Annales Univ. Sci. Budapest., Sect. Comp. 21 (2004) 41-56

INTERPOLATION AND QUADRATURE FORMULAE FOR RATIONAL SYSTEMS ON THE UNIT CIRCLE

Z. Szabó (Budapest, Hungary)

Abstract. Over the last years a general theory has been developed for the construction and analysis of rational orthonormal basis functions, often called generalized orthonormal basis functions in the engineering literature. These investigations motivate the interest in the examination of the approximation properties of the rational orthonormal systems generated by a given set of poles. These basis can be viewed as an extension of the trigonometric system on the unit circle, that corresponds to the special choice when all of the poles are located at the origin. The aim of this paper is to present a computationally effective quadrature method based on a rational interpolation formula on the unit circle. The paper provides a generalization of the Marcinkiewicz classical $\mathbb{L}^p$ norm convergence theorem of the trigonometric interpolation on equidistant nodes on the unit circle, see [23], to the rational interpolation process generated by the case where the underlying orthonormal basis is a rational one that contains the trigonometric basis as a special case.

1. Introduction

The first mention of rational orthonormal systems seems to have occured in the mid 20-th in the work of Takenaka and Malmquist [20, 11]. The context of this early work was application to approximation via interpolation. The wide ranging work of Walsh studied further the application of these bases for approximation on the unit disk and on the half plane [22].

Supported by FKFP under the grant 0198/1999 and OTKA T032719 which is gratefully acknowledged.

Although rational functions are to be considered in this paper, it has to be mentioned the role of the polynomials in the genesis of the ideas. The simplest way to see what it is meant by orthogonal rational functions is to consider them as generalizations of orthogonal polynomials [19, 7].

It is an important problem in numerical analysis to compute integrals. Most quadrature formulae approximate the integral by a weighted combination of the function values in certain nodes. It is known that interpolation processes lead in a natural way to such quadrature formulae. For the polynomial case, concerning convergence properties and domain of validity, an adequate selection of the nodes of the interpolation turns out to be fundamental. For the orthogonal polynomials on the unit circle p_n, also called Szegő polynomials, one can introduce the so called paraorthogonal polynomials q_n, defined as $q_n(z) = p_n(z) + wp_n^*(z)$, where $|w| = 1$ is fixed and $p_n^*(z) = z^n\overline{p}\left(\frac{1}{\overline{z}}\right)$ is the reciprocal polynomial. The zeros of q_n are simple and contained on $\mathbb{T}$. The Lagrange interpolation based on these zeros give rise to a Gauss quadrature formula, valid for all the Laurent polynomials of $\mathcal{P}_{\pm n}$, of degree less then n, [9]. This idea was extended to the case of rational orthonormal functions on the unit circle [2]. The drawback of these methods that it is needed to compute the zeros of recursively defined polynomials in order to obtain the set of desired interpolation nodes.

This paper is to present a more transparent and computationally much more attractive method to determine the interpolation nodes, hence the quadrature formula, if one restricts the problem to the classical situation on the unit circle, when the underlying measure is the Lebesgue measure.

In contrast to the algebraic polynomial interpolation, where a large number of different node systems were used, for the interpolation of functions defined on $\mathbb{T}$, most of the results make use only of the roots of unity as interpolation nodes, see e.g. [16]. The classical Erdős-Turán theorem tells us that if one considers the set u_{2n+1} of roots of unity and the symmetric interpolation polynomial $\mathcal{P}_n$ based on these nodes, then for every f continuous on $\mathbb{T}$ one has

$$\lim_{n\to\infty} \|f - \mathcal{P}_n f\|_2 = 0.$$

A similar result is true for the interpolating polynomial defined by the n-th roots of unity on the disc algebra $\mathbb{A}$. Moreover, it is true the mean convergence of the interpolation process on the roots of unity for $\mathbb{A}$, see e.g. [15] and [3] for the Marcinkiewicz type inequalities.

In [17] a generalization of the Marcinkiewicz-Zygmund inequalities and the classical $\mathbb{L}^p$ norm convergence theorem was given to the rational interpolation process generated by the case where the underlying system of nodes are defined by a periodic set of poles that determine the rational orthonormal basis. This paper extends these results to the general situation, i.e. for the nonperiodic case. Applications of the presented methods in system identification and control theory can be found in [13, 14, 18].

The structure of the presentation will be the following. First, the basic notations must be fixed. An important tool will be the monotone increasing, invertable and differentiable function $\beta_{(n)}$, for a finite Blaschke product B_n of order n defined as a mapping of the interval $[-\pi, \pi)$ onto itself, such that $B_n(e^{it}) = e^{in\beta_{(n)}(t)}$. After this, a rational interpolation operator with nodes on the unit circle will be defined and the properties of the quadrature formula induced by this operator will be investigated. Finally a Marcinkiewicz type theorem and a theorem for $\mathbb{L}^p$ norm convergence of these rational interpolation operators will be proved.

The paper is concluded with a small numerical example that illustrates the efficiency of the quadrature method.

2. Basic notations

In this chapter we are concerned with complex function theory on the unit circle. Let us denote by $\mathbb{R}$ the set of real numbers, by $\mathbb{C}$ the set of complex numbers and let $\mathbb{Z}$ be the set of integers. The open unit disc, its boundary will be denoted by

$$\mathbb{D} := \{z \in \mathbb{C} |\, |z| < 1\} \quad \text{and} \quad \mathbb{T} := \{z \in \mathbb{C} |\, |z| = 1\}.$$

Let us denote by $\mathcal{I}$ the integral mean operator on $\mathbb{T}$, i.e.

$$\mathcal{I}(f) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) dt.$$

By $\mathbb{L}^p,\ 1 < p < \infty$ will be denoted the classical $\mathbb{L}^p(\mathbb{T})$ Banach space endowed with the norm

$$\|f\|_p := \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{it})|^p dt \right)^{\frac{1}{p}}, \quad f \in \mathbb{L}^p,\ 1 < p < \infty,$$

and $\mathbb{L}^\infty$ is the Banach space of essentially bounded functions on the unit circle. The scalar product considered in $\mathbb{L}^2$ is the usual one, i.e.

$$\langle f, g\rangle := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})\overline{g}(e^{it})dt = \mathcal{I}(f\overline{g}), \qquad f, g \in \mathbb{L}^2.$$

$\mathbb{H}^2$ will be the Hardy space of square integrable functions on $\mathbb{T}$ whose continuation through the Cauchy integral formula on the unit disc is analytic. The disc algebra $\mathbb{A} = \mathbb{H}^\infty \cap C(\mathbb{T})$, where $C(\mathbb{T})$ is the space of continuous functions on the unit circle. For a classical introduction in $\mathbb{H}^p$ theory see e.g. [8, 12, 4]. A more advanced treatise of the topic can be found in [6].

If otherwise is not stated, throughout this paper it will be supposed $z \in \mathbb{T}$, i.e. $z := e^{it}$, $t \in \mathbb{R}$. Let B_n be a finite Blaschke product of order n written in the form

$$B_n := \prod_{j=1}^{n} b_{\alpha_j}, \quad \text{where} \quad b_{\alpha_j}(z) := \frac{z-\alpha_j}{1-\overline{\alpha}_j z}, \quad |\alpha_j| < 1.$$

For further usage let us introduce the notation

$$B_{\overline{n}}(z) := \overline{B}_n(z), \quad z \in \mathbb{T}.$$

If $\mathcal{P}_k$ denotes the space of polynomials of degree at most

$$k, \eta(z) := \prod_{i=1}^{n}(1-\overline{\alpha}_i z) \quad \text{and} \quad w(z) := \prod_{i=1}^{n}(z-\alpha_i),$$

then set

$$\mathcal{R}_n := \left\{\frac{p}{\eta} \mid p \in \mathcal{P}_{n-1}\right\}, \quad \mathcal{R}_{\overline{n}} := \left\{\frac{p}{w} \mid p \in \mathcal{P}_{n-1}\right\},$$

respectively. Accordingly, one can set

$$\mathcal{R}_{\pm n} := \left\{\frac{p}{\eta w} \mid p \in \mathcal{P}_{2n-1}\right\}.$$

If these sets include the constants, then they will be denoted by $\mathcal{R}_n^0$ and $\mathcal{R}_{\pm n}^0$. If $\varphi_j = \dfrac{d_j}{1-\overline{\alpha}_j z}$, where $d_j = \sqrt{1-|\alpha_j|^2}$, one has that the system

$$\Phi_n = \left\{\phi_j = \varphi_j B_{j-1}, B_{j-1} := \prod_{k=1}^{j-1} b\alpha_j \mid j = 1, \ldots, n\right\} \tag{1}$$

forms an orthonormal basis in $\mathcal{R}_n$. This is the so-called Takenaka-Malmquist system.

As a starting point, let us mention the fact that for a finite Blaschke product B_n of order n there exists a strictly monotone increasing and differentiable function $\beta_{(n)}$, mapping the interval $[-\pi, \pi)$ onto itself, see [13], so that

$$B_n(e^{it}) = e^{in\beta_{(n)}(t)}.$$

Denote by $\gamma_{(n)}$ the inverse function $\beta_{(n)}^{-1}$.

Let us denote the phase function of a single term by β_k, i.e. $b_{\alpha k}(e^{it}) = = e^{i\beta_k(t)}$, then the function $\beta_{(n)}$ can be expressed as

$$\beta_{(n)}(t) := \frac{1}{n}\sum_{k=1}^{n} \beta_k(t).$$

For the derivatives one has, see [14],

$$\beta_k'(t) = \frac{1-|\alpha_k|^2}{|1-\overline{\alpha}_k e^{it}|^2}. \tag{2}$$

Since $1-|\alpha_k| \le |1-\overline{\alpha}_k e^{it}| \le 1+|\alpha_k|$, one can obtain the bounds

$$\frac{1-|\alpha_k|}{1+|\alpha_k|} \le \beta_k'(t) \le \frac{1+|\alpha_k|}{1-|\alpha_k|},$$

and hence

$$\frac{1}{n}\sum_{k=1}^{n}\frac{1-|\alpha_k|}{1+|\alpha_k|} \le \beta_{(n)}'(t) \le \frac{1}{n}\sum_{k=1}^{n}\frac{1+|\alpha_k|}{1-|\alpha_k|}.$$

It follows that the derivative of the inverse is bounded by

$$\frac{n}{\sum\limits_{k=1}^{n}\frac{1+|\alpha_k|}{1-|\alpha_k|}} \le \gamma_{(n)}'(t) \le \frac{n}{\sum\limits_{k=1}^{n}\frac{1-|\alpha_k|}{1+|\alpha_k|}}.$$

If there is a constant $0<c<1$ such that $|\alpha_k|<c,\ k=1,\dots,n$, then one has the uniform bounds

$$\frac{1-c}{2} \le \beta_{(n)}'(t) \le \frac{2}{1-c} \quad \text{and} \quad \frac{1-c}{2} \le \gamma_{(n)}'(t) \le \frac{2}{1-c}.$$

The reproducing kernel $K : \mathbb{T} \times \mathbb{T} \to \mathbb{C}$ of a subspace $V \subset \mathbb{L}^2$ is defined by its reproducing property, i.e. $\forall f \in V \quad f(\mu) = \langle f, K(\cdot, \mu)\rangle, \ \mu \in \mathbb{T}$. If an orthonormal basis is considered in the finite dimensional subspace V say $\{\psi_j(z) \mid j = 1, \dots, n\}$, where $n = dimV$, then the reproducing or Dirichlet kernel is given by

$$K_n(z, \mu) = \sum_{k=1}^{n} \psi_k(z)\overline{\psi}_k(\mu), \quad z, \mu \in \mathbb{T},$$

and it is independent of the choice of the orthonormal system, see [1]. Applying this to the subspace $\mathcal{R}_n$ one can obtain $K_n(z, w) = \sum\limits_{k=1}^{n} \phi_k(z)\overline{\phi}_k(w)$ as a reproducing kernel, that can be expressed in a compact form by the following Christoffel-Darboux formula, see [5, 10].

Lemma 2.1.

$$K_n(z, w) := \sum_{k=1}^{n} \phi_k(z)\overline{\phi}_k(w) = \frac{1 - B_n(z)\overline{B}_n(w)}{1 - z\overline{w}}, \quad z, \mu \in \mathbb{D} \cup \mathbb{T}. \tag{3}$$

Using the expression for the derivative of the function $\beta_{(n)}$ derived from (1) and (2) one has

$$K_n(e^{it}, e^{it}) := \sum_{k=1}^{n} |\phi_k(e^{it})|^2 = n\beta'_{(n)}(t). \tag{4}$$

2.1. A discrete rational orthonormal system on the unit circle

Let us denote the set of equidistant nodes on the unit circle, i.e. the n-th roots of unity, by

$$\mathbb{U}_n = \left\{ e^{i\nu_k} \;\middle|\; \nu_k = \frac{2k\pi}{n}, \ k = 0, \dots, n-1 \right\}$$

and by

$$\mathbb{W}_n = \left\{ \zeta_k = e^{i\gamma_k} \;\middle|\; \gamma_k = \beta_{(n)}^{-1}(\nu_k), \ k = 0, \dots, n-1 \right\}$$

the image of the roots of unity through the $\gamma_{(n)} = \beta_{(n)}^{-1}$ function.

Considering as nodes the set $\mathbb{W}_n$, one can introduce the following rational interpolation operator

$$(5)\qquad (\mathcal{L}_n f)(z) := \sum_{\zeta\in\mathbb{W}_n} \frac{K_n(z,\zeta)}{K_n(\zeta,\zeta)} f(\zeta),$$

where f is a continuous function on $\mathbb{T}$ and $z \in \mathbb{T}$. Let us denote by

$$\mathbf{l}_{n,\zeta}(z) := \frac{K_n(z,\zeta)}{K_n(\zeta,\zeta)}, \quad \zeta \in \mathbb{W}_n.$$

From the definition of $\mathbb{W}_n$ and by (3) it follows that for $0 \le k, l < n,\ k \ne l$, one has

$$\mathbf{l}_{n,\zeta_k}(\zeta_l) = \frac{1 - e^{in(\beta_n(\gamma_k)-\beta_n(\gamma_l))}}{1-\zeta_l\overline{\zeta}_k} = \frac{1-e^{2\pi(k-l)}}{1-\zeta_l\overline{\zeta}_k} = 0.$$

Consequently, for $0 \le k, l < n$,

$$(6)\qquad \mathbf{l}_{n,\zeta_k}(\zeta_l) = \delta_{k,l},$$

i.e. $\mathbf{l}_{n,\zeta},\ \zeta \in \mathbb{W}_n$, are the Lagrange functions corresponding to the system $\{\phi_i, \mid i = 1,\ldots,n\}$.

This implies that $\mathcal{L}_n f$ interpolates f at the points of $\mathbb{W}_n$, i.e. $\mathcal{L}_n f(\zeta) = f(\zeta), \zeta \in \mathbb{W}_n$. It is also clear that $\mathcal{L}_n f = f$ for $f \in \mathcal{R}_n$, and $\{l_{n,\zeta} \mid \zeta \in \mathbb{W}_n\}$ is a basis in $\mathcal{R}_n$.

Let us define the discrete scalar product

$$[f,g]_n := \sum_{\zeta\in\mathbb{W}_n} \frac{f(\zeta)\overline{g}(\zeta)}{K_n(\zeta,\zeta)} = \sum_{\zeta\in\mathbb{W}_n} \frac{f(\zeta)\overline{g}(\zeta)}{n\beta'_{(n)}(\gamma)},$$

where $\zeta = e^{i\gamma}$.

For the classical case, i.e. when $\alpha_1 = \ldots = \alpha_n = 0$, the β function is the identity and this scalar product is exactly the discrete Fourier scalar product defined by

$$[f,g]_n := \frac{1}{n} \sum_{\zeta\in\mathbb{U}_n} f(\zeta)\overline{g}(\zeta).$$

Using this discrete scalar product the interpolation operator can be written as

$$(\mathcal{L}_n f)(z) = [f, K_n(\cdot, z)]_n$$

for $f \in \mathbb{A}$. Using this fact and by (6) follows that for $\zeta, \xi \in \mathbb{W}_n$ one has

$$\mathcal{I}(\mathbf{l}_{n,\zeta}\overline{\mathbf{l}_{n,\xi}}) = \delta_{\zeta,\xi}. \tag{7}$$

It is easy to see using the reproducing property of the kernel that

$$\langle \mathcal{L}_n f, \mathcal{L}_n g\rangle = [f, g]_n,$$

and follows that every orthonormal system $\{\psi_k \mid k = 1, \dots, n\}$ on the subspace defined by the reproducing kernel is also discrete orthonormal, i.e.

$$[\psi_k, \psi_l]_n = \delta_{k,l} \quad \text{for} \quad 1 \le k, l \le n.$$

Let us denote by $\mathbb{W}_n^0$ the set of nodes that corresponds to zB_{n-1} and by K_n^0 and $\mathcal{L}_n^0$ the corresponding kernel interpolation operator, respectively.

2.2. A quadrature formula for rational functions on the unit circle

Using the interpolation operator $\mathcal{L}_n$, see (5), one can introduce a quadrature formula as

$$\mathcal{I}_n(f) := \sum_{\zeta \in \mathbb{W}_n} \rho_\zeta^{(n)} f(\zeta), \quad \text{where} \quad \rho_\zeta^{(n)} := \mathcal{I}\left(\frac{K_n(\cdot, \zeta)}{K_n(\zeta,\zeta)}\right), \tag{8}$$

where $\mathcal{I}$ denotes the integral mean operator on $\mathbb{T}$. Then it is clear that $\mathcal{I}_n(f) = \mathcal{I}(f)$ for all $f \in \mathcal{R}_n$.

To get $\rho_\zeta^{(n)}$ it have been used the fact that for any $g \in \mathbb{A}$ one has $\mathcal{I}(g) = g(0)$. Thus by (3) for $\zeta \in \mathbb{W}_n$ one has

$$\mathcal{I}(K_n(\cdot,\zeta)) = \sum_{k=1}^{n} \mathcal{I}(\phi_k)\overline{\phi}_k(\zeta) = \sum_{k=1}^{n} \phi_k(0)\overline{\phi}_k(\zeta) = 1 - B_n(0)\overline{B}_n(\zeta) = 1 - B_n(0).$$

Consequently by (8) the coefficients of the quadrature formula are of the form

$$\rho_\zeta^{(n)} := \frac{1 - B_n(0)}{K_n(\zeta,\zeta)} = \frac{1 - B_n(0)}{n\beta'_{(n)}(\gamma)}, \quad \text{where} \quad \zeta = e^{i\gamma}. \tag{9}$$

If one of the zeros of the Blaschke product is zero, say, $\alpha_n = 0$, hence $b_{\alpha_n}(z) = z$, one has $B_n(0) = 0$. Follows that in this case the coefficients of the quadrature formula

$$\rho_\zeta^{(n)} = \frac{1}{n\beta'_{(n)}(\gamma)} > 0, \quad \text{where} \quad \zeta = e^{i\gamma},$$

are positive.

For every $g \in \mathcal{R}_{\overline{n}}$ one has $\mathcal{I}(g) = 0$, and $g = hB_{\overline{n}}$ for some $h \in \mathcal{R}_n$. It follows, that $\mathcal{I}_n(g) = \mathcal{I}_n(h) = \mathcal{I}(h) = h(0)$, i.e. in general one cannot expect $\mathcal{I}(g) = \mathcal{I}_n(g)$. But $\mathcal{I}(g) = \mathcal{I}_n(g)$ if $g \in \mathcal{R}_{\overline{n}} \cap z\mathcal{R}_{\overline{n}}$.

One can obtain a completely analogous result as for the polynomial case, if one chose the quadrature formula induced by $\mathcal{L}_n^0$, based on the interpolation nodes $\mathbb{W}_n^0$.

Theorem 2.2. *Let us introduce the Gauss type quadrature formula*

$$\mathcal{I}_n^0(f) := \sum_{\zeta \in \mathbb{W}_n^0} \frac{f(\zeta)}{K_n^0(\zeta,\zeta)},$$

then $\mathcal{I}_n^0(f) = \mathcal{I}(f)$ *for all* $f \in \mathcal{R}_{\pm n}^0$.

Proof. In what follows, we make explicit the argument above. By

$$\mathcal{I}\left(\frac{K_n^0(\cdot,\zeta)}{K_n^0(\zeta,\zeta)}\right) = \frac{1}{K_n^0(\zeta,\zeta)}$$

one has $\mathcal{I}(\mathcal{L}_n^0 f) = \mathcal{I}_n^0(f)$, i.e. $\mathcal{I}_n^0(f) = \mathcal{I}(f)$ for all $f \in \mathcal{R}_n^0$.

One can consider that $\alpha_n = 0$, i.e. $\mathcal{R}_n^0 = \mathcal{R}_{n-1} \oplus \mathbb{C}$. Let us consider a basis $\{\phi_k\}$ in $\mathcal{R}_{n-1}$, then $\{\phi_k B_{\overline{n-1}}\}$ is a basis in $\mathcal{R}_{\overline{n-1}}$. Using the fact that $\zeta B_{n-1}(\zeta) = 1$ for $\zeta \in \mathbb{W}_n^0$, i.e. $\overline{\zeta} B_{\overline{n-1}}(\zeta) = 1$, follows

$$\mathcal{I}_n^0(\phi_k B_{\overline{n-1}}) = \sum_{\zeta \in \mathbb{W}_n^0} \frac{\phi_k(\zeta) B_{\overline{n-1}}(\zeta)}{K_n^0(\zeta,\zeta)} = \sum_{\zeta \in \mathbb{W}_n^0} \frac{\zeta\phi_k(\zeta)}{K_n^0(\zeta,\zeta)} = \mathcal{I}_n^0(z\phi_k) = \mathcal{I}(z\phi_k) = 0.$$

It follows that $\mathcal{I}_n^0(f) = \mathcal{I}(f)$ for all $f \in \mathcal{R}_{\pm n}^0$.

Since $1 \in \mathcal{R}_{\pm n}^0$, one has

$$\sum_{\zeta \in \mathbb{W}_n^0} \frac{1}{K_n^0(\zeta,\zeta)} = 1.$$

One can show that in fact the quadrature formula introduced above has a maximal domain of validity.

Theorem 2.3. *Let us consider an n-point quadrature formula of interpolatory type, i.e. a set of distinct nodes $\mathbf{X}_n = \{\xi_i \in \mathbb{T},\ i = 1, 2, \dots, n\}$ and $\mathcal{I}_n f := := \sum_{\xi \in X_n} \kappa_\xi f(\xi)$. Then, there does not exist a formula that has $\mathcal{R}_{\pm n}$ as a domain of validity.*

Proof. The proof follows the ideas presented in [9] for the polynomial case. Let us consider $w_n = \prod_{\xi \in \mathbf{X}_n} (z - \xi)$ and $r(z) := \dfrac{w_n(z)}{\eta_n} \in \mathbb{H}^2$. For every $f \in \mathcal{R}_n$, one has $f\overline{r} \in \mathcal{R}_{\pm n}$. Suppose now that the quadrature formula is true for $\mathcal{R}_{\pm n}$. It follows, that $\mathcal{I}(f\overline{r}) = \langle f, r\rangle = \mathcal{I}_n(fr) = 0$, i.e. r is orthogonal to $\mathbb{R}_n$. Therefore $r \in B_n\mathbb{H}^2$, a contradiction.

Let us return for a moment to the quadrature formula induced by $\mathcal{L}_n$. Then one has

Theorem 2.4. *For every $n \in \mathbb{N},\ n \geq 2$,*

$$\sum_{\zeta \in \mathbb{W}_n} \frac{1}{K_n(\zeta, \zeta)} = \frac{1 - |B_n(0)|^2}{|1 - B_n(0)|^2} \tag{10}$$

and consequently for the norm of the functionals $\mathcal{I}_n$ one has

$$\|\mathcal{I}_n\| = \sum_{\zeta \in \mathbb{W}_n} \left|\rho_\zeta^{(n)}\right| < \frac{2}{1 - |\alpha_1|}. \tag{11}$$

Moreover, if $\sum_{k=1}^{\infty} (1 - |\alpha_k|) = \infty$, then

$$\lim_{n\to\infty} \sum_{\zeta \in \mathbb{W}_n} \frac{1}{K_n(\zeta, \zeta)} = \lim_{n\to\infty} \sum_{\zeta \in \mathbb{W}_n} \left|\rho_\zeta^{(n)}\right| = 1. \tag{12}$$

Proof. Let us consider the function $g \in \mathcal{R}_n$ defined as

$$g(z) := \sum_{\zeta \in \mathbb{W}_n} \frac{K_n(z, \zeta)}{K_n(\zeta, \zeta)}.$$

Then, it is clear that $\mathcal{I}_n(g) = \mathcal{I}(g)$, i.e. by (8) and (9)

$$\begin{aligned}(1 - B_n(0)) \sum_{\zeta \in \mathbb{W}_n} \frac{1}{K_n(\zeta, \zeta)} &= \sum_{k=1}^{n} \mathcal{I}(\phi_k) \overline{\sum_{\zeta \in \mathbb{W}_n} \frac{\phi_k(\zeta)}{K_n(\zeta, \zeta)}} = \\ &= \frac{1}{1 - \overline{B_n(0)}} \sum_{k=1}^{n} \mathcal{I}(\phi_k)\overline{\mathcal{I}(\phi_k)} = \frac{K_n(0, 0)}{1 - \overline{B_n(0)}},\end{aligned}$$

and (10) is proved. To see (11) observe that $|B_n(0)| = \prod_{k=1}^{n} |\alpha_k| < |\alpha_1| < 1$, and consequently by (10) one has

$$\sum_{\zeta\in\mathbb{W}_n} \left|\rho_\zeta^{(n)}\right| = (1-|B_n(0)|)\sum_{\zeta\in\mathbb{W}_n} \frac{1}{K_n(\zeta,\zeta)} = \\ = \frac{1-|B_n(0)|^2}{|1-B_n(0)|} \le \frac{2}{1-|B_n(0)|} \le \frac{2}{1-|\alpha_1|},$$

and (11) is proved. It is known that $\sum_{k=1}^{\infty}(1-|\alpha_k|) = \infty$, implies $\lim_{n\to\infty} |B_n(0)| = = \lim_{n\to\infty} \prod_{k=1}^{n} |\alpha_k| = 0$, see [12], and (12) follows from (10).

As a corollary one has

Theorem 2.5. *If* $\sum_{k=1}^{\infty}(1-|\alpha_k|) = \infty$, *then for every* $f \in \mathbb{A}$ *one has*

$$\lim_{n\to\infty} \mathcal{I}_n(f) = \mathcal{I}(f).$$

Proof. Since $\sum_{k=1}^{\infty}(1-|\alpha_k|) = \infty$, then the orthonormal system is complete and using (11) by a usual density argument follows the assertion.

Based on this result, one has the following generalization of the Erdős-Turán theorem for $\mathcal{L}_n$ on $\mathbb{A}$.

Theorem 2.6. *Consider the interpolation operator*

$$(\mathcal{L}_n f)(z) = \sum_{\zeta\in\mathbb{W}_n} \frac{K_n(z,\zeta)}{K_n(\zeta,\zeta)} f(\zeta).$$

If $\sum_{k=1}^{\infty}(1-|\alpha_k|) = \infty$, *then for every* $f \in \mathbb{A}$, *one has*

$$\lim_{n\to\infty} \|f - \mathcal{L}_n f\|_2 = 0.$$

Proof. The proof follows the method of [21]. Let us consider the function $g_n \in \mathcal{R}_n$ as the best uniform approximant of f, and denote by $e_n := f - g_n$ and by $E_n(f) := \|f - g_n\|_\infty$. It is known that $\lim_{n\to\infty} E_n(f) = 0$. Then

$$\|f-\mathcal{L}_n f\|_2 \le \|f-g_n\|_2 + \|\mathcal{L}_n f - \mathcal{L}_n g_n\|_2 \le E_n(f) + \mathcal{I}\left(\left|\sum_{\zeta\in\mathbb{W}_n} e_n(\zeta)\mathbf{l}_{n,\zeta}\right|^2\right)^{\frac{1}{2}}.$$

Since for $\zeta, \xi \in \mathbb{W}_n$ one has $\mathcal{I}(\mathbf{l}_{n,\zeta}\overline{\mathbf{l}_{n,\xi}}) = 0$ if $\zeta \neq \xi$, see (7), the assertion follows by

$$\|f - \mathcal{L}_n f\|_2 \leq E_n(f)\left(1 + \left(\sum_{\zeta \in \mathbb{W}_n} \frac{1}{K_n(\zeta,\zeta)}\right)^{\frac{1}{2}}\right) \leq CE_n(f).$$

2.3. $\mathbb{L}^p$ norm convergence of certain rational interpolation operators on the unit circle

In what follows an extension of the Marcinkiewicz-Zygmund type inequalities will be given for the interpolation operator $\mathcal{L}_n$ on $\mathbb{A}$. Based on this result the mean convergence of this interpolation operator will be proved.

Theorem 2.7. *Let $f \in \mathcal{R}_n$ and $1 - |\alpha_k| > \delta > 0,\ k \in \mathbb{N}$. Then there exist constants $C_1, C_2 > 0$ depending only on p, such that for $1 < p < \infty$ one has*

$$C_1\|f\|_p \leq [\mathcal{I}_n(|f|^p)]^{\frac{1}{p}} \leq C_2\|f\|_p.$$

Proof. For the first part of the assertion let us consider the identity

$$f(z)K_n(z,\zeta) = \langle fK_n(\cdot,\zeta), K_n(\cdot,z) + B_n(\cdot)\overline{B}_n(z)K_n(\cdot,z)\rangle$$

for $f \in \mathcal{R}_n$. Let us introduce the kernel

$$T_n(z,w) = \frac{|K_n(z,w)|^2}{K_n(w,w)}.$$

It follows that

$$|f(\zeta)| \leq 2\langle |f|, T_n(\cdot,\zeta)\rangle.$$

Since $T_n > 0$ and $\mathcal{I}(T_n(\cdot,\zeta)) = 1$ by the Jensen inequality, see e.g. [12], one has

$$|f(\zeta)|^p \leq 2^p\langle |f|^p, T_n(\cdot,\zeta)\rangle.$$

Using the fact that $\displaystyle\sum_{\zeta \in \mathbb{W}_n} \frac{|K_n(\zeta,w)|}{K_n(\zeta,\zeta)} = K_n(w,w)$ follows

$$\sum_{\zeta \in \mathbb{W}_n} |f(\zeta)|^p \leq 2^p\left\langle |f|^p, \frac{\max K_n(\cdot,\cdot)}{\min K_n(\zeta,\zeta)}\right\rangle,$$

and by $\dfrac{\delta}{2} \le K_n(w,w) \le \dfrac{2}{\delta}$, one has

$$[\mathcal{I}_n(|f|^p)]^{\frac{1}{p}} \le C\|f\|_p.$$

The proof of the second part uses the fact that for $f \in \mathbb{L}^p$ exists $g \in \in \mathbb{L}^q$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$, such that $\|g\|_q = 1$ and $\|f\|_p = \langle f, g\rangle$. Using the Hölder inequality and the previous result one can obtain

$$\begin{aligned}\|f\|_p = \langle f, g\rangle = \langle f, S_n g\rangle = [f, S_n g]_n^\beta \le \\ \le [\mathcal{I}_n(|f|^p)]^{\frac{1}{p}}[\mathcal{I}_n(|S_n g|^q)]^{\frac{1}{q}} \le C_2[\mathcal{I}_n(|f|^p)]^{\frac{1}{p}},\end{aligned}$$

where we have used the fact that $\|S_n g\|_q \le C\|g\|_q$.

One can observe that for the case when $B(z) = z$ one can obtain the classical Marcinkiewicz theorems.

By using these results one can prove the following mean convergence theorem.

Theorem 2.8. *If $1 - |\alpha_k| > \delta > 0$, $k \in \mathbb{N}$, then for every $f \in \mathbb{A}$ and $1 < p < \infty$ one has*

$$\|f - L_n f\|_p \le C E_n(f),$$

and consequently,

$$\lim_{n\to\infty} \|f - \mathcal{L}_n f\|_p = 0.$$

Proof. The proof is almost the same as the one for the Erdős-Turán theorem. Let us consider the function $g_n \in \mathcal{R}_n$ as the best uniform approximant of f, and denote by $e_n := f - g_n$. Then by Theorem 2.7

$$\begin{aligned}\|f - \mathcal{L}_n f\|_p \le \|f - g_n\|_p + \|\mathcal{L}_n f - \mathcal{L}_n g_n\|_p \le E_n(f) + C_2[\mathcal{I}(|e_n|^p)]^{\frac{1}{p}} \le \\ \le E_n(f)\left(1 + C_2\left(\sum_{\zeta\in\mathbb{W}_n} \frac{1}{K_n(\zeta,\zeta)}\right)^{\frac{1}{p}}\right) \le C E_n(f),\end{aligned}$$

that is exactly the assertion of the theorem.

3. A numerical example

In this section a small numerical example is given to illustrate how the accuracy of the quadrature formula depends on the location of the poles. Let us consider the analytic function $f(z) = \frac{1}{z - 0.9}$, whose integral on $\mathbb{T}$ is to be computed. In the simulations the following poles were used to generate the orthonormal basis: $\{0\}, \{0.3\}, \{0.5\}, \{0.7\}, \{0.8\}, \{0.9\}$, and the number of points n was chosen to be $\{5, 10, 25, 50, 100\}$. Table 1 summarizes the results of the numerical experiments:

	5	10	25	50	100
0	1.6022	0.5948	0.0859	0.0058	0.0000
0.3	0.9878	0.3161	0.0263	0.0006	0.0000
0.5	0.5900	0.1517	0.0056	0.0000	0.0000
0.7	0.2247	0.0311	0.0001	0.0000	0.0000
0.8	0.0838	0.0038	0.0000	0.0000	0.0000
0.9	0.0577	0.0000	0.0000	0.0000	0.0000

Table 1. Errors for different poles

References

[1] **Aronszajn N.**, Theory of reproducing kernels, *Transactions of the American Math. Soc.*, (1950), 337-404.

[2] **Bultheel A., González-Vera P., Hendriksen E. and Njåstad O.**, *Orthogonal rational functions,* Cambridge Monographs on Applied and Computational Mathematics, Cambridge, 1994.

[3] **Chui C.K. and Zhong L.**, Polynomial interpolation and Marcinkiewicz-Zygmund inequalities on the unit circle, *J of Math. Anal. and Applications,* **233** (1999), 387-405.

[4] **Duren P.L.**, *Theory of $\mathbb{H}^p$ spaces,* Academic Press, 1970.

[5] **Džrbašian M.M.**, Translation of the Russian paper "Orthogonal systems of rational functions on the unit circle", Technical report by K. Müller and A. Bultheel TW253, Department of Computer Science, K.U. Leuven, 1997.

[6] **Garnett J.B.,** *Bounded analytic functions,* Academic Press, 1981.

[7] **Grenander U. and Szegő G.,**: *Toeplitz forms and their applications.* University of California Press, 1958.

[8] **Hoffman K.,** *Banach spaces of analytic functions,* Prentice-Hall, New Jersey, 1962.

[9] **Jones W.B., Njåstad O. and Thron W.J.,** Moment theory, orthonogal polynomials, quadrature and continued fractions associated with the unit circle, *Bull. London Math. Soc.,* **21** (1989), 113-152.

[10] **Lorentz G.G., v.Golitschek M. and Makovoz Y.,** *Constructive approximation,* Springer, 1996.

[11] **Malmquist F.,** Sur la détermination d'une classe de fonctions analytiques par leurs valeurs dans un ensemble donné de points, *Compte Rendus de Sixième Congrés des Mathématiciens Scandinaves, Copenhague, Denmark, 1925,* 253-259.

[12] **Rudin W.,** *Real and complex analysis,* McGraw-Hill, New York, 1966.

[13] **Schipp F., Gianone L., Bokor J. and Szabó Z.,** Identification in generalized orthonormal basis - a frequency domain approach, *Proc. IFAC World Congress, San Francisco, 1996*, 387-392.

[14] **Schipp F. and Bokor J.,** Approximate identification in Laguerre and Kautz bases, *Automatica,* **34** (1998), 463-468.

[15] **Sharma A. and Vértesi P.,** Mean convergence and interpolation on the roots of unity, *SIAM J. Math. Anal.,* **14** (1983), 800-806.

[16] **Szabados J. and Vértesi P.,** A survey on mean convergence of interpolatory processes, *J. of Comp. and Appl. Math.,* **43** (1992), 3-18.

[17] **Szabó Z.,** $\mathbb{L}^p$ norm convergence of rational operators on the unit circle, *Mathematica Pannonica,* **9** (1998), 281-292.

[18] **Szabó Z., Bokor J. and Schipp F.,** Identification of rational approximate models in $\mathbb{H}^\infty$ using generalized orthonormal basis, *IEEE Trans. Autom. Control,* **44** (1999), 153-154.

[19] **Szegő G.,** *Orthogonal polynomials,* AMS Colloquium Publications **XXIII**, AMS, 1939.

[20] **Takenaka S.,** On the orthogonal functions and a new formula of interpolations, *Japanese J. of Mathematics,* **2** (1925), 129-145.

[21] **Walsh J. and Sharma A.,** Least squares and interpolation on the roots of unity, *Pacific J. Math.,* **14** (1964), 727-730.

[22] **Walsh J.,** *Interpolation and approximation by rational functions in the complex domain,* AMS Colloquium Publications **XX**, AMS, 1935.

[23] **Zygmund A.,** *Trigonometric series, Vol. 1,2,* Cambridge Univ. Press, Cambridge, 1977.

(Received February 20, 2001)

Z. Szabó
Computer and Automation Research Institute
Hungarian Academy of Sciences
Kende u. 13-17.
H-1502 Budapest, Hungary
szaboz@decst.scl.sztaki.hu