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BETA DISTRIBUTION IN THE POLYNOMIAL SEMIGROUP

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This paper is dedicated to Professor K.-H. Indlekofer

Abstract. We consider the asymptotical behaviour of arithmetic processes defined in the polynomial semigroup.

Introduction

The sets of natural, integer, real and complex numbers we denote by $\mathcal{N}, \mathcal{Z}, \mathcal{R}, \mathcal{C}$, respectively. The cardinality of a finite set $\mathcal{A}$ is denoted by $|\mathcal{A}|$.

Functional limit distributions related to arithmetical functions, which are defined in terms of the natural divisor functions were considered in [3] and [6].

The following sequence of the arithmetical processes was investigated in [3]. Let $\tau(m,v)$ be the number of natural divisors of $m \in \mathcal{N}$ which do not exceed v, $v \geq 1$, and $\tau(m) = \tau(m,m)$. In the mentioned paper was obtained that

$$\frac{1}{n}\sum_{m=1}^{n}\frac{\tau(m,n^t)}{\tau(m)} = \frac{2}{\pi}\arcsin\sqrt{t} + o(1)$$

uniformly in $t \in [0,1]$, as $n \to \infty$. This result can be found in [7, p. 207], too. Let $f(d)$ be a nonnegative multiplicative function satisfying the conditions: $f(p) = \xi > 0$ and $f(p^k) \geq 0$, here p being the prime number. Put

$$F(m,v) = \sum_{d|m, d\leq v} f(d), \quad F(m,m) = F(m),$$

here $m, d \in \mathcal{N}$. Furthermore, for $t \in [0,1],\ m, n \in \mathcal{N}$ define

$$X_n := X_n(m,t) = \frac{F(m, n^t)}{F(m)} \in D[0,1],$$

here $D[0,1]$ is the space of real-valued functions on $[0,1]$ which are right-continuos and have left-hand limits. In this space the Skorokhod topology is introduced, $\mathcal{D}$ is the Borel σ-algebra.

In [6] the following assertion was proved.

The sequence $\{X_n\}$ converges weakly to a limit process defined on $\mathcal{D}$, as $n \to \infty$.

Functional limit distributions related to multiplicative functions, which are defined in the polynomial semigroup and more general semigroups, were studied in [1] and [2]. In the present paper we consider special form of the arithmetical processes to obtain the Beta distribution as a limit.

Let $GF[q,x]$ be the ring of polynomials over a finite field with q elements, q being a prime power. Let $\mathcal{M}$ be the multiplicative semigroup consisting of primary polynomials $m \in GF[q,x]$ and let $\mathcal{P} \subset \mathcal{M}$ be the set of all primary irreducible polynomials. Each polynomial $d, k, l, m, \ldots \in \mathcal{M}$ uniquely factors in $\mathcal{P}$. It is well known that

$$|\{m \in \mathcal{M}, \partial(m) = n\}| = q^n.$$

In what follows $c_i, i \in \mathcal{N}$ being absolute constants. By D we denote some expressions, which depend upon various parameters. The absolute value of D is bounded by an absolute constant.

Suppose that $f : \mathcal{M} \to \mathcal{R}$ be some multiplicative function satysfying the condition

$$f(p) = k > 0,\ \ f(p^\alpha) \geq 0,\ \alpha \geq 2,\ p \in \mathcal{P}.$$

Write

$$T(m, tn) = \sum_{\substack{d|m, \partial(m)=n, \\ \partial(d) \leq tn}} f(d), \quad t \in [0,1], \quad n \in \mathcal{N}, \quad m \in \mathcal{M}$$

and

$$T(m) = \sum_{d|m} f(d), \quad m \in \mathcal{M}.$$

Introduce the sequence of the functions

$$S_n(m,t) = \frac{T(m,nt)}{T(m,n)}, \quad t \in [0,1], \ \ n \in \mathcal{N}, \ \ m \in \mathcal{M}.$$

Let us consider the asymptotical behaviour of the sequence

$$G_n(t) = \frac{q-1}{q^{n+1}} \sum_{\substack{l\in\mathcal{M},\\ \partial(l)\le n}} S_n(l,t), \quad t\in[0,1], \quad n\in\mathcal{N}.$$

Auxilliary lemmas

Lemma 1 *Let $g:\mathcal{M}\to\mathcal{C}$ be some multiplicative function satisfying the condition: there exists constant $\xi\in\mathcal{C}$ such that*

$$\sum_{k\le n} q^k \sum_{\substack{\partial(p)=k,\\ p\in\mathcal{P}}} (g(p)-\xi) = \rho(n),$$

where $\rho(u)=c_1 q^u r(u)$ and $r(u)$ is decreasing function for which

$$\int_{-\infty}^{\infty} \frac{r(u)}{u}\mathrm{d}u < \infty.$$

Then

$$\frac{1}{q^n}\sum_{\substack{m\in M,\\ \partial(m)=n}} g(m) = n^{\xi-1}H\left(\frac{1}{q},g\right) + D\min\big\{\ln n, \{1-|Re\xi|^{-1}\}\big\}R(n),$$

where

$$R(n) = \max\left\{\frac{1}{n}, r(n), \frac{1}{n}\int_1^n r(u)\mathrm{d}u, \int_n^{\infty}\frac{r(u)}{u}\mathrm{d}u\right\},$$

$$H\left(\frac{1}{q},g\right) = \prod_{p\in P}\left(\sum_{\alpha=0}^{\infty}\frac{g(p^\alpha)}{q^{\alpha\partial(p)}}\right)\left(1-\frac{1}{q^{\partial(p)}}\right)^{\xi}$$

and D depends upon g and q.

Proof of this lemma can be found in [5].

Lemma 2. *Suppose that $\sigma\in\mathcal{R}$. Then*

$$\sum_{m\le n} m^\sigma q^m = \frac{q}{q-1}n^\sigma q^n + Dn^{\sigma-1}q^n,$$

where $D \le 2$.

Proof of this lemma can be found in [4, p. 86].

Lemma 3. *Suppose that the function $u(n),\ n \in \mathcal{N}$ is increasing. Then*

$$\sum_{n \le y} u(n) = \int_0^y u(x)\mathrm{d}x + Bu(x).$$

Let the function $v(n),\ n \in \mathcal{N}$ be monotonically decreasing. Then

$$\sum_{n \le y} v(n) = \int_1^y v(x)\mathrm{d}x + A + Bv(x),$$

where A is some absolute constant.

Proof of this lemma can be found in [7, p.4].

Results

Set

$$\beta = \frac{1}{k+1}, \quad \alpha = \frac{k}{k+1}.$$

Theorem. *Uniformly for $t \in [0,1]$ and $x \ge 2,\ x \in \mathcal{N}$ we have*

$$G_x(t) = \frac{q-1}{q^{x+1}} \sum_{\substack{m \in \mathcal{M}, \\ \partial(m) \le x}} S_x(m,t) = B\big(\alpha, \beta, t\big) + D\left\{\frac{\ln x}{x^\alpha} + \frac{1}{x^\beta}\right\},$$

where

$$B(\alpha, \beta, t) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t u^{\alpha-1}(1-u)^{\beta-1}\mathrm{d}u, \quad t \in [0,1].$$

Here D depends upon function f.

Corollary. *Suppose, that the multiplicative function $f : \mathcal{M} \to \mathcal{R}$ is defined by $f(m) \equiv 1,\ m \in \mathcal{M}$. Then uniformly for $x \ge 2,\ x \in \mathcal{N},\ t \in [0,1]$ we have*

$$\frac{q-1}{q^{x+1}} \sum_{\substack{m \in \mathcal{M}, \\ \partial(m) \le n}} S_x(m,t) = \frac{2}{\pi}\mathrm{arcsin}\sqrt{t} + \frac{D \ln x}{\sqrt{x}}.$$

Set

$$h_k^0(d) = \left(\frac{2}{k+1}\right)^{\omega(d)}, \quad h_k^1(d) = \prod_{p|d}\left(\frac{1}{k+1} + \frac{2}{(k+1)q^{\partial(p)}}\right)$$

and

$$h_k(d,x) = \frac{h_k^0(d)}{x} + \frac{h_k^1(d)}{x^\beta},\ x \in \mathcal{N},$$

where $\omega(d)$ equals the total number of different irreducible polynomials dividing $d \in \mathcal{M}$.

Lemma 4. *Uniformly in $n \in \mathcal{N}$, $\partial(d) \geq 0$ we have*

$$\frac{1}{q^n}\sum_{\substack{m\in\mathcal{M},\\ \partial(m)=n}} \frac{1}{T(md)} = \frac{H_1(k)}{n^\alpha}\big(g(d) + D\cdot h_k(d,n)\big),$$

where

$$H_1(k) = \frac{1}{\Gamma(\beta)}\prod_{p\in P}\left(\sum_{j=0}^{\infty}\frac{1}{T(p^j)q^{j\partial(p)}}\right)\left(1-\frac{1}{q^{\partial(p)}}\right)^\beta.$$

Moreover, the multiplicative function $g(d),\ d\in\mathcal{M}, n\in\mathcal{N}$ satisfies the equality

$$\frac{1}{q^n}\sum_{\substack{m\in\mathcal{M},\\ \partial(m)=n}} g(m) = \frac{H\left(\frac{1}{q},g\right)}{n^{-\alpha}\Gamma(\beta)} + \frac{D}{n}.$$

Proof of Lemma 4. Introduce the generating series of the function $1/T(md)$ by

$$\psi_d(s) = \sum_{n=0}^{\infty}\frac{1}{T(md)q^{s\partial(m)}}.$$

Applying the multiplicity of the function $T(m),\ m\in\mathcal{M}$ we derive

$$\frac{1}{T(md)} = \frac{1}{\prod_{p^\gamma||md} T(p^\gamma)} = \prod_{p|d}\frac{T(p^{\gamma_p(m)})}{T(p^{\gamma_p(m)+\gamma_p(d)})}\prod_{p|m}\frac{1}{T(p^{\gamma_p(m)})},$$

where

$$\gamma_p(m) = \begin{cases}\gamma, & p^\gamma||m,\\ 0, & p^\gamma \nmid m.\end{cases}$$

The last equality enables us to express the function $\psi_d(s)$ in the form of a product of Eulerian type:

$$\psi_d(s) = \prod_{p|d} \left(\left(\sum_{\gamma=0}^{\infty} \frac{1}{T(p^{\gamma+\gamma_p(d)}) q^{s\partial(p^\gamma)}} \right) \left(\sum_{\gamma=0}^{\infty} \frac{1}{T(p^\gamma) q^{s\partial(p^\gamma)}} \right)^{-1} \right) \cdot \tag{1}$$
$$\cdot \prod_{p\in\mathcal{P}} \left(\sum_{\gamma=0}^{\infty} \frac{1}{T(p^\gamma) q^{s\partial(p^\gamma)}} \right) =: g_d(s)\psi_1(s).$$

Set $s = \sigma + it$ and

$$\sigma_0 = \begin{cases} 1-\beta^2, & k \geq 1, \\ 1-\alpha^2, & k < 1. \end{cases}$$

Then uniformly for $p \in \mathcal{P}$ we have

$$\left| \sum_{\gamma=0}^{\infty} \frac{1}{T(p^\gamma) q^{s\partial(p^\gamma)}} \right| \geq c_0(k) > 0.$$

Further, the function $g_d(s)$, for each fixed $d \in \mathcal{M}$ is a finite product of ratios of series, each of which absolutely converges for $\sigma > 0$. Thus the function $g_d(s)$ is analytic for $\sigma > 0$. In what follows we assume that $\sigma > \sigma_0$.

Set

$$x(p,s) = \sum_{\alpha=1}^{\infty} \frac{1}{T(p^\alpha) q^{s\partial(p^\alpha)}}.$$

We have that for $\sigma > \sigma_0$

$$g_d(\sigma) \leq h_k^0(d), \quad g_d(1) \leq h_k^1(d). \tag{2}$$

Thus we can define the multiplicative function $g(d) := g_d(1)$.

Introduce the Dirichlet series of the functions defined in (1). We write

$$\psi_d(s) = \sum_{n=0}^{\infty} \frac{a_1(n)}{q^{ns}}, \quad \psi_1(s) = \sum_{n=0}^{\infty} \frac{a_2(n)}{q^{ns}}, \quad g_d(s) = \sum_{n=0}^{\infty} \frac{a_3(n)}{q^{ns}}.$$

It therefore follows from the last equalities that

$$a_1(n) = \sum_{j=0}^{n-1} a_2(n-j)a_3(j) + a_3(n). \tag{3}$$

Lemma 1 implies that

$$a_2(n) = \frac{q^n}{n^\alpha \Gamma(\beta)} \prod_{p\in\mathcal{P}} x(p,1)\left(1-\frac{1}{q^{\partial(p)}}\right)^\beta + \frac{Dq^n}{n} =: \frac{q^n}{n^\alpha} H_1(k) + \frac{D}{n} q^n. \tag{4}$$

We have

$$\left(1-\frac{j}{n}\right)^{-\alpha} = 1 + \frac{Dj}{n-j}, \quad D \le 1, \quad 0 \le j \le n-1.$$

Using Lemma 1 and combining the above equality with the equalities (4) and (3), we then obtain

$$a_1(n) = \frac{q^n H_1(k)}{n^\alpha} \sum_{j=0}^{n-1} \frac{a_3(j)}{q^j (1-\frac{j}{n})^\alpha} + a_3(n) + D\frac{q^n}{n} \sum_{j=1}^{n-1} \frac{a_3}{(1-\frac{j}{n})q^j} =$$

$$= \frac{q^n H_1(k)}{n^\alpha} \Bigg\{ g(d) + D\Bigg(\sum_{j>n} \frac{a_3(j)}{q^j} + \frac{a_3(n) n^\alpha}{q^n} +$$

$$+ \frac{1}{n^\beta}\Bigg(\sum_{j=0}^{n-1} \frac{a_3(j)}{q^j} + \frac{1}{n^\alpha} \sum_{j=0}^{n^\beta} \frac{a_3(j)}{q^j} + \sum_{n^\beta < j < n-1} \frac{a_3(j) j}{q^j (n-j)} \Bigg) \Bigg) \Bigg\}.$$

Using the inequality (2) we arrive at the relation

$$a_1(n) = \frac{q^n H_1(k)}{n^\alpha} \big(g(d) + D \cdot h_k(d,n) \big).$$

This implies the stated formula. Lemma 4 is proved.

Proof of Theorem. Initialy let us prove the following

Lemma 5. *For each $0 \le t \le 0.5$ and $x \in \mathcal{N}$ we have*

$$G_x(t) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t u^\alpha (1-u)^{-\beta} \mathrm{d}u + D\left(\frac{1}{x^\beta} + \frac{\ln x}{x^\alpha}\right).$$

Proof of Lemma 5. We have

$$G_x(t) = \frac{q-1}{q^{x+1}} \sum_{n=0}^{x} \sum_{\substack{m\in\mathcal{M},\\ \partial(m)=n}} \frac{1}{T(m,n)} \sum_{\substack{d|m,\partial(m)=n,\\ \partial(d)\le nt}} f(d) =$$

$$(5) \qquad = \frac{q-1}{q^{x+1}} \sum_{n=0}^{x} \sum_{\substack{m\in\mathcal{M},\\ \partial(m)=n}} \frac{T(m,xt)}{T(m,n)} - R(t,x) =: S(t,x) - R(t,x).$$

First let us consider the remainder term of the last equality. Therefore

$$\begin{aligned} R(x,t) \le & \frac{1}{q^x} \sum_{n=0}^{x} \sum_{\substack{m\in\mathcal{M},\\ \partial(m)=n}} \frac{T(m,xt)-T(m,nt)}{T(m,n)} = \\ = & \frac{1}{q^x} \sum_{n=0}^{x} \sum_{\substack{m\in\mathcal{M},\\ \partial(m)=n}} \frac{1}{T(m,n)} \sum_{nt\le\partial(d)\le xt} f(d) \le \\ \le & \frac{1}{q^x} \sum_{\partial(d)\le xt} f(d) \sum_{\partial(k)\le\partial(d)\frac{(1-t)}{t}} \frac{1}{T(kd,n)}. \end{aligned}$$

It follows from Lemmas 4 and 2 that

$$(6) \qquad R(t,x) \le \frac{c_1 H_1(k)}{q^x} \times$$

$$\sum_{\partial(d)\le xt} f(d) \left(q^{\partial(d)(\frac{1-t}{t})} \frac{t^\alpha}{\partial^\alpha(d)} g(d) + \frac{q^{\partial(d)(\frac{1-t}{t})} t^{\alpha+1}}{\partial^{\alpha+1}(d)} h_k^0(d) + \frac{q^{\partial(d)(\frac{1-t}{t})} t}{\partial(d)} h_k^1(d) \right).$$

Set

$$\Phi(u) = \sum_{\partial(d)\le u} f(d) g(d),$$

$$\Psi_0(u) = \sum_{\partial(d)\le u} f(d) h_k^0(d), \quad \Psi_1(u) = \sum_{\partial(d)\le u} f(d) h_k^1(d).$$

We deduce from Lemma 1 that

$$\Phi(u) = Dq^u/(1+u)^\beta, \ \ \Phi_1(u) = Dq^u/(1+u)^\beta, \ \ \Phi_0(u) = Dq^u/(1+u)^{\alpha-\beta}.$$

By means of Lemmas 1, 2 and equalities above we deduce that

$$R(t,x) \le \frac{c_2}{q^x} \left(\frac{q^{x(1-t)} H_1(k) q^{xt}}{x^\alpha (1+xt)^\beta} H\left(\frac{1}{q}, fg\right)\right) +$$

$$+ \frac{c_3}{q^{xt} x^{1+\alpha}} \sum_{\partial(d)\le xt} f(d) h_k^0(d) + \frac{c_4}{q^{xt} x} \sum_{\partial(d)\le xt} f(d) h_k^1(d) \le \frac{c_5}{x^\alpha}.$$

Hence we get from (5) that

$$G_x = S(t,x) + \frac{D}{x^{\alpha}}. \tag{7}$$

Consider now the main term of the equality (7). We have

$$\frac{q}{q-1}S(t,x) = \frac{1}{q^x}\sum_{\substack{m\in\mathcal{M},\\ \partial(m)=n}}\frac{T(m,tx)}{T(m,n)} = \frac{1}{q^x}\sum_{n=0}^{x}\sum_{\substack{m\in\mathcal{M},\\ \partial(m)=n}}\frac{1}{T(m,n)}\sum_{\substack{d|m,\\ \partial(d)\le tx}} f(d) =$$

$$= \frac{1}{q^x}\sum_{\partial(d)\le tx} f(d)\sum_{j=0}^{x-\partial(d)}\sum_{\partial(k)=j}\frac{1}{T(kd,n)}.$$

Taking account of Lemmas 4 and 2 we obtain

$$\frac{q}{q-1}S(t,x) = \frac{1}{q^x}\sum_{\partial(d)\le tx} f(d)\sum_{j=1}^{x-\partial(d)}\frac{q^j H_1(k)}{j^{\alpha}\Gamma(\beta)}\Big(g(d) + Dh_k(d,j)\Big) =$$

$$= \frac{H_1(k)}{q^x\Gamma(\beta)}\sum_{\partial(d)\le tx} f(d)g(d)\sum_{j=1}^{x-\partial(d)}\frac{q^j}{j^{\alpha}} + D\frac{H_1(k)}{q^x}\sum_{\partial(d)\le xt} f(d)h_k^1(d)\sum_{j=1}^{x-\partial(d)}\frac{q^j}{j} +$$

$$+ D\frac{H_1(k)}{q^x}\sum_{\partial(d)\le xt} f(d)h_k^0(d)\sum_{j=1}^{x-\partial(d)}\frac{q^j}{j^{1+\alpha}} =:$$

$$=: \frac{q}{q-1}\frac{H_1(k)}{\Gamma(\beta)}\sum_{\partial(d)\le xt}\frac{f(d)g(d)}{(x-\partial(d))q^{\partial(d)}} + R_1(t,x) =:$$

$$=: \frac{q}{q-1}\frac{H_1(k)}{\Gamma(\beta)}S_1(t,x) + R_1(t,x). \tag{8}$$

Estimate the remainder term of (8). We therefore obtain that

$$R_1(t,x) \le c_6\int_0^{tx}\Psi_1(u)\mathrm{d}\big((x-u)^{-1}q^{-u}\big) + \Psi_1(u)(x-u)^{-1}q^{-u}\big|_0^{tx}\big) +$$

$$+c_7 \int\limits_0^{tx} \Psi_0(u)\mathrm{d}\big((x-u)^{-1-\alpha}q^{-u}\big) + \Psi_0(u)(x-u)^{-1-\alpha}q^{-u}|_0^{tx}) \le \frac{c_8}{x^\beta}.$$

Thus the inequality (9) and relation (8) yield

$$S(t,x) = \frac{H_1(k)}{\Gamma(\beta)} S_1(t,x) + \frac{D}{x^\beta}. \tag{10}$$

Now let us consider the main term of (10). We have

$$S_1(t,x) \sum_{\partial(d)\le tx} f(d)g(d)\frac{(x-\partial(d))^{-\alpha}}{q^u} = \int\limits_0^{xt} \frac{(x-u)^{-\alpha}}{q^u}\mathrm{d}\Phi(u).$$

By partial integration, it follows that

$$S_1(t,x) =$$

$$= \frac{(x-u)^{-\alpha}}{q^u}\Phi(u)|_0^{xt} - \alpha\int\limits_0^{xt}\Phi(u)\frac{(x-u)^{-\alpha-1}}{q^u}\mathrm{d}u + \int\limits_0^{xt}\Phi(u)\frac{(x-u)^{-\alpha}}{q^u}\ln q\mathrm{d}u =:$$

$$=: R_{21} + R_{22} + S_2.$$

Noting that $\Phi(u) = Dq^u/(1+u)^\beta$ we obtain that $R_{21} + R_{22} \le c_9/x^\alpha$. This implies that

$$S_1(t,x) = S_2 + \frac{D}{x^\alpha}. \tag{11}$$

Putting the equality (11) in to (10), we deduce

$$S(t,x) = \frac{H_1(k)}{\Gamma(\beta)} S_2 + \frac{D}{x^\alpha}.$$

Furthermore

$$S_2 = \int\limits_0^{xt}\Phi(u)\frac{(x-u)^{-\alpha}}{q^u}\ln q\mathrm{d}u = \ln q\int\limits_0^{xt}\sum_{l\le u}\sum_{\partial(d)=l} f(d)g(d)\frac{(x-u)^{-\alpha}}{q^u}\mathrm{d}u =$$

$$= \ln q\frac{H(\frac{1}{q},fg)}{\Gamma(\alpha)}\int\limits_0^{xt}\sum_{l\le u}\frac{q^l}{l^\beta}(x-u)^{-\alpha}q^{-u}\mathrm{d}u + D\ln q\int\limits_0^{xt}\sum_{l\le u}\frac{q^l}{l+1}(x-u)^{-\alpha}q^{-u}\mathrm{d}u$$

$$=: \frac{\ln q}{\Gamma(\alpha)} H\left(\frac{1}{q}, fg\right) S_{22} + R_3. \tag{12}$$

Estimate the remainder term of the equality (12). It is clear that

$$R_3 \leq c_{10} \ln q \int\limits_0^{xt} \sum_{l\leq u} \frac{q^l}{(l+1)(x-u)^\alpha q^u} \mathrm{d}u \leq \frac{c_{11}\ln x}{x^\alpha}. \tag{13}$$

Considering the relation S_{22} we make use of the formula of partial integration. Thus

$$S_{22} = \frac{1}{\ln q} \sum_{l\leq xt} \frac{(-1)q^l}{l^\beta} \left((x-u)^{-\alpha} q^{-u}\Big|_l^{xt} - \alpha \int\limits_l^{xt} \frac{\mathrm{d}u}{(x-u)^{\alpha+1} q^u} \right) =$$

$$= \frac{1}{\ln q} \sum_{l\leq xt} \frac{(-1)q^l}{l^\beta} \left((x-l)^{-\alpha} q^{-l} - \alpha \int\limits_l^{xt} \frac{\mathrm{d}u}{(x-u)^{\alpha+1} q^u} \right) + \frac{D}{x} =:$$

$$=: \frac{1}{\ln q} S_3 + R_4 + \frac{D}{x^\alpha}. \tag{14}$$

It is clear, that

$$R_4 = c_{12} \sum_{l\leq xt} \frac{q^l}{l^\beta} \int\limits_l^{xt} \frac{\mathrm{d}u}{(x-u)^{\alpha+1} q^u} \leq \frac{c_{13}}{x}.$$

Substituting the last estimate into (14) we obtain

$$S_{22} = \frac{1}{\ln q} S_3 + \frac{D}{x^\alpha}. \tag{15}$$

The main term of (15) can be written as

$$S_3 = (-1) \sum_{l\leq xt} \frac{q^l}{l^\beta} \frac{(x-l)^{-\alpha}}{q^l} = \sum_{l\leq xt} \frac{1}{(x-l)^\alpha l^\beta}.$$

It is not difficult to see that the function

$$u^{-\beta}(x-u)^{\beta-1}$$

is monotone in the intervals $\left(0, \dfrac{x}{k+1}\right)$ and $\left(\dfrac{x}{k+1}, 1\right)$.

In view of the Lemma 3 we can thus write

$$S_3 = \int_0^{xt} \frac{\mathrm{d}u}{u^\beta (x-u)^\alpha} + \frac{D}{x^\alpha} = \int_0^{t} \frac{\mathrm{d}u}{u^\beta (1-u)^\alpha} + \frac{D}{x^\alpha}.$$

Substituting the last equality into (15) and combining (15), (13), (12), (11) and (10) we deduce from (7) that

$$G_x(t) = \frac{H_1(t)H(\frac{1}{q}, fg)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^{t} \frac{\mathrm{d}u}{u^\beta (1-u)^\alpha} + D\left(\frac{1}{x^\beta} + \frac{\ln x}{x^\alpha}\right).$$

The last equality and the relation

$$H_1(k)H\left(\frac{1}{q}, fg\right) =$$

$$= \prod_{p\in\mathcal{P}} \left(\sum_{0}^{\infty} \frac{1}{T(p^\alpha) q^{\partial(p^\alpha)}}\right)\left(1 + \sum_{\alpha=1}^{\infty} \frac{f(p^\alpha)g(p^\alpha)}{q^{\partial(p^\alpha)}}\right)\left(1 - \frac{1}{q^{\partial(p)}}\right) =$$

$$\prod_{p\in\mathcal{P}} \left(x(p) + \sum_{\alpha=1}^{\infty} \frac{1}{q^{\partial(p^\alpha)}} - \big(x(p)-1\big)\right)\left(1 - \frac{1}{q^{\partial(p)}}\right) = 1$$

complete the proof of Lemma 5. Lemma 5 is proved.

In order to complete the proof of Theorem, it remains to show that the equality of Lemma 5 is valid for $t \in [0.5, 1]$. We have

$$G_x(1) = \frac{q-1}{q^{x+1}} \sum_{\substack{m\in\mathcal{M},\\ \partial(m)\le n}} \frac{1}{T(m,n)} \sum_{\substack{d|m,\\ \partial(m)\le n}} f(d) = 1 - \frac{1}{q^x}.$$

Thus for each $0.5 < t \le 1$ we can write

$$G_x(t) = 1 - G_x(1-t) - \frac{1}{q^{x+1}} =$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \frac{\mathrm{d}u}{u^{\beta-1}(1-u)^{\alpha-1}} - \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^{1-t} \frac{\mathrm{d}u}{u^{\beta-1}(1-u)^{\alpha-1}} -$$

$$-\frac{1}{q^{x+1}} + D\left(\frac{1}{x^\beta} + \frac{\ln x}{x^\alpha}\right) =$$

$$= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^t \frac{\mathrm{d}u}{u^{\beta-1}(1-u)^{\alpha-1}} + D\left(\frac{1}{x^\beta} + \frac{\ln x}{x^\alpha}\right).$$

The desired assertion then follows from the last equality and Lemma 5. Theorem is proved.

The Corollary is a direct conclusion of Theorem. It is sufficient to choose $f(d) \equiv 1,\ d \in \mathcal{M}$.

References

[1] **Bareikis G.,** Analytic and probabilistic methods in number theory, *Proceedings of the Third International Conference in Honour of J. Kubilius, Palanga, Lithuania, 2002,* 1-10.

[2] **Bareikis G. and Indlekofer K.-H.,** Arithmetical processes in the semigroups, *Lith. Math. J.,* **43** (to appear)

[3] **Deshouillers J.M., Dress F. and Tenenbaum G.,** Lois de répartition des diviseurs I., *Acta Arith.,* **34** (1979), 7-19.

[4] **Knopfmacher J.,** *Analytic arithmetic of algebraic function fields,* Lecture Notes in Pure and Appl. Math. **50**, 1979.

[5] **Manstavičius E. and Skrabutėnas R.,** An analytic problem in combinatorial structures, *Lith. Math. J.,* **40** (2000), spec. number, 61-67.

[6] **Manstavičius E. and Timofeev N.M.,** A functional limit theorem related to natural divisors, *Acta Math. Hungar.,* **75** (1-2) (1997), 1-13.

[7] **Tenenbaum G.,** *Intoduction to analytic and probabilistic number theory,* Cambridge Univ. Press, Cambridge, 1995.

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