$\varphi(u) = au + b, \quad \psi(u) = cu + d \quad (ac \neq 0)$$

for all $u \in J$. Whence, with the notation $f := f_3$, $f \in \mathcal{CM}(I)$, and since $\varphi = f_1 \circ f_3^{-1} = f_1 \circ f^{-1}$, $\psi = f_2 \circ f_3^{-1} = f_2 \circ f^{-1}$, we have $f_1 = \varphi \circ f, f_2 = \psi \circ f$, where φ and ψ are given in (5.1). Thus, $M_i(x, y; \lambda) = A_f(x, y; \lambda)$ is valid for all $x, y \in I$ and $i = 1, 2, 3$.

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