ON THE SOLUTIONS OF $\sigma_2(n) = \sigma_2(n+\ell)$

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Abstract. For each integer $n \geq 1$, let $\sigma_2(n) = \sum_{d|n} d^2$. We show that if a famous conjecture of Schinzel is true, then $\sigma_2(n) = \sigma_2(n+2)$ has an infinite number of solutions. We also examine the solutions of the more general equation $\sigma_2(n) = \sigma_2(n+\ell)$, where ℓ is a fixed positive integer.

1. Introduction

For each integer $n \geq 1$, let $\sigma_2(n) = \sum_{d|n} d^2$. It is mentioned in the book of R.Guy [1], page 68, that Paul Erdős "doubts that

$$\sigma_2(n) = \sigma_2(n+2) \tag{1}$$

has infinitely many solutions". We shall show that if a famous conjecture of Schinzel often called *Hypothesis H* is true, then (1) has an infinite number of solutions. We will also show how to construct such an infinite family of solutions and provide all 24 solutions $< 10^9$.

We also study the more general equation

$$\sigma_2(n) = \sigma_2(n+\ell), \tag{2}$$

where ℓ is a fixed positive integer. In particular, we will show that if ℓ is odd, (2) has only a finite number of solutions, while if ℓ is even, a large family of solutions of (2) can be derived from those of (1) .

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2. The case ℓ odd

Given a positive odd integer ℓ, we will show that

$$\sigma_2(n) = \sigma_2(n+\ell) \tag{3}$$

has only a finite number of solutions, and in some cases none at all.

Actually we shall show that, given a fixed odd positive integer ℓ,

$$\sigma_2(n) < \sigma_2(n+\ell) \quad \text{if} \quad n \quad \text{is odd,} \tag{4}$$

$$\sigma_2(n) > \sigma_2(n+\ell) \quad \text{if} \quad n \quad \text{is even and large enough.} \tag{5}$$

First assume n is odd. Define the positive integer α implicitely by

$$n+\ell = 2^\alpha \cdot \frac{n+\ell}{2^\alpha} \quad \text{with} \quad \left(2^\alpha, \frac{n+\ell}{2^\alpha}\right) = 1. \tag{6}$$

The function $\sigma_2(n)$ being multiplicative, it follows from (6) that

$$\sigma_2(n+\ell) = \sigma_2(2^\alpha)\sigma_2\left(\frac{n+\ell}{2^\alpha}\right) = \frac{4^{\alpha+1}-1}{3}\sigma_2\left(\frac{n+\ell}{2^\alpha}\right) >$$

$$> \frac{4^{\alpha+1}-1}{3}\left(\frac{n+\ell}{2^\alpha}\right)^2 \geq \frac{5}{4}(n+\ell)^2.$$

On the other hand, since n has no even divisors,

$$\sigma_2(n) < n^2 + \frac{n^2}{3^2} + \frac{n^2}{5^2} + \ldots = \frac{\pi^2}{8}n^2.$$

Since $\frac{5}{4} > \frac{\pi^2}{8}$, inequality (4) holds indeed for all odd $n \geq 1$.

Assume now that n is even and choose α so that

$$n = 2^\alpha \cdot \frac{n}{2^\alpha} \quad \text{with} \quad \left(2^\alpha, \frac{n}{2^\alpha}\right) = 1.$$

Then

$$\sigma_2(n) = \sigma_2(2^\alpha)\sigma_2\left(\frac{n}{2^\alpha}\right) = \frac{4^{\alpha+1}-1}{3}\sigma_2\left(\frac{n}{2^\alpha}\right) > \frac{4^{\alpha+1}-1}{3}\left(\frac{n}{2^\alpha}\right)^2 \geq \frac{5}{4}n^2,$$

while

$$\sigma_2(n+\ell) < (n+\ell)^2 + \frac{(n+\ell)^2}{3^2} + \frac{(n+\ell)^2}{5^2} + \ldots = \frac{\pi^2}{8}(n+\ell)^2.$$

Since $\frac{5}{4} > \frac{\pi^2}{8}$ and ℓ is fixed, $\frac{5}{4}n^2 > \frac{\pi^2}{8}(n+\ell)^2$ if n is large enough, which proves (5).

Note that it is easy to show that $n = 6$ is the only solution of (3).

Using a computer one can easily check that (3) has no solution if $\ell = 3$, 9, 15, 27, 33, 35, 39, 45, 51, 57, 69, 75, 81, 87, 93 or 99.

On the other hand, it is easy to show that if $(\ell, 6) = (\ell, 7) = 1$, then $n = 6\ell$ is a solution of (3). Indeed, since $\sigma_2(6) = \sigma_2(7)$, we have

$$\sigma_2(n) = \sigma_2(6\ell) = \sigma_2(6)\sigma_2(\ell) = \sigma_2(7)\sigma_2(\ell) = \sigma_2(7\ell) = \sigma_2(n+\ell).$$

3. The case $\ell = 2$

We shall first look for odd solutions n of

$$\sigma_2(n) = \sigma_2(n+2) \tag{7}$$

which satisfy

$$n = pq, \qquad n+2 = rs, \tag{8}$$

where $p > q$ and $r < s$ are odd primes.

It follows from (7) and (8) that

$$\begin{aligned} 1 + p^2 + q^2 + n^2 &= 1 + r^2 + s^2 + (n+2)^2 \\ p^2 + q^2 &= r^2 + s^2 + 4n + 4 \\ p^2 + q^2 - 2pq &= r^2 + s^2 + 2pq + 4 \\ p - q &= r + s. \end{aligned}$$

Hence we shall look for distinct odd primes p, q, r, s such that

$$\begin{cases} p - q = r + s, & p > q, \\ pq + 2 = rs, & r < s. \end{cases} \tag{9}$$

For such primes, we must have $pq + 2 = r(p - q - r) = pr - (q + r)r$, from which it follows that $r > q$ and hence that

$$q < r < s. \tag{10}$$

Now, using $pq + 2 = rs$ and (10), we have

$$q(r + s + q) + 2 = rs \tag{11}$$

and therefore $2qs + q^2 + 2 > q(r + s + q) + 2 = rs$, from which we obtain $2qs + (q^2 + 2) - rs > 0$ and hence

$$r < 2q + \frac{q^2 + 2}{s} < 2q + \frac{q^2 + 2}{r}.$$

It follows that $r^2 - 2qr - (q^2 + 2) < 0$, which yields

$$q < r < q + \sqrt{2(q^2 + 1)}.$$

Hence if we set $\Delta = r - q$, we have, using (11),

$$s = \frac{(r + q)q + 2}{r - q} = \frac{(r + q)q + 2}{\Delta} = \frac{(2q + \Delta)q + 2}{\Delta} = q + \frac{q^2 + 1}{\Delta/2}.$$

First consider the case $\Delta = 2$. In this case,

$$s = q^2 + q + 1 \quad \text{and} \quad p = q + r + s = q + (q + 2) + (q^2 + q + 1) = q^2 + 3q + 3.$$

Hence, if we can find infinitely many q's such that

$$q, \quad q + 2, \quad q^2 + q + 1 \quad \text{and} \quad q^2 + 3q + 3 \qquad \text{are all primes,} \tag{12}$$

then equation (7) has infinitely many solutions. But it follows from the following conjecture of Schinzel that there exist infinitely many such quadruples of primes.

HYPOTHESIS H (A.Schinzel and W.Sierpinski [2]) *Let $k \geq 1$ and $f_1(x), \ldots, f_k(x)$ be irreducible polynomials with integer coefficients with positive leading coefficients. Assume that there exists no integer > 1 dividing the products $f_1(n) \ldots f_k(n)$ for all integers n. Then there exist infinitely many positive integers m such that all numbers $f_1(m), \ldots, f_k(m)$ are primes.*

If $q = 5$, then the prime quadruple $(5,\ 7,\ 31,\ 43)$ yields the solution $n = pq = 43 \cdot 5 = 215$. The next quadruple of the form (12) is $(1\,091,\ 1\,093,\ 1\,191\,373,\ 1\,193\,557)$ which provides the solution

$$n = pq = 1\,193\,557 \cdot 1\,091 = 1\,302\,170\,687.$$

But there are smaller solutions!

Small solutions $n = pq$ of (7) will be obtained if p and q are relatively small. Since

$$p = q + r + s, \qquad r = q + \Delta \qquad \text{and} \qquad s = q + \frac{q^2+1}{\Delta/2},$$

the size of p will be contained if s is not too large. Hence, searching for solutions of (7) using a computer, we need to consider those "admissible" values of Δ which are big enough to keep $2(q^2+1)/\Delta$ small, but not too big so that $r = q + \Delta$ remains small. This will be accomplished if $\Delta + \dfrac{2(q^2+1)}{\Delta}$ is as small as possible. Clearly this happens if $\Delta \approx q\sqrt{2}$. Moreover it is clear that Δ must also satisfy $q^2 \equiv -1 \pmod{\Delta/2}$ which sets the further restriction $\left(\dfrac{-1}{\Delta/2}\right) = 1$. It turns out that Δ is "admissible" if

$$\Delta = 2^\alpha \prod_{\substack{p^\beta \| \Delta \\ p \geq 5}} p^\beta \qquad (\alpha = 1, 2, \quad p \equiv 1 \pmod 4).$$

The first values of Δ are therefore 2, 10, 26, 34, 50, 122, 130, 202, ...

Besides the even solutions $n = 24$ and $n = 280$, we found, using the above algorithm, 78 solutions of $\sigma_2(n) = \sigma_2(n+2)$ below 10^{12}. Below, we give all 24 solutions smaller than 10^9.

We believe that, besides $n = 24$ and $n = 280$, all solutions of $\sigma_2(n) = \sigma_2(n+2)$ are of the type described in our algorithm, but we could not prove this.

4. The case of even $\ell \geq 4$

Let $\ell \geq 4$ be an even integer. It is clear that the method outlined in Section 3 produces all solutions of (7) of the form $n = pq$, where p and q are

24	=	$2^3 \cdot 3$
215	=	$5 \cdot 43$
280	=	$2^3 \cdot 5 \cdot 7$
1 079	=	$13 \cdot 83$
947 519	=	$163 \cdot 5\,813$
1 362 239	=	$467 \cdot 2\,917$
2 230 271	=	$463 \cdot 4\,817$
14 939 999	=	$1\,279 \cdot 11\,681$
19 720 007	=	$457 \cdot 43\,151$
32 509 439	=	$1\,783 \cdot 18\,233$
45 581 759	=	$827 \cdot 55\,117$
45 841 247	=	$607 \cdot 75\,521$

49 436 927	=	$2\,843 \cdot 17\,389$
78 436 511	=	$2\,503 \cdot 31\,337$
82 842 911	=	$2\,903 \cdot 28\,537$
101 014 631	=	$2\,473 \cdot 40\,847$
166 828 031	=	$4\,363 \cdot 38\,237$
225 622 151	=	$4\,217 \cdot 53\,503$
225 757 799	=	$2\,801 \cdot 80\,599$
250 999 559	=	$6\,553 \cdot 38\,303$
377 129 087	=	$6\,959 \cdot 54\,193$
554 998 751	=	$3\,727 \cdot 148\,913$
619 606 439	=	$6\,977 \cdot 88\,807$
846 765 431	=	$7\,853 \cdot 107\,827$

odd distinct primes. Let $n = pq$ be such a solution of $\sigma_2(n) = \sigma_2(n+2)$, but which also satisfies $(\ell, n) = (\ell, n+2) = 1$. Clearly such a solution exists. We claim that $m = \frac{\ell}{2}n = \frac{\ell}{2}pq$ is a solution of $\sigma_2(m) = \sigma_2(m+\ell)$. This follows immediately from

$$\sigma_2(m) = \sigma_2\left(\frac{\ell}{2}pq\right) = \sigma_2\left(\frac{\ell}{2}\right)\sigma_2(pq) = \sigma_2\left(\frac{\ell}{2}\right)\sigma_2(pq+2) =$$
$$= \sigma_2\left(\frac{\ell}{2}pq + \ell\right) = \sigma_2(m+\ell).$$

This shows in particular that $\sigma_2(n) = \sigma_2(n+\ell)$, with ℓ even, has an infinite number of solutions if $\sigma_2(n) = \sigma_2(n+2)$ has an infinite number of solutions.

This shows in particular that if $\sigma_2(n) = \sigma_2(n+2)$ has an infinite number of solutions, then so does $\sigma_2(n) = \sigma_2(n+\ell)$ for each even integer $\ell \geq 4$.

References

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[2] **Schinzel A. and Sierpinski W.**, Sur certaines hypothèses concernant les nombres premiers, *Acta Arith.*, **4** (1958), 185-208., *Corrigendum: ibid.* **5** (1959), 259.

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