

ON LINEAR COMBINATIONS OF q -ADDITIVE FUNCTIONS

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1. Let $q \geq 2$, $A = \{0, 1, \dots, q-1\}$, $\varepsilon(n) (\in A)$ be the sequence of the digits in the q -ary expansion of n ,

$$(1.1) \quad n = \sum \varepsilon_j(n)q^j.$$

Let \mathcal{A}_q be the set of real valued q -additive functions. We say that $f : \mathbb{N}_0 (= \mathbb{N} \cup \{0\}) \rightarrow \mathbb{R}$ belongs to \mathcal{A}_q , if $f(0) = 0$ and

$$(1.2) \quad f(n) = \sum_{j=0}^{\infty} f(\varepsilon_j(n)q^j)$$

holds for every n .

Let $1 \leq a_1 < a_2 < \dots < a_k (< q)$, $(a_i, q) = 1$, $(a_i, a_j) = 1$ for every i and $j \neq i$. Let furthermore $f_1, \dots, f_k \in \mathcal{A}_q$,

$$(1.3) \quad l(n) := f_1(a_1n) + f_2(a_2n) + \dots + f_k(a_kn).$$

We say that a function $g : \mathbb{N} \rightarrow \mathbb{R}$ is "tough" if there is a sequence E_N such that

$$(1.4) \quad \begin{cases} \limsup_{N \rightarrow \infty} \frac{1}{q^N} \# \{n < q^N \mid |f(n) - E_N| > K\} := c(K), \\ c(K) \rightarrow 0 \quad (K \rightarrow \infty). \end{cases}$$

We say that g is "bounded in mean" if (1.4) holds with $E_N = 0$.

Let $\alpha > 0$. We say that g belongs to the class L^α , if

$$(1.5) \quad \limsup_{x \rightarrow \infty} \frac{1}{x} \sum_{n < x} |g(n)|^\alpha < \infty.$$

The following assertions are quite obvious.

Theorem 1. *The function $f \in \mathcal{A}_q$ is tough if and only if*

$$(1.6) \quad \sum_{j=0}^{\infty} \sum_{b \in A} f^2(bq^j) < \infty.$$

Theorem 2. *Let $f \in \mathcal{A}_q$. f is bounded in mean if and only if (1.6) holds and*

$$(1.7) \quad E_N^* := \frac{1}{q} \sum_{j=0}^{N-1} \sum_{b \in A} f(bq^j)$$

is bounded.

Theorem 3. *Let $f \in \mathcal{A}_q$. If f is bounded in mean, then $f \in L^\alpha$ for each $\alpha > 0$.*

Remark. The opposite assertion is clear. Theorems 1,2 are well-known in probability theory.

Let

$$m_j := \frac{1}{q} \sum_{b=0}^{q-1} f(bq^j), \quad E_N^* = \sum_{j=0}^{N-1} m_j,$$

X_0, X_1, \dots be a sequence of independent random variables,

$$P(X_j = f(bq^j) - m_j) = 1/q \quad (b \in \mathbb{A}) \quad b = 0, \dots, q-1$$

and let

$$Y_N = X_0 + \dots + X_{N-1}.$$

Let us prove Theorem 3. Assume that f is bounded in mean, that is (1.6) holds and (1.7) is bounded.

It is clear that EX_j , the mean value of X_j , is zero, furthermore that $X_j \rightarrow 0$ ($j \rightarrow \infty$) in measure.

Let $k \geq 1$. Then

$$Y_N^{2k} = \sum_{\substack{\alpha_1 + \dots + \alpha_r = 2k \\ \alpha_\nu \geq 1}} c(\alpha_1, \dots, \alpha_r) \sum X_{i_1}^{\alpha_1} \dots X_{i_r}^{\alpha_r},$$

where the coefficient $c(\alpha_1, \dots, \alpha_r)$ depends only on k . From the independency of X_0, \dots, X_{N-1} we obtain that $E(X_{i_1}^{\alpha_1} \dots X_{i_r}^{\alpha_r}) = 0$ if $\alpha_\nu = 1$ for some ν . Furthermore

$$|E(X_{i_1}^{\alpha_1} \dots X_{i_r}^{\alpha_r})| \leq C \cdot E(X_{i_1}^2) \dots E(X_{i_r}^2)$$

with some constant C which may depend on k .

Consequently from (1.6) we have that

$$E(Y_N^{2k}) \ll \sum_{r \leq k} \left(\sum_{j=0}^{N-1} E^2(X_j) \right)^r \ll C_k.$$

The proof is completed.

2. In our paper [2] we proved the following theorem which we state here as

Lemma 1. *Assume that $l(n)$ is defined by (1.3), and the conditions, stated there, are satisfied. Then $l(n)$ is "tough" if and only if there exist $\gamma_1, \dots, \gamma_k \in \mathbb{R}$ such that $a_1\gamma_1 + \dots + a_k\gamma_k = 0$, and for $\psi_l(n) = f_l(n) - \gamma_l n$*

$$(2.1) \quad \sum_j \sum_b \psi_l^2(bq^j) < \infty \quad (l = 1, \dots, k).$$

Let

$$E_N^{(l)} = \frac{1}{q} \sum_{j=0}^{N-1} \sum_{b \in \mathbb{A}} \psi_l(bq^j),$$

$$(2.2) \quad E_N = E_N^{(1)} + \dots + E_N^{(k)}.$$

Furthermore, $l(n)$ is bounded in mean, if (2.1) is satisfied and (2.2) is bounded.

By using the argument which was applied by the proof of Theorem 3, we obtain

Theorem 4. *Let $l(n)$ as is (1.3). If $l(n)$ is bounded in mean, then so are $f_j(n)$ ($j = 1, \dots, k$), and $l \in L^\alpha$ for every $\alpha > 0$.*

3. The $\log n$ is a very special function among the additive arithmetical functions. Somehow its role is played by cn among the q -additive functions.

The function $f(n) = n$ has a simple distribution, if we normalize appropriately:

$$(3.1) \quad \lim_{x \rightarrow \infty} \frac{1}{x} \# \left\{ n < x \mid \frac{n}{x} < y \right\} = G(y),$$

$$G(y) = \begin{cases} y & \text{if } y \in [0, 1], \\ 1 & \text{if } y \geq 1, \\ 0 & \text{if } y \leq 0. \end{cases}$$

Theorem 5. *Assume that $f \in \mathcal{A}_q$, and*

$$(3.3) \quad \lim_{N \rightarrow \infty} \frac{1}{x} \left\{ n < q^N \mid \frac{f(n)}{x} < y \right\} = G(y).$$

Then $f(n) = n + h(n)$, and $\frac{h(n)}{n} \rightarrow 0$ ($n \rightarrow \infty$).

Proof. Assume that (3.3) holds. Let

$$F_x(y) := \frac{1}{x} \# \left\{ n < x \mid \frac{f(n)}{x} < y \right\}.$$

Since G is continuous, therefore the convergence of $F_{q^N}(y) \rightarrow G(y)$ is uniform in y .

First we prove that

$$\begin{aligned} \limsup_{N \rightarrow \infty} \frac{f(bq^N)}{bq^N} &\leq 1, \quad b = 1, \dots, q-1, \\ \liminf_{N \rightarrow \infty} \frac{f(bq^N)}{bq^N} &\geq -1/b, \quad b = 1, \dots, q-1. \end{aligned}$$

Let $\delta > 0$ be an arbitrary small constant, and $N \rightarrow \infty$.

From (3.3) we obtain that for all but $o(x)$ of the integers $\nu \leq x$,

$$f(\nu) \in [-\delta x, (1 + \delta)x].$$

Let us consider the integers $n \in [bq^N, (b+1)q^N]$. A typical integer m can be written as $m = bq^N + \nu$, $\nu < q^N$. Since

$$-\delta q^N < f(m) \leq (b+1+\delta)q^N$$

for all but $o(q^N)$ integers, and for $\varepsilon > 0$ the number of the integers ν satisfying $\frac{f(\nu)}{q^N} \in (1 - \varepsilon, 1 + \varepsilon)$ has a positive proportion, therefore

$$\frac{f(bq^N)}{q^N} + 1 \in [-\delta - \varepsilon, b + 1 + \delta + \varepsilon],$$

and so

$$\frac{f(bq^N)}{bq^N} < 1 + \frac{\delta + \varepsilon}{b},$$

if N is large enough. Thus

$$\limsup_{N \rightarrow \infty} \frac{f(bq^N)}{bq^N} \leq 1.$$

Similarly, we can prove that

$$\liminf \frac{f(bq^N)}{bq^N} \geq -1/b.$$

Let $c \in \{1, \dots, q\}$. Count the integers $n < cq^N$ for which $\frac{f(n)}{cq^N} < y$. We subdivide the set of those n according to the leading digit.

If $n = bq^N + \nu$, then

$$\frac{f(n)}{cq^N} < y \quad \text{if and only if} \quad \frac{f(\nu)}{cq^N} < y - \frac{f(bq^N)}{cq^N},$$

i.e. if

$$\frac{f(\nu)}{q^N} < cy - \frac{f(bq^N)}{q^N}.$$

Hence we obtain that

$$cq^N F_{cq^N}(y) = \sum_{b=0}^{c-1} q^N F_{q^N} \left(cy - \frac{f(bq^N)}{q^N} \right),$$

and from (3.3) we obtain that

$$(3.4) \quad cG(y) = \sum_{b=0}^{c-1} G \left(cy - \frac{f(bq^N)}{q^N} \right) + O(\varepsilon_N),$$

where $\varepsilon_N \rightarrow 0$ as $N \rightarrow \infty$. The relation (3.4) is valid uniformly in y . Thus

$$\sum_{b=0}^{c-1} G \left(\frac{-f(bq^N)}{q^N} \right) \rightarrow 0 \quad (N \rightarrow \infty)$$

which proves that

$$\liminf \frac{f(bq^N)}{bq^N} \geq 0.$$

Assume that $\liminf_{N \rightarrow \infty} \frac{f(bq^N)}{bq^N} < 1$ holds for some b . Let b be the smallest digit with this property. Let

$$\begin{aligned} \frac{f(bq^{N_j})}{bq^{N_j}} &\rightarrow 1 - \Delta \quad (j \rightarrow \infty), \\ \frac{f(sq^{N_j})}{sq^{N_j}} &\rightarrow 1 \quad \text{as } s = 1, \dots, b-1, \quad j \rightarrow \infty. \end{aligned}$$

Choose $c = b + 1$. Then

$$cG(y) = G(cy - (c-1)(1-\Delta)) + \sum_{s=0}^{c-2} G(cy - s).$$

Put now $y = \frac{c-1}{c}$. Then

$$c-1 = cG\left(\frac{c-1}{c}\right) = G(\Delta(c-1)) + \sum_{s=0}^{c-2} G(c-1-s).$$

The sum on the right is $c-1$, $G(c-1-s) = 1$ for $s = 0, \dots, c-2$, consequently $G(\Delta(c-1)) = 0$, and so $\Delta(c-1) \leq 0$, which by $\Delta \geq 0$ implies that $\Delta = 0$. The proof is ready.

4. Let a_1, \dots, a_k , q be as earlier, f_1, \dots, f_k be integer valued q -additive functions. For some integers m_1, \dots, m_k let

$$\begin{aligned} (4.1) \quad \delta_{m_1, \dots, m_k}(u_1, \dots, u_k) &= \\ &= \lim_{x \rightarrow \infty} \frac{1}{x} \#\{n \leq x \mid f_j(a_j n) \equiv u_j \pmod{m_j}, \quad j = 1, \dots, k\}, \end{aligned}$$

assuming that the limit exists.

Let \mathcal{P} be the set of "generating elements",

$$\mathcal{P} = \{bq^j \mid b \in A, \quad j = 0, 1, 2, \dots\}.$$

Theorem 6. Assume that $(m_i, m_j) = 1$, $(a_i, m_i) = 1$ for $i = 1, \dots, k$, $j \neq i$. Let Δ_j be the largest divisor of m_j for which $\Delta_j \mid f_j(\pi)$ holds for all but finitely many $\pi \in \mathcal{P}$.

The relation

$$(4.2) \quad \delta_{m_1, \dots, m_k}(u_1, \dots, u_k) = \frac{1}{m_1 \dots m_k}$$

holds for every $u_j = 0, \dots, m_j - 1$ ($j = 1, \dots, k$) if and only if $\Delta_1 = \Delta_2 = \dots = \Delta_k = 1$.

Proof. I. Assume that the limit in (4.1) exists for every u_1, \dots, u_k . Let $m_1^* \mid m_1, \dots, m_k^* \mid m_k$. Then $\delta_{m_1^*, \dots, m_k^*}(v_1, \dots, v_k)$ exists for every v_1, \dots, v_k . If the distribution is uniform for $\{m_1, \dots, m_k\}$, (i.e. if (4.2) holds), then it is uniform for $\{m_1^*, \dots, m_k^*\}$ as well.

II. If the distribution of $\{f_1(a_1n) \pmod{m_1}, \dots, f_k(a_kn) \pmod{m_k}\}$ is uniform, and $\{i_1, \dots, i_h\}$ is a subset of $\{1, \dots, k\}$, then the distribution of

$$\{f_{i_l}(a_{i_l}n) \pmod{m_{i_l}} \mid l = 1, \dots, h\}$$

is uniform. Especially

$$\{f_i(a_in) \pmod{m_i}\}$$

is distributed uniformly.

III. Let $m > 1$, $a \in \mathbb{N}$, $(a, m) = (m, q) = (a, q) = 1$, $f \in \mathcal{A}_q$, $f(n) \in \mathbb{Z}$ ($n \in \mathbb{N}$). Assume that for some $l_0 \in \mathbb{N}_0$, $f(nq^{l_0}) \equiv 0 \pmod{m}$. Then the limit distribution of $f(an) \pmod{m}$ exists and it is non-uniform.

Indeed, let us write n as $n = n_0 + q^{l_0}n_1$, $s(n) = n_1$, $T(n) = n_1 = \left\lfloor \frac{n}{q^{l_0}} \right\rfloor$.

Then $an = s(an) + q^{l_0}T(an)$, $an_0 = s(an_0) + q^{l_0}T(an_0)$, consequently $s(an) \equiv s(an_0) \pmod{q^{l_0}}$, and

$$(4.3) \quad f(an) \equiv f(s(an_0)) \pmod{m}.$$

The density of the integers $n \equiv u \pmod{q^{l_0}}$ is $1/q^{l_0}$, for every $u \pmod{q^{l_0}}$. Therefore

$$\delta_m(v) := \lim_{x \rightarrow \infty} \frac{1}{x} \sum_{\substack{f(an) \equiv v \pmod{m} \\ n \leq x}} 1 = \sum_{h=0}^{q^{l_0}-1} \tau_h(v),$$

where

$$\tau_h(v) = \lim_{x \rightarrow \infty} \frac{1}{x} \sum_{\substack{n \leq x \\ n \equiv h \pmod{q^{l_0}} \\ f(an) \equiv v \pmod{m}}} 1.$$

Since the fulfilment of $f(an) \equiv v \pmod{m}$ does depend only on $n \pmod{q^{l_0}}$, therefore $\tau_h(v)$ equals 0 or $1/q^{l_0}$. Consequently $\delta_m(v) = \frac{\text{integer}}{q^{l_0}}$, so $\delta_m(v) = 1/m$ cannot hold.

IV. If (4.2) holds, then $\Delta_1 = \dots = \Delta_k = 1$. This is a direct consequence of III. Assume that $\Delta_1 > 1$. If (4.2) holds, then by II., $\{f_1(a_1n) \pmod{m_1}\}$ is distributed uniformly, and from I., $\{f_1(a_1n) \pmod{\Delta_1}\}$ is distributed uniformly. But this is impossible due to III.

V. From now we assume that $\Delta_1 = \dots = \Delta_k = 1$. The fulfilment of (4.2) is equivalent to

$$(4.4) \quad \frac{1}{x} \sum_{n < x} e \left(\frac{h_1}{m_1} f_1(a_1n) + \dots + \frac{h_k}{m_k} f_k(a_kn) \right) \rightarrow 0$$

for each choice of $h_j \pmod{m_j}$ ($j = 1, \dots, k$) except the case when $h_j \equiv 0 \pmod{m_j}$ for every j . Assume that (4.2) does not hold. Then there exists h_1, \dots, h_k , $h_j \in \{0, \dots, m_j - 1\}$, $(h_1, \dots, h_k) \neq (0, \dots, 0)$ such that

$$(4.5) \quad \limsup \frac{1}{x} \left| \sum e \left(\frac{h_1 f_1(a_1n)}{m_1} + \dots + \frac{h_k f_k(a_kn)}{m_k} \right) \right| > 0.$$

From our theorem [2] we have: there exist $\gamma_1, \dots, \gamma_k \in \mathbb{R}$ such that

$$a_1 \gamma_1 + \dots + a_k \gamma_k = \frac{E}{q^{l_0}},$$

$l_0 \geq 0$, $E \in \mathbb{Z}$, and

$$(4.6) \quad \sum_{\pi \in \mathcal{P}} \left\| \frac{h_j f_j(\pi)}{m_j} - \gamma_j \pi \right\|^2 < \infty.$$

It implies that

$$(4.7) \quad \sum_{\pi \in \mathcal{P}} \|m_j \gamma_j \pi\|^2 < \infty.$$

Hence we obtain that $m_j \gamma_j$ is rational. Indeed, assume that $\lambda = m_j \gamma_j$ is irrational. We shall show that for every c there is an $l > c$ for which $\|\lambda q^l\| > 1/q^2$, and this is enough.

Assume that $q^{-T} < \|\lambda q^h\| < q^{-T+1}$ (for $j = 0, \dots, T-2$), and so $\|\lambda q^{h+(T-2)}\| > 1/q^2$. Let $\lambda = \frac{U_j}{V_j}$, $(U_j, V_j) = 1$. From (4.7) we obtain that

$V_j \mid q^{l_0}$ for some suitable integer l_0 . If $V_j = P_j Q_j$, $(Q_j, q) = 1$, $P_j \mid q^{l_0}$, $Q_j > 1$, then $\|\lambda q^l\| \geq \frac{1}{Q_j}$, which contradicts to (4.7).

Thus we can write

$$\gamma_j = \frac{U_j}{V_j m_j}, \quad V_j \mid q^{l_0} \quad (j = 1, \dots, k)$$

and from (4.6), that

$$\frac{h_j f_j(\pi \cdot q^{l_0})}{m_j} - \frac{U_j \left(\frac{q^{l_0}}{V_j}\right) \pi}{m_j} \equiv 0 \pmod{1},$$

whence

$$\begin{aligned} h_j f_j(\pi q^{l_0}) &\equiv D_j \pi \pmod{m_j} \\ D_j &= U_j \cdot \left(\frac{q^{l_0}}{V_j}\right), \\ \frac{a_1 U_1}{m_1 \cdot V_1} + \dots + \frac{a_k U_k}{m_k V_k} &= \frac{E}{q^{l_1}}. \end{aligned}$$

We may assume that $l_1 \geq l_0$. Then

$$\begin{aligned} \frac{a_1 U_1}{m_1} W_1 + \dots + \frac{a_k U_k}{m_k} W_k &= E, \\ W_j &= \frac{q^{l_1}}{V_j} \in \mathbb{Z}. \end{aligned} \tag{4.8}$$

Let us multiply (4.8) with m_1, \dots, m_k . Then m_j is a divisor of

$$a_j U_j W_j \prod_{\substack{\nu=1 \\ \nu \neq j}}^k m_\nu.$$

Since $(m_j, m_\nu) = 1$, $(m_j, q) = 1$ and $W_j \mid q^{l_1}$, therefore $(m_j, W_j) = 1$, $(m_j, a_j) = 1$ holds by the assumptions, therefore $m_j \mid U_j$ ($j = 1, \dots, k$). Consequently $\gamma_j \pi = \frac{(U_j \mid m_j)}{V_j} \pi = \text{integer}$ if $q^{l_0} \mid \pi$, i.e. for all but finitely many $\pi \in \mathcal{P}$. Therefore

$$\sum_{\pi \in \mathcal{P}} \left\| \frac{h_j f_j(\pi)}{m_j} \right\|^2 < \infty \quad (j = 1, \dots, k). \tag{4.9}$$

Let j be such an index for which $h_j \not\equiv 0 \pmod{m_j}$. From (4.9) we obtain that

$$(4.10) \quad h_j f_j(\pi) \equiv 0 \pmod{m_j}$$

for all but finitely many $\pi \in \mathcal{P}$. Then $(h_j, m_j) < m_j$, $\Delta_j := \frac{m_j}{(m_j, h_j)} > 1$.

(4.10) implies that

$$f_j(\pi) \equiv 0 \pmod{\Delta_j}$$

for all but finitely many $\pi \in \mathcal{P}$.

This contradicts to our assumption. The theorem is proved.

5. Kym proved that for $f \in \mathcal{A}_q$ $f(n) \bmod 1$ is distributed uniformly if and only if for every nonzero integer k either

$$\sum_{b=0}^{q-1} e(kf(bq^{l_k})) = 0$$

for some l_k , or

$$(5.1) \quad \sum_{\pi \in \mathcal{P}} \|kf(\pi)\|^2 = \infty.$$

Let $s_n^{(h)} := f(n \cdot q^h) \bmod 1$. From Kym's result we have: the sequences $\{s_n^{(h)}\}_{n=0}^{\infty}$ are distributed uniformly for every $h = 0, 1, 2, \dots$ if and only if (5.1) holds.

Let $l(n)$ be defined by (1.3). We are interested in the following question. Under what condition is true that $\{l(nq^j) \bmod 1\}_{n=0}^{\infty}$ is distributed uniformly for every j .

Theorem 7. *The sequences $\{l(n \cdot q^j) \bmod 1\}_{n=0}^{\infty}$ are mod 1 uniformly distributed for every j in each case, except when there is an integer $m \neq 0$, $\gamma_1, \dots, \gamma_k \in \mathbb{R}$, $l \in \mathbb{N}_0$ such that*

$$q^l(a_1\gamma_1 + \dots + a_k\gamma_k) \equiv 0 \pmod{1},$$

and

$$(5.2) \quad \sum_{\pi \in \mathcal{P}} \|mf_j(\pi) - \gamma_j\pi\|^2 < \infty \quad (j = 1, \dots, k)$$

hold.

The assertion is a direct consequence of Weyl’s theorem and our result in [2] which we quote now as Lemma 1.

Let $t(n) = g_1(a_1n) \dots g_k(a_kn)$, $t_j(n) = t(n \cdot q^j)$, where $g_\nu(n)$ are q -multiplicative functions, $|g_\nu(n)| = 1$ ($n \in \mathbb{N}_0$, $\nu = 1, \dots, k$). Assume furthermore that a_1, \dots, a_k satisfies the conditions stated in Section 1. Let

$$M_j(x) = \sum_{n < x} t_j(n), \quad m_j(N) = \frac{1}{q^N} M_j(q^N),$$

$$\alpha_j = \liminf |m_j(N)|, \quad \beta_j = \limsup |m_j(N)|.$$

Lemma 2. *Assume that $\beta_j > 0$ for some j . Then $\alpha_l = \beta_l \rightarrow 1$ as $l \rightarrow \infty$, furthermore there exist suitable real numbers $\gamma_1, \dots, \gamma_k$ and some $j_0 \geq 0$ such that*

$$q^{j_0}(\gamma_1 a_1 + \dots + \gamma_k a_k) \equiv 0 \pmod{1},$$

and in the notation $h_j(n) := e(-\gamma_j n)g_j(n)$,

$$\sum_{j=0}^{\infty} \sum_{c \in \mathcal{A}_q} \operatorname{Re}(1 - h_l(cq^j)) < \infty \quad (l = 1, \dots, k).$$

Proof of Theorem 7. Let $m \neq 0$, $g_\nu(n) = e(mf_\nu(n))$, $t(n) = g_1(a_1n) \dots g_k(a_kn)$.

By Weyl’s famous theorem $\{l(nq^j) \pmod{1}\}$ is distributed uniformly mod 1 for every j , if and only if $\frac{M_j(x)}{x} \rightarrow 0$ ($x \rightarrow \infty$) for every j , and every $m \neq 0$. If $\frac{M_j(x)}{x} \rightarrow 0$ ($x \rightarrow \infty$), then $\alpha_l = \beta_l = 0$ for every l and m . This proves the necessity of the conditions. Let us assume now that for some $m(\neq 0)$, $\beta_j = 0$ ($j = 0, 1, \dots$).

Let $q^N < x \leq q^{N+1}$. Consider the sequences $A^M = \{(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{M-1}), \varepsilon_\nu \in \mathbb{A}\}$. We classify them according to the following rule. We say that $(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{M-1}) \in \mathcal{B}_0$, if $\varepsilon_{M-1} = 0$, and that $(\varepsilon_0, \dots, \varepsilon_{M-1}) \in \mathcal{B}_h$, if $\varepsilon_{M-h-1} = 0$, and $\varepsilon_{M-h}, \varepsilon_{M-h+1}, \dots, \varepsilon_{M-1}$ are nonzero. Let finally \mathcal{B}^* be the set of those elements for which $\varepsilon_\nu \neq 0$ ($\nu = M-1, M-2, \dots, 0$).

Observe that $\#\mathcal{B} = q^{M-1}$, $\#\mathcal{B}_1 = q^{M-2}(1 - 1/q)$, $\#\mathcal{B}_2 = q^{M-1} \cdot (1 - 1/q)^2, \dots, \#\mathcal{B}_h = q^{M-1}(1 - 1/q)^h$, and $\#\mathcal{B}^* = q^M(1 - 1/q)^M$.

Let us write each $n < x$ as $n_0 + q^M u = n$, where $n_0 \in \{0, 1, \dots, q^M - 1\}$, $u \in \left\{0, \dots, \left[\frac{x}{q^M}\right]\right\}$. If $u < \left[\frac{x}{q^M}\right]$, then $n < x$ holds for each $n_0 \in \{0, 1, \dots, q^M - 1\}$. $u = \left[\frac{x}{q^M}\right]$ occurs for $O(q^M)$ distinct integers n .

Thus

$$\begin{aligned} M_j(x) &= \sum_{u=0}^{\left[\frac{x}{q^M}\right]-1} \sum_{n_0=0}^{q^M-1} t_j(n_0 + q^M u) + O(q^M) = \\ &= \sum_h \sum_{n_0 \in \tilde{\mathcal{B}}_h} t_j(n_0 + q^M u) + O(q^M) + O\left(x(1-1/q)^M\right). \end{aligned}$$

Here

$$\sum_h = \sum_{u=0}^{\left[x/q^M\right]-1} \sum_{n_0 \in \tilde{\mathcal{B}}_h} t_j(n_0 + q^M u),$$

where $\tilde{\mathcal{B}}_h$ is the set of those nonnegative integers $n_0 < q^M$ for which $(\varepsilon_0(n_0), \dots, \varepsilon_{M-1}(n_0))$ belongs to \mathcal{B}_h .

Let $n_0 \in \tilde{\mathcal{B}}_h$. Then $n_0 = \nu + q^{M-h}\mu$, where $0 \leq \nu < q^{M-h-1}$, $0 \leq \mu < q^h$, and each digit of μ differs from zero. The opposite assertion is true as well. Let $\nu \in [0, q^{M-h-1} - 1]$, and $\mu \in [0, q^h - 1]$ such that $\varepsilon_\nu(\mu) \neq 0$ ($\nu = 0, \dots, h-1$). Let us observe furthermore that

$$t_j(n_0 + q^M u) = t_j(\nu) t_j(\mu q^{M-h} + q^M u).$$

Thus

$$\sum_h = M_j(q^{M-h-1}) \sum_{\mu, u} t_j(\mu q^{M-h} + q^M u),$$

and so

$$(5.3) \quad \left| \sum_h \right| \leq |M_j(q^{M-h-1})| \frac{x}{q^M} (q-1)^h.$$

Thus

$$(5.4) \quad \left| \frac{1}{x} \sum_h \right| \leq m_j(M-h-1) (1-1/q)^h.$$

By using (5.4) for $h = 0, \dots, K$, and the trivial inequality

$$\left| \frac{1}{x} \sum_h \right| \leq (1-1/q)^h$$

for $h > K$, we obtain that

$$\left| \frac{M_j(x)}{x} \right| \leq \sum_{h=0}^K m_j(M-h-1) \left(1 - \frac{1}{q}\right)^h + \sum_{k>K+1} (1-1/q)^h + O\left(\frac{q^M}{x}\right) + O\left(\left(1 - \frac{1}{q}\right)^M\right).$$

We shall prove that $\limsup \left| \frac{M_j(x)}{x} \right| = 0$. Indeed, let $K > 0$ be fixed, $M = N - K$, $N = \left\lceil \frac{\log x}{\log q} \right\rceil$. Then for $x \rightarrow \infty$, $M \rightarrow \infty$, thus $m_j(M-h-1) \rightarrow 0$ as $x \rightarrow \infty$. Therefore

$$\limsup \left| \frac{M_j(x)}{x} \right| = O(q^{-K}) + O\left(\left(1 - \frac{1}{q}\right)^K\right).$$

Since the inequality holds for every K , therefore it holds for $K \rightarrow \infty$, consequently

$$\limsup \frac{|M_j(x)|}{x} = 0.$$

References

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