## LAUDATIO TO

## Professor Karl-Heinz Indlekofer

by I. Kátai

Karl-Heinz Indlekofer was born on January 2, 1943 in Wertheim (Main) in Germany. He studied mathematics and physics at University of Freiburg, where he defended his Ph.D. (Dr. rer. nat.) under the supervision of Prof. Wolfgang Schwarz in 1970. He habilitated at University Frankfurt am Main in 1974. From 1974 he is a professor at University Paderborn. He is married since 1972, his wife Irmgard. They have two children: Dorothee (1978), Thomas (1985).

Indlekofer has very intensive, close connection with Hungarian scientists, some artists and politicians, therefore his 60th anniversary was celebrated in Hungary, in the village Noszvaj in the De la Motte Castle, in the frame of an international conference called

**Numbers, Functions, Equations 2003**
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He headed several TEMPUS projects with the participation of several (6) Hungarian universities, several research projects financed by DFG and DAAD, Erasmus program. We, staff members and students in Hungary, benefitted from these very much, indeed.

He invited to Paderborn some famous choirs, and organized an exhibition of Hungarian painters. He came to Hungary with his choir, and they gave some concerts here.

In appreciation of his services the following precious awards have been given to him:

**Order of Merit of Eötvös Loránd University, 1992,**
**Doctor Honoris Causa, Kossuth Lajos University, 1992,**
**Doctor Honoris Causa, Janus Pannonius University, 1996.**

Professor Indlekofer is an acknowledged mathematician. He wrote more than 100 papers, mainly in number theory.

We shall concentrate ourselves to his most interesting results, which we shall subdivide as follows:

1. Equivalent power series. 2. Mean-values of multiplicative functions.

3. Sets of uniqueness.

4. An elementary proof of Halász's theorem.

5. Additive arithmetical semigroups.

6. On a theorem of H. Daboussi.

7. Arithmetical functions on the set of shifted primes. Collaboration with N. Timofeev.

8. Computational number theory. World records. Collaboration with A. Járai.

9. Some other topics.

**1. Equivalent power series** Let

$$D = \left\{ z \,\middle|\, |z| < 1 \right\},\ \overline{D} = \{z \mid |z| \le 1\},$$
$$A(D) = \{f \mid f \text{ regular in } D\}.$$

$(0 \neq)\ \xi_0 \in D,\ \phi(z) = \dfrac{z - \xi_0}{1 + \overline{\xi}_0 z}$.

Then $\phi : D \to D$ is invertible,

$$\phi^{-1}(z) = \frac{z + \xi_0}{1 + \overline{\xi}_0 z}.$$

$f_1, f_2 \in A(D)$ are equivalent, if $\exists\ \xi_0 \in D$, such that $f_2(z) = f_1(\phi(z))$. Turán (1958) proved: for every $\xi_0 \in D$ there exists $f_1(z) = \sum a_n z^n \in$ $\in A(D)$ such that $\sum a_n$ is convergent, but the power series $f_2(z) = f_1(\phi(z)) =$ $= \sum b_n(\xi_0) z^n$ is divergent in the point $z = e^{i\varphi}$, where $e^{i\varphi} = \phi^{-1}(1)$.

These type of questions were investigated by J.Clunie, W.Schwarz, L.Alpár, G.Halász and Indlekofer.

Alpár proved: if $\sum a_n$ is $(C, k)$ summable, then $\sum b_n(\xi_0) e^{i\varphi}$ is $(C, k+\delta)$ summable for $\delta > 1/2$.

Indlekofer proved (in his doctorate thesis): (similar assertion is true for the absolute summability) if $\sum a_n$ is $|(C, k)|$ summable, then $\sum b_n(\xi_0) e^{i\varphi}$ is

$|(C, k+\delta)|$ summable, if $\delta > 1/2$, and this assertion does not hold in general for $\delta < 1/2$. See [1], [2], [8].

Clunie: the Turán-type phenomenon can occur even if $f$ is continuous on $\overline{D}$.

Indlekofer: it can be assumed that the continuity module $\omega$ of $\varphi(t) = = f(e^{it})$ satisfies

$$\omega(\varphi, h) = O\left(\left(\log \frac{2\pi}{h}\right)^{-1}\right) \qquad h \to 0. \tag{1.1}$$

Furthermore, he proved: if $\omega$ is somewhat smoother than the right hand side of (1.1) then such a function does not exist.

In [13] he proved the following assertion.

Let

$$A_a(D) := \left\{ f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad \sum |a_n| < \infty \right\}.$$

Let $w : [0, \pi] \to \mathbb{R},\ 0 < w(h) \downarrow 0\ (h \to 0+)$, $w(h_1 + h_2) \leq w(h_1) + w(h_2)$. Let

$$A_{a,w} := \left\{ f \in A_a(D), \sup_{h \in (0,\pi)} \frac{\omega(f,h)}{w(h)} < \infty \right\}.$$

**I.** If $\sum 2^{i/2} w(2^{-i}) = \infty$, then each automorphism $\tau$ of the algebra $A_{a,w}$ is of the form

$$(\tau f)(z) = f(e^{i\alpha} z), \quad \alpha = \alpha(\tau) \in \mathbb{R}.$$

**II.** If $\sum 2^{i/2} w(2^{-i}) < \infty$, then for each automorphism $\tau$, there is some Moebius transform= $\phi$ such that

$$(\tau f)(z) = f(\phi(z)) \qquad (z \in \overline{D}). \tag{1.2}$$

Furthermore, for each $\phi$, the right hand side of (1.2) is always an automorphism of $A_{a,w}$.

## 2. Mean-values of multiplicative functions

Let $g$ be multiplicative,

$$M(g; x) := \frac{1}{x} \sum_{n \leq x} g(n). \tag{2.1}$$

Assume $|g(n)| \le 1$.

The asymptotical behaviour is completely characterized by H.Delange, E.Wirsing, G.Halász.

For some $f : \mathbb{N} \to \mathbb{C}$, and $q \ge 1$ let

$$\|f\|_q := \left\{ \limsup_{x\to\infty} \frac{1}{x} \sum_{n\le x} |f(n)|^q \right\}^{1/q}$$

(if the right hand side is finite).

Let

$$\mathcal{L}^q = \left\{ f \;\middle|\; \|f\|_q < \infty \right\}.$$

Elliott and Daboussi characterized those multiplicative functions $f \in \mathcal{L}^q$ for which

$$M(f) := \lim_{x\to\infty} M(f,x) \ne 0.$$

Indlekofer introduced the notion of ”*uniformly summable functions*” [35].

**Definition.** A function $f \in \mathcal{L}^1$ is said to be uniformly summable if

$$\lim_{K\to\infty} \left( \sup_{N>1} N^{-1} \sum_{\substack{n\le N \\ |f(n)|>K}} |f(n)| \right) = 0.$$

$\mathcal{L}^*$ := space of uniformly summable functions.

Observe: $\mathcal{L}^* = \|\cdot\|_1$ closure of $l^\infty$.

Indlekofer proved:

**A.** Let $g : \mathbb{N} \to \mathbb{C}$ be multiplicative, and $q \ge 1$. Then the following assertions (i), (ii) hold.

(i) If $g \in \mathcal{L}^* \cap \mathcal{L}^q$, and $M(g)$ exists, and $\ne 0$, then the series

$$\sum \frac{g(p)-1}{p} \qquad \text{is convergent,}$$

$$\sum \frac{|g(p)-1|^2}{p} < \infty,$$

$$\sum_{|g(p)-1|\ge \frac{1}{2}} \frac{|g(p)|^q + |g(p)|}{p} < \infty, \tag{2.1}$$

$$\sum_p \sum_{k\ge 2} \frac{|g(p^k)|^q}{p^k} < \infty,$$

and, for each prime $p$,

$$1+\sum_{k=1}^{\infty}\frac{g(p^k)}{p^k}\neq 0. \tag{2.2}$$

(ii) If (2.1) holds, then $g\in\mathcal{L}^*\cap\mathcal{L}^q$ and the mean values $M(g),\ M(|g|^\lambda)$ $\lambda\in[1,q]$ exist. If (2.2) holds, then $M(g)\neq 0$.

**B.** Let $g\in\mathcal{L}^*$ be real valued multiplicative. Then, the existence of $M(|g|)$ implies that of $M(g)$.

**C.** If $g:\mathbb{N}\to\mathbb{C}$ is multiplicative, $g\in\mathcal{L}^*$, and for each $t\in\mathbb{R}$, the series

$$\sum_{||g(p)|-1|\le 1/2}\frac{1}{p}\left(1-\frac{\operatorname{Re} g(p)}{|g(p)|\cdot p^{it}}\right)$$

diverges, then $M(g)=0$.

## 3. Sets of uniqueness

We say that $E\subseteq\mathbb{N}$ is a set of uniqueness for the class of completely additive functions, if $f\in\mathcal{A}^*$, $f(E)=0$ implies that $f(\mathbb{N})=0$.

D. Wolke proved: $E$ is a set of uniqueness, if and only if, for every $n\in\mathbb{N}$

$$n=e_1^{r_1}\dots e_k^{r_k}$$

can be solved so that $e_1,\dots,e_k\in E,\ r_1,\dots,r_k\in\mathbb{Q}$.

We say that $\mathbb{E}$ is a set of uniqueness $\bmod 1$, if $f\in\mathcal{A}^*$, $f(E)\equiv 0 \pmod 1$ implies that $f(\mathbb{N})\equiv 0 \pmod 1$.

Indlekofer [23], furthermore Dress and Volkman, and P.Hoffman, independently proved that $E$ is a set of uniqueness $\bmod 1$ iff every $n\in\mathbb{N}$ can be written as

$$n=e_1^{l_1}\dots e_k^{l_k},$$

where $e_1,\dots,e_k\in E,\ l_1,\dots,l_k\in\mathbb{Z}$.

In a joint paper with Fehér and Timofeev [101] they proved that

$$E=\{x^2+y^2+a\mid x,y\in\mathbb{Z}\}$$

is a set of uniqueness $\bmod 1$ if $a$ is a sum of two squares.

A quite new unpublished result of Indlekofer and Timofeev is the following:

Let $a, b \in \mathbb{N},\ c \in \mathbb{Z},\ c \neq 0,\ (a,b) = 1,\ (ab, 2c) = 1$. Let

$$S(x) := \#\{n \leq x \mid a(n+c) = b(m+c); \quad n, m \in \mathcal{B}\},$$
$$\mathcal{B} = \{u^2 + v^2 \mid u, v \in \mathbb{Z}\}.$$

Then

$$S(x) \geq \Theta \frac{x}{\log x},$$

for every $x \geq x_0$, where $\Theta = \Theta(a,b,c) > 0$. Hence, especially follows: for odd $a, b \in \mathbb{N}$ $\quad \frac{a}{b} = \frac{n+1}{m+1}$ hold for infinitely many $n, m \in \mathcal{B}$.

**4. An elementary proof of Halász's theorem**

Hildebrand gave an elementary proof for the theorem of Wirsing. He used the prime number theorem in his proof. Daboussi gave an elementary proof for the prime number theorem, completely else than that of Selberg.

Daboussi and Indlekofer, in their nice paper [47] gave an elegant elementary proof for the theorem of Halász.

**5. Additive arithmetical semigroups** J. Knopfmacher introduced this notion.

An additive arithmetical semigroup is a commutative semigroup $G$ with identity element 1 together with a subset $\mathcal{P}$ and a mapping $\delta : G \to \mathbb{N}_0$, such that the following conditions hold:

(i) every element $a \neq 1$ in $G$ admits a factorization into a finite product of elements of $\mathcal{P}$ which is unique, except for the order of the factors,

(ii) $\delta(ab) = \delta(a) + \delta(b)$ for all $a, b \in G$,

(iii) $\delta(1) = 0$ and $\delta(a) \neq 0$ if $a \neq 1$,

(iv) $\pi(n) := \#\{p \in \mathcal{P} : \delta(p) = n\} < \infty$ for every $n \in \mathbb{N}$. Let

$$\gamma(n) := \#\{a \in G : \delta(a) = n\},$$
$$Z(y) = \sum_{n=0}^{\infty} \gamma(n) y^n.$$

Assume the fulfilment of the following condition which is called

Axiom $A^*$ : $\quad \gamma(n) = Aq^n + O(q^{\delta n})$ with some $\delta < 1$. Here $A > 0$.

Under this condition, $Z(y) = \dfrac{A}{1-qy} + H_1(y)$, where $H_1(y)$ is convergent on a disc around zero with radius bigger than $1/q$.

In a paper [50], written jointly with Manstavičius and Warlimont, it is proved that

$$\pi(n) = \varepsilon(n)\frac{q^n}{n} + O\left(q^{\Theta n}\right)$$

with some $\Theta < 1$, where $\varepsilon(n) = 1 \quad (n \in \mathbb{N})$ if $Z(y) \neq 0$ on the circle $|z| = 1/q$, while in the case when $Z(y^*) = 0$, $|y^*| = 1/q$, then $y^* = -1/q$ and $\varepsilon(2n) = 2,\ \varepsilon(2n-1) = 0 \quad (n \in \mathbb{N})$.

In a joint paper with Manstavičius [63] they proved the analogon of Halász's theorem for arithmetical semigroups.

## 6. On a theorem of Daboussi

Daboussi (1974) proved: If $f \in \mathcal{M},\ |f(n)| \leq 1, \alpha =$ irrational, then

$$\frac{1}{x}\sum_{n\leq x} f(n)e^{2\pi i n\alpha} \to 0 \qquad (x \to \infty). \tag{6.1}$$

Indlekofer extended this for $f \in \mathcal{L}^*$.

We proved some assertions:

**I.** Let $\mathcal{P}_k =$ set of integers $n$ with $\omega(n) = k$. $\omega(n)$ is the number of prime factors of $n$. Then, for each irrational $\alpha$ and bounded $f \in \mathcal{M}$,

$$\sup_{\delta \leq \frac{k}{\log\log x} < 2-\delta} \frac{1}{\pi_k(x)} \left| \sum_{\substack{n\leq x \\ n\leq x\in\mathcal{P}_k}} f(n)e^{2\pi i n\alpha} \right| \to 0$$

as $\quad x \to \infty$.

**II.** If $P(u) = \alpha_k x^k + \ldots + \alpha_1 x$, and at least one of $\alpha_1, \ldots, \alpha_k$ is irrational, then

$$\frac{1}{x}\sum_{n\leq x} f(n)e^{2\pi i P(n)} \to 0$$

whenever $f \in \mathcal{M} \cap \mathcal{L}^*$.

**III.** Let $f \in \mathcal{L}^* \cap \mathcal{M}$, and assume that (6.1) holds for every $\alpha \in \mathbb{R}$. Let $g$ be a $q$-multiplicative function, $|g(n)| = 1 \quad (n \in \mathbb{N})$.

Then

$$\frac{1}{x} \sum_{n \le x} f(n) g(n) \to 0.$$

## 7. Arithmetical functions on the set of shifted primes. Collaboration with N. Timofeev

In [72] they proved: Let $f_i \in \mathcal{M},\ (i = 1, \ldots, k), \quad \alpha_i \in \mathbb{C}$,

$$A(n) := \sum_{i=1}^{k} \alpha_i f_i(n) \ge 0 \qquad (n \in \mathbb{N}).$$

Assume

$$\begin{aligned} &\sum_{n \le x} |f_i(n)|^2 \le A_1 x (\log x)^c, \quad c \ge 0, \\ &\sum_{p \le x} |f_i(p)|^2 \le A_2 x (\log x)^{-\rho} \end{aligned} \tag{7.1}$$

with some $\rho \in (0, 1]$. Then

$$\frac{1}{\pi(x)} \sum_{p \le x} A(p+1) \ll \frac{\log\log x}{x} \sum_{n \le x} A(n) + \frac{(\log\log 10x)}{(\log x)^{\rho/2}}.$$

If (7.1) holds with $c = 0$, then

$$\frac{1}{\pi(x)} \sum_{p \le x} A(p+1) \ll \frac{(\log v)}{x} \sum_{n \le x} A(n) + \frac{1}{v^{\rho_1}} + \frac{1}{(\log x)^{\rho_1}}$$

is satisfied for some constant $\rho_1 > 0$ and for every $v \in [3, (\log x)^A]$.

They gave a lot of interesting consequences of their theorem.

In [73] they proved:

Let $f \in \mathcal{L}_2 \cap \mathcal{M}$,

$$\sum_{p \le x} |f(p)| \ll \frac{x}{(\log x)^\rho},$$

with some $\rho > 0$. If

$$\lim \frac{1}{x} \sum_{n \le x} f(n)\chi_d(n)$$

exists and nonzero, where $\chi_d(n)$ is a character, then the limit

$$m_{\mathcal{P}}(f) := \lim_{x \to \infty} \frac{1}{\pi(x)} \sum_{p \le x} f(p+1)\chi_d(p+1)$$

exists, $m_{\mathcal{P}}(f) \ne 0$, and they gave it explicitily.

Some other (yet unpublished) results of them:

**1.**

$$\sum_{p \le x} f(p+1) \sim C(f)x, \tag{7.2}$$

if $f(p) \sim 2$ for primes, in some sense.

**2.** If $\chi_d^{(1)}, \chi_d^{(2)}$ are real characters $\bmod d$, $f(p) \sim \chi_d^{(1)}(p) + \chi_d^{(2)}(p)$ in some sense, then (7.2) is true.

## 8. Computational number theory. Collaboration with A. Járai

Indlekofer headed a research group in number theory at the University of Paderborn. Járai worked in this group. They worked out very effective parallel algorithms for computing very large twin primes, Sophie Germain primes. They achieved a lot of world records in giving such numbers. See [67].

## 9. Some other topics

a. Mean-values of $q$-multiplicative and the distribution of $q$-additive functions (jointly with Kátai): [91], [95], [100].

b. Generalized number systems and fractal geometry: [60], [65].