

## A CHARACTERIZATION OF SOME UNIMODULAR MULTIPLICATIVE FUNCTIONS II.

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*Dedicated to the memory of Professor Péter Kiss*

### 1. Introduction and results

Let  $\mathcal{N}$  denote the set of all positive integers. Let  $\mathcal{M}$  ( $\mathcal{M}^*$ ) be the set of complex-valued multiplicative (completely multiplicative) functions. A function  $g$  is said to be unimodular if  $g$  satisfies the condition  $|g(n)| = 1$  for all positive integers  $n$ . In the following we shall denote by  $\mathcal{M}(1)$  and  $\mathcal{M}^*(1)$  the class of all unimodular functions  $g \in \mathcal{M}$  and  $g \in \mathcal{M}^*$ , respectively.

More than 15 years ago I.Kátai stated as a conjecture that  $f \in \mathcal{M}$ ,  $\Delta f(n) = f(n+1) - f(n) = o(1)$  as  $n \rightarrow \infty$  imply that either  $f(n) = o(1)$  or  $f(n) = n^s$  ( $n \in \mathcal{N}$ ),  $0 \leq \text{Re } s < 1$ . This was proved by Wirsing in 1984 and some years later independently by Tang and Shao. The joint paper of Wirsing, Tang and Shao [5] contains two different proofs. It is not hard to deduce from Wirsing's theorem that if  $f, g \in \mathcal{M}$ ,  $g(n+1) - f(n) = o(1)$  as  $n \rightarrow \infty$ , then either  $f(n) = o(1)$ , or  $f(n) = g(n)$  ( $n \in \mathcal{N}$ ), and in the last case  $f(n) = n^s$  ( $n \in \mathcal{N}$ ),  $0 \leq \text{Re } s < 1$ .

Recently, improving these results we showed in [2] that if  $k \in \mathcal{N}$  is given and  $f, g \in \mathcal{M}$  satisfy the condition

$$g(n+k) - f(n) = o(1) \quad \text{as } n \rightarrow \infty,$$

then either  $f(n) = o(1)$  as  $n \rightarrow \infty$  or there are  $F, G \in \mathcal{M}$  and a complex constant  $s$  such that

$$f(n) = n^s F(n), \quad g(n) = n^s G(n), \quad 0 \leq \text{Re } s < 1$$

and  $G(n+k) = F(n)$  are satisfied for all  $n \in \mathcal{N}$ . In [1] the equation

$$G(n+k) = F(n)$$

for functions  $F, G \in \mathcal{M}$  is solved completely.

The general case concerning the characterization of those  $f, g \in \mathcal{M}$  for which

$$g(an + b) - Cf(An + B) = o(1) \quad \text{as } n \rightarrow \infty,$$

where  $a > 0$ ,  $b, A > 0$ ,  $B$  are fixed integers with  $Ab - aB \neq 0$  and  $C$  is a non-zero complex constant, seems to be a hard problem and a complete solution is not known.

For unimodular multiplicative functions there are few partial results concerning this problem. In [3] and [4] we obtained a generalization of E. Wirsing's theorem, namely we proved that if functions  $g_1 \in \mathcal{M}(1)$  and  $g_2 \in \mathcal{M}(1)$  satisfy the condition

$$g_1(an + b) - dg_2(cn) = o(1) \quad \text{as } n \rightarrow \infty$$

for some positive integers  $a, b, c$  and non-zero complex number  $d$ , then there are a real number  $\tau$  and functions  $G_1, G_2 \in \mathcal{M}(1)$  such that

$$g_1(n) = n^{i\tau}G_1(n), \quad g_2(n) = n^{i\tau}G_2(n), \quad G_1(an + b) - d\frac{c^{i\tau}}{a^{i\tau}}G_2(cn) = 0$$

for all positive integers  $n$ . Furthermore, it is shown in [4] that if  $a, b, c$  are positive integers and  $d$  is a non-zero complex number, then the functions  $g_1 \in \mathcal{M}(1)$  and  $g_2 \in \mathcal{M}(1)$  satisfy the condition

$$\sum_{n \leq x} \frac{1}{n} |g_1(an + b) - dg_2(cn)| = o(\log x) \quad \text{as } x \rightarrow \infty$$

if and only if there are functions  $g \in \mathcal{M}^*(1)$  and  $G_1, G_2 \in \mathcal{M}(1)$  such that

$$g_1(n) = g(n)G_1(n), \quad g_2(n) = g(n)G_2(n), \quad G_1(an + b) - d\frac{g(c)}{g(a)}G_2(cn) = 0$$

are satisfied for all positive integers  $n$ , furthermore

$$\sum_{n \leq x} \frac{1}{n} |g(n+1) - g(n)| = o(\log x) \quad \text{as } x \rightarrow \infty.$$

Trivially, this relation holds for functions of the type  $g(n) = n^{i\tau}$ , where  $\tau$  is a real number, and it has been conjectured that these are the only multiplicative functions of modulus 1 that satisfy the last relation. This conjecture remains open.

Our purpose in this paper is to prove the following

**Theorem 1.** *Let  $a > 0$ ,  $b$ ,  $A > 0$ ,  $B$  be integers with  $(a, A) = 1$ ,  $\Delta = Ab - aB \neq 0$  and let  $C$  be a non-zero complex number. If  $g \in \mathcal{M}^*(1)$  satisfies the condition*

$$(1) \quad \sum_{n \leq x} \frac{1}{n} |g(an + b) - Cg(An + B)| = o(\log x) \quad \text{as } x \rightarrow \infty,$$

then

$$(2) \quad \sum_{n \leq x} \frac{1}{n} |g(n+1) - g(n)| = o(\log x) \quad \text{as } x \rightarrow \infty.$$

**Theorem 2.** *Let  $a > 0$ ,  $b$ ,  $A > 0$ ,  $B$  be integers with  $(a, A) = 1$ ,  $\Delta = Ab - aB \neq 0$  and let  $C$  be a non-zero complex number. If  $g \in \mathcal{M}^*(1)$  satisfies the condition*

$$(3) \quad g(an + b) - Cg(An + B) = o(1) \quad \text{as } n \rightarrow \infty,$$

then there is a real-number  $\tau$  such that

$$(4) \quad g(n) = n^{i\tau} \quad \text{for all } n \in \mathbb{N}.$$

We note that if  $h(n)$  is a real-valued completely function, then  $f(n) := e^{2\pi i h(n)}$  is a complex-valued completely multiplicative function of modulus 1. By using the fact

$$\|x\| \ll |e^{2\pi i x} - 1| \ll \|x\|,$$

where  $\|x\|$  denotes the distance of a real number  $x$  to the nearest integer, the following corollary follows directly from Theorem 2

**Corollary.** *If a real-valued completely additive function  $h(n)$  satisfies*

$$\|h(an + b) - h(An + B) - D\| = o(1) \quad \text{as } n \rightarrow \infty$$

with integers  $a > 0$ ,  $b$ ,  $A > 0$ ,  $B$ ,  $(a, A) = 1$ ,  $\Delta = Ab - aB \neq 0$  and some real number  $D$ , then there exists a real number  $\tau$  for which

$$\|h(n) - \tau \log n\| = 0 \quad \text{for all } n \in \mathbb{N}.$$

## 2. Proof of Theorem 1

We may suppose without loss of generality that  $(A, B) = (a, b) = 1$ ,  $\Delta = Ab - aB > 0$ , and so by our assumption we have  $(a, A) = (a, \Delta) = (A, \Delta) = 1$ . Let

$$M := (aA + 1)^{\max(A-1, a-1)}.$$

First we prove that

$$(5) \quad \sum_{\substack{n \leq x \\ (n, A)=1}} \frac{1}{n} |g(Mn + A\Delta) - g(M)g(n)| = o(\log x)$$

and

$$(6) \quad \sum_{\substack{m \leq x \\ (m, a)=1}} \frac{1}{m} |g(Mm - a\Delta) - g(M)g(m)| = o(\log x)$$

as  $x \rightarrow \infty$ .

Let  $Q = aA + 1$  and  $k \in \mathbb{N}$ . We start from the relation

$$Q^k(an + b) = a \left( Q^k n + b \frac{Q^k - 1}{a} \right) + b,$$

whence by (1), we have

$$\sum_{n \leq x} \frac{1}{n} \left| g \left( A \left( Q^k n + b \frac{Q^k - 1}{a} \right) + B \right) - g(Q)^k g(An + B) \right| = o(\log x).$$

Consequently

$$(7) \quad \sum_{n \leq x} \frac{1}{n} \left| g \left( Q^k (An + B) + A\Delta \frac{Q^k - 1}{Q - 1} \right) - g(Q)^k g(An + B) \right| = o(\log x).$$

Repeating the argument that was used above, we also have

$$(8) \quad \sum_{m \leq x} \frac{1}{m} \left| g \left( Q^k (am + b) - a\Delta \frac{Q^k - 1}{Q - 1} \right) - g(Q)^k g(am + b) \right| = o(\log x)$$

as  $x \rightarrow \infty$ .

Since  $Q = aA + 1$ , therefore  $\frac{Q^{k-1}}{Q-1} \equiv k \pmod{A}$ . This shows that for each positive integer  $l$ ,  $(l, A) = 1$ , there is a positive integer  $k = k(l) < A$  such that  $lk(l) \equiv B \pmod{A}$ , therefore

$$\frac{Q^{k(l)} - 1}{Q - 1} l \equiv B \pmod{A}.$$

Hence, if we write  $k(l)$  in place of  $k$  and

$$\frac{Q^{k(l)} - 1}{Q - 1} t + \frac{1}{A} \left( \frac{Q^{k(l)} - 1}{Q - 1} l - B \right)$$

in place of  $n$ , then

$$An + B = \frac{Q^{k(l)} - 1}{Q - 1} (At + l),$$

and so we can write the condition (7) in the form

$$\sum_{t \leq x} \frac{1}{t} \left| g \left[ Q^{k(l)} (At + l) + A\Delta \right] - g(Q)^{k(l)} g(At + l) \right| = o(\log x).$$

Since

$$M = Q^{k(l)} Q^{\max(A-1, a-1)-k(l)} \quad \text{and} \quad Q^{\max(A-1, a-1)-k(l)} \equiv 1 \pmod{A},$$

we infer from the above relation that

$$\sum_{t \leq x} \frac{1}{t} |g[M(At + l) + A\Delta] - g(M)g(At + l)| = o(\log x)$$

holds for each positive integer  $l$ ,  $(l, A) = 1$ . Hence, (5) immediately follows.

By using (8), the proof of (6) is similar and we omit it. Thus, we have proved (5) and (6).

In the following, we denote by  $\mathcal{S}(A)$  the set of those positive integers which are products of the prime factors of  $A$ . Next, we prove that there is a positive integer  $P = P(a, A)$  such that

$$(9) \quad \sum_{\substack{n \leq x \\ (n, A)=1}} \frac{1}{n} |g(aPsn + \Delta) - g(aPs)g(n)| = o(\log x)$$

for all  $s \in \mathcal{S}(A)$ .

Let  $s \in \mathcal{S}(A)$ . Since  $(A, M) = (a, A) = 1$ , there are positive integers  $x_s, y_s$ , such that

$$Asx_s + a = My_s, \quad (x_sy_s, aA) = 1 \quad \text{and} \quad x_s < aAM.$$

Since  $x_s < aAM$  and  $(x_s, aA) = 1$  for all  $s \in \mathcal{S}(A)$ , we can define  $P$  as

$$P := \text{LCM}_{s \in \mathcal{S}(A)}[x_s].$$

Hence, if  $m = (asx_s n + \Delta)y_s$ , then we have

$$(aMy_s n + A\Delta)sx_s = Mm - a\Delta$$

and

$$(m, a) = ((asx_s n + \Delta)y_s, a) = 1 \quad \text{for all} \quad n \in \mathbb{N}.$$

Consequently, (5) and (6) imply

$$\sum_{\substack{n \leq x \\ (n, A) = 1}} \frac{1}{n} |g(asx_s n + \Delta) - g(asx_s)g(n)| = o(\log x),$$

which, using the definition of  $P$  and the fact  $(P, aA) = 1$ , proves (9). Thus, the relation (9) is proved.

Now we prove Theorem 1. We shall deduce from (9) that

$$(10) \quad \frac{1}{\log x} \sum_{n \leq x} \frac{1}{n} |g(aPn + \Delta) - g(aPn)| = o(1) \quad \text{as} \quad x \rightarrow \infty.$$

Let  $A = \pi_1^{\alpha_1} \cdot \dots \cdot \pi_r^{\alpha_r}$ ,  $r \geq 1$ ,  $A_1 = \pi_2^{\alpha_2} \cdot \dots \cdot \pi_r^{\alpha_r}$ . To show (10), we shall prove that we can reduce  $A$  to  $A_1$  in (9), i.e.

$$(11) \quad \frac{1}{\log x} \sum_{\substack{n \leq x \\ (n, A_1) = 1}} \frac{1}{n} |g(aPs_1 n + \Delta) - g(aPs_1)g(n)| = o(1)$$

for all  $s_1 \in \mathcal{S}(A_1)$ . Repeating this argument we conclude that (10) holds.

Let  $s_1 \in \mathcal{S}(A_1)$  and  $\pi = \pi_1$ . For each integer  $\gamma \geq 0$  let

$$S_\gamma(x) := \frac{1}{\log x} \sum_{\substack{n \leq x, \\ \pi^\gamma \parallel n}} \frac{1}{n} |g(aPs_1 n + \Delta) - g(aPs_1 n)|,$$

where  $\pi^\gamma \parallel n$  denotes that  $\pi^\gamma | n$  and  $(\frac{n}{\pi^\gamma}, \pi) = 1$ . One can get from (9) that  $S_\gamma(x) = o(1)$  as  $x \rightarrow \infty$  is satisfied for each  $\gamma \geq 0$ . This relation together with  $|g| \equiv 1$  implies that for each positive integer  $\mu$ , we have

$$\begin{aligned} & \frac{1}{\log x} \sum_{\substack{n \leq x \\ (n, A_1)=1}} \frac{1}{n} |g(aPs_1n + \Delta) - g(aPs_1)g(n)| \leq \\ & \leq \sum_{0 \leq j \leq \mu-1} S_j(x) + \frac{1}{\log x} \sum_{\substack{n \leq x \\ \pi^\mu | n}} \frac{2}{n} \ll o(\mu) + \frac{2}{\pi^\mu}, \end{aligned}$$

and so

$$\limsup_{x \rightarrow \infty} \frac{1}{\log x} \sum_{\substack{n \leq x \\ (n, A_1)=1}} \frac{1}{n} |g(aPs_1n + \Delta) - g(aPs_1)g(n)| \ll \pi^{-\mu}.$$

This with  $\mu \rightarrow \infty$  shows that

$$\frac{1}{\log x} \sum_{\substack{n \leq x \\ (n, A_1)=1}} \frac{1}{n} |g(aPs_1n + \Delta) - g(aPs_1)g(n)| = o(1) \quad \text{as } x \rightarrow \infty,$$

as asserted in (11). Thus, the proof of (10) is complete.

Finally, it is shown in [4, Lemma 2] that if  $g \in \mathcal{M}^*(1)$  satisfies the condition (10), then (2) holds, and so the proof of Theorem 1 is finished.

### 3. Proof of Theorem 2

Assume that the function  $g \in \mathcal{M}^*(1)$  satisfies the condition (3). We will go along a thought line similar to the proof of (10) to deduce that there is a positive integer  $P = P(a, A)$  such that

$$(12). \quad g(aPn + \Delta) - g(aPn) = o(1) \quad \text{as } n \rightarrow \infty.$$

Finally, it is shown in the proof of Theorem 3 [4] that if the function  $g \in \mathcal{M}^*(1)$  satisfies the condition (12), then

$$g(n + 1) - g(n) = o(1) \quad \text{as } n \rightarrow \infty.$$

This with Wirsing's theorem [5] implies that there is a real number  $\tau$  such that (4) is true. Thus, Theorem 2 is proved.

### References

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