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THE SOLUTION OF LINEAR PROGRAMMING PROBLEMS WITH QUASI-TRIANGULAR FUZZY NUMBERS IN CAPACITY VECTOR

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Abstract. We propose a new solution concept for fuzzy linear programming problems. This is similar to the concept proposed by J.J. Buckley in [3], but is differs from that, because in this paper we define the feasible set and solution set by using the degree of possibility. First we define this concept and obtain the basic properties of the new solution set and the fuzzy optimal value of the objective function. After that we present in details a method to resolve linear programming problems with quasi-triangular fuzzy numbers in capacity vector. Finally we apply this method for a special problem.

1. Introduction

In a lot of practical problems, the quantities may only be uncertainly estimated. If these quantities are coefficients of linear programming problems, then they are possible to be characterized with fuzzy numbers. That linear programming problem, in which at least one coefficient is a fuzzy number, we call a fuzzy linear programming problem. Optimal solutions and optimal value of this problem was formulated by J.J. Buckley in [3].

In practice, when one or more coefficients have uncertain values, then the optimal value will be uncertain. As a conclusion, the optimal value of fuzzy linear programming problem has to be a fuzzy quantity. In order to reach the α-level of optimal value we must make an optimal decision. This decision must be exact. Therefore the set of α-optimal solution of a fuzzy linear programming problem contains vectors of real numbers.

In the present paper we define a new optimal solution concept and new optimal value concept of fuzzy linear programming problems, such that they fulfil the above expectations. In the third section we present the elementary properties of these concepts, if the coefficients of fuzzy linear programming problems are fuzzy numbers or quasi-triangular fuzzy numbers. This concept of quasi-triangular fuzzy numbers was introduced by M. Kovács in [9]. In the fourth section is presented a solving method of fuzzy linear programming problems with quasi-triangular fuzzy numbers in the capacity vector. In the last section we resolve an example of linear programming problem with quasi-triangular fuzzy numbers in the capacity vector.

2. Preliminaries

In this section we collect those definitions and basic propositions which will be needed in the present paper.

2.1. Fuzzy number

Definition 1. Let be X a set. A mapping $\mu : X \to [0,1]$ is called membership function, and the set $A = \{(x, \mu(x)) \;/\; x \in X\}$ is called fuzzy set on X. The membership function of A is denoted by μ_A. The collection of all fuzzy sets on X is denoted by $\mathcal{F}(X)$.

Definition 2. The support of A is the subset of X given by

$$\text{supp } A = \{x \in X \;/\; \mu_A(x) > 0\}. \tag{1}$$

Definition 3. The α-cut of A is

$$[A]^\alpha = \begin{cases} \{x \in X \;/\; \mu_A(x) \geq \alpha\}, & \text{if } \alpha > 0, \\ cl(\text{supp } A), & \text{if } \alpha = 0, \end{cases} \tag{2}$$

where $cl(\text{supp } A)$ is closure of the support of A. The height of A is

$$hgt(A) = \sup_{x \in X} \mu_A(x). \tag{3}$$

Definition 4. A fuzzy set A on X is convex, if all α-cuts are convex subsets of X, and it is normal if $[A]^1 \neq \emptyset$.

Definition 5. A convex, normal fuzzy set on the real line R with upper semicontinuous membership function will be called fuzzy number. The set of all fuzzy numbers will be denoted by $\mathcal{N}$.

Let $g : [0,1] \to [0,\infty]$ be a continuous, strictly decreasing function with the boundary properties $g(1) = 0$ and $\lim_{t\to 0} g(t) = g_0 \leq \infty$. The concept of quasi-triangular fuzzy numbers generated by g was introduced first by M. Kovács [9] and defined, as follows.

Definition 6. If λ and d are real numbers, with $d > 0$, let $\mathcal{F}_g$ the subset of fuzzy numbers with the membership function

$$\mu(t) = \begin{cases} g^{(-1)}\left(\frac{|t-\lambda|}{d}\right), & \text{if } d > 0, \\ \chi_{\{\lambda\}}(t), & \text{if } d = 0 \end{cases} \tag{4}$$

for all $t \in R$, where

$$g^{(-1)}(t) = \begin{cases} g^{-1}(t), & \text{if } t \in [0, g(0)), \\ 0, & \text{if } t \geq g(0) = g_0. \end{cases} \tag{5}$$

Here and in the following χ_A denotes the characteristic function of the set A. The elements of $\mathcal{F}_g$ will be called quasi-triangular fuzzy numbers generated by g with center λ and spread d and we will denote them with (λ, d). The following results are given in [7], [8], [9].

Proposition 1. *Let A be a fuzzy number. Let us introduce the lower and upper bound functions $a_1, a_2 : [0,1] \to R \cup \{-\infty, +\infty\}$ of its α-cuts, namely $a_1(\alpha) = \min[A]^\alpha$ and $a_2(\alpha) = \max[A]^\alpha$. Then (i) $a_1(\alpha) \leq a_2(\alpha)$ for all $\alpha \in [0,1]$; (ii) a_1 is increasing and a_2 is decreasing functions; (iii) a_1 is lower semicontinuous and a_2 is upper semicontinuous functions; (iv) $[A]^\alpha = [a_1(\alpha), a_2(\alpha)]$; (v) $[A]^0 = cl(\text{supp}A) = [a_1(0), a_2(0)]$.*

Proposition 2. *If $(\lambda, d) \in \mathcal{F}_g$ and $d > 0$, then for all $\alpha \in [0,1]$*

$$[(\lambda, d)]^\alpha = [\lambda - dg(\alpha), \ \lambda + dg(\alpha)]. \tag{6}$$

If $d = 0$, then for all $\alpha \in [0,1]$

$$[(\lambda, d)]^\alpha = \{\lambda\}. \tag{7}$$

Let X be a vector space. Using the Zadeh's extension principle we define extended addition, extended substraction, extended opposite and extended multiplication of a fuzzy set by a scalar in $\mathcal{F}(X)$, as follows.

Definition 7. (Extended Addition) Suppose A and B are fuzzy sets on X. If using the extension principle, then the membership function of fuzzy set $A+B$ is

$$\mu_{A+B}(x) = \sup_{t_1+t_2=x} \min\{\mu_A(t_1), \mu_B(t_2)\}. \tag{8}$$

Definition 8. (Extended Opposite) Suppose A is a fuzzy set on X. If using the extension principle, then the membership function of fuzzy set $-A$ is

$$\mu_{-A}(x) = \mu_A(-x). \tag{9}$$

Definition 9. (Extended Substraction) Suppose A and B are fuzzy sets on X. If using the extension principle, then the membership function of fuzzy set $A-B$ is

$$\mu_{A-B}(x) = \sup_{t_1-t_2=x} \min\{\mu_A(t_1), \mu_B(t_2)\}. \tag{10}$$

Definition 10. (Extended Multiplication) Suppose A is a fuzzy set on X and c is a real number. If using the extension principle, then the membership function of fuzzy set cA is

$$\mu_{cA}(x) = \begin{cases} \mu_A\left(\frac{x}{c}\right), & \text{if} \quad c \neq 0, \\ hgt(A), & \text{if} \quad c = 0 \text{ and } x = 0, \\ 0, & \text{if} \quad c = 0 \text{ and } x \neq 0. \end{cases} \tag{11}$$

Proposition 3. *Suppose A and B are fuzzy numbers with $[A]^\alpha = [a_1(\alpha), a_2(\alpha)]$ and $[B]^\alpha = [b_1(\alpha), b_2(\alpha)]$ for all $\alpha \in [0,1]$. Then for all $\alpha \in [0,1]$ and for all $c \in R$*

$$(i) \qquad [A+B]^\alpha = [a_1(\alpha) + b_1(\alpha), \quad a_2(\alpha) + b_2(\alpha)]; \tag{12}$$

$$(ii) \qquad [A-B]^\alpha = [a_1(\alpha) - b_2(\alpha), \quad a_2(\alpha) - b_1(\alpha)]; \tag{13}$$

$$(iii) \qquad [cA]^\alpha = \begin{cases} [ca_1(\alpha), ca_2(\alpha)], & \textit{if} \quad \alpha \geq 0, \\ [ca_2(\alpha), ca_1(\alpha)], & \textit{if} \quad \alpha < 0. \end{cases} \tag{14}$$

2.2. Possibility

Definition 11. Let A and B be fuzzy numbers. The degree of possibility that the proposition „A is less than or equal to B" is true we denote by $\mathrm{Pos}(A \leq B)$ and define by the extension principle as

$$\mathrm{Pos}(A \leq B) = \sup_{x \leq y} \min\{\mu_A(x), \mu_B(y)\}. \tag{15}$$

Definition 12. Let A and B be fuzzy numbers. The degree of possibility that the proposition „A is greater than or equal to B" is true we denote by $\mathrm{Pos}(A \geq B)$ and define by the extension principle as

$$\mathrm{Pos}(A \geq B) = \sup_{x \geq y} \min\{\mu_A(x), \mu_B(y)\}. \tag{16}$$

Definition 13. Let A and B be fuzzy numbers. The degree of possibility that the proposition „A is equal to B" is true we denote by $\mathrm{Pos}(A = B)$ and define by the extension principle as

$$\mathrm{Pos}(A = B) = \sup_{x} \min\{\mu_A(x), \mu_B(x)\}. \tag{17}$$

Proposition 4. *Let A and B be fuzzy numbers. Then*

$$\mathrm{Pos}(A = B) = \min\{\mathrm{Pos}(A \geq B), \mathrm{Pos}(A \leq B)\}. \tag{18}$$

Proposition 5. *Let A and B be fuzzy numbers. Then*

$$\mathrm{Pos}(A \leq B) = \tag{19}$$

$$= \begin{cases} \sup\{\alpha \in [0,1] \ / \ \min[A]^\alpha \leq \max[B]^\alpha\}, & if \ \min[A]^0 < \max[B]^0, \\ 0, & if \ \min[A]^0 \geq \max[B]^0. \end{cases}$$

Proposition 6. *Let $A = (\lambda_1, d_1) \in \mathcal{F}_g$ and $B = (\lambda_2, d_2) \in \mathcal{F}_g$. If $d_1 > 0$ or $d_2 > 0$, then*

$$Pos(A \leq B) = \begin{cases} 1, & if \quad \lambda_1 \leq \lambda_2, \\ g^{(-1)}\left(\dfrac{\lambda_1 - \lambda_2}{d_1 + d_2}\right), & if \quad \lambda_1 > \lambda_2, \end{cases} \tag{20}$$

and

$$(21) \qquad Pos(A = B) = g^{(-1)}\left(\frac{|\lambda_1 - \lambda_2|}{d_1 + d_2}\right).$$

If $A = (\lambda_1, 0) \in \mathcal{F}_g$ *and* $B = (\lambda_2, 0) \in \mathcal{F}_g$, *then*

$$(22) \qquad Pos(A \leq B) = \begin{cases} 1, & if \quad \lambda_1 \leq \lambda_2, \\ 0, & if \quad \lambda_1 > \lambda_2, \end{cases}$$

and

$$(23) \qquad Pos(A = B) = \begin{cases} 1, & if \quad \lambda_1 = \lambda_2, \\ 0, & if \quad \lambda_1 \neq \lambda_2. \end{cases}$$

3. Fuzzy optimal value of fuzzy linear programming problems

In this section we present the new solution concept and the fuzzy optimal value concept of fuzzy linear programming problems.

A fuzzy linear programming problem is

$$(24) \qquad \begin{cases} Z = \bar{c}x \rightarrow \max\,(\text{or min})\,; \\ \bar{A}_i x \leq \bar{b}_i \quad i \in I, \\ \bar{A}_j x \geq \bar{b}_j \quad j \in J, \\ \bar{A}_k x = \bar{b}_k \quad k \in K, \\ x \geq 0, \end{cases}$$

where $\bar{c} = (C_1, C_2, ..., C_n)$ is a $1 \times n$ vector of fuzzy numbers, B_l are fuzzy numbers for all $l \in I \cup J \cup K$, $\bar{A}_l = (A_{l,1}, A_{l,2}, ..., A_{l,n})$ is a $1 \times n$ vector of fuzzy numbers for any $l \in I \cup J \cup K$.

Definition 14. Let $\alpha \in [0,1]$. The α-feasible set of problem (24) is

$$(25) \qquad \begin{aligned} \mathcal{F}_\alpha\left(\bar{A}, \bar{b}\right) = \{x \geq 0 \;/\; & \mathrm{Pos}\,(A_i x \leq B_i) \geq \alpha,\ \forall i \in I \\ \text{and} \quad & \mathrm{Pos}\,(A_j x \geq B_j) \geq \alpha,\ \forall j \in J \\ \text{and} \quad & \mathrm{Pos}\,(A_k x = B_k) \geq \alpha,\ \forall k \in K\}. \end{aligned}$$

Definition 15. Let $\alpha \in [0,1]$. The α-optimal solution set is

$$\mathcal{S}_\alpha\left(\bar{A},\bar{b},\bar{c}\right) = \tag{26}$$

$$= \left\{x \in \mathcal{F}_\alpha\left(\bar{A},\bar{b}\right) \,/\, \operatorname{Pos}\left(cx \geq cy\right) \geq \alpha, \quad \forall y \in \mathcal{F}_\alpha\left(\bar{A},\bar{b}\right)\right\},$$

if we are searching the maximum in the problem (24), and

$$\mathcal{S}_\alpha\left(\bar{A},\bar{b},\bar{c}\right) = \tag{27}$$

$$= \left\{x \in \mathcal{F}_\alpha\left(\bar{A},\bar{b}\right) \,/\, \operatorname{Pos}\left(cx \leq cy\right) \geq \alpha, \quad \forall y \in \mathcal{F}_\alpha\left(\bar{A},\bar{b}\right)\right\},$$

if we are searching the minimum in the problem (24).

We define next the fuzzy set M which will represent the fuzzy optimal value of the objective function in the problem (24).

Definition 16. Let

$$\mathcal{P}_\alpha = \Big\{ vx \,/\, x \in \mathcal{S}_\alpha\left(\bar{A},\bar{b},\bar{c}\right) \text{ and } v = (v_1, v_2, \ldots, v_n), \text{ where } v_i \in [C_i]^\alpha, \forall i = 1,\ldots,n \Big\}, \tag{28}$$

$0 \leq \alpha \leq 1$. M will be a fuzzy set on R, defined by its membership function

$$\mu_M(t) = \tag{29}$$

$$= \begin{cases} \sup\{\alpha \in [0,1] \,/\, t \in \mathcal{P}_\alpha\} & \text{if } \exists \alpha \in (0,1] \text{ such that } t \in \mathcal{P}_\alpha, \\ 0, & \text{else.} \end{cases}$$

Proposition 7. *(i) For all $i \in I$ and $\alpha \in (0,1]$ we have*

$$\{x \geq 0 \,/\, \operatorname{Pos}(A_i x \leq B_i) \geq \alpha\} = \tag{30}$$

$$= \left\{x \geq 0 \,/\, \sum_{l=1}^{n} \min\,[A_{il}]^\alpha x_l \leq \max\,[B_i]^\alpha\right\}.$$

(ii) For all $j \in J$ and $\alpha \in (0,1]$ we have

$$\{x \geq 0 \,/\, \operatorname{Pos}(A_j x \geq B_j) \geq \alpha\} = \tag{31}$$

$$= \left\{ x \geq 0 \;/\; \sum_{l=1}^{n} \max\,[A_{jl}]^{\alpha}\, x_l \geq \min\,[B_j]^{\alpha} \right\}.$$

(iii) For all $k \in K$ and $\alpha \in (0,1]$ we have

(32)
$$\{x \geq 0 \;/\; \mathrm{Pos}\,(A_k x = B_k) \geq \alpha\} =$$

$$= \left\{ x \geq 0 \;/\; \sum_{l=1}^{n} \max\,[A_{kl}]^{\alpha}\, x_l \geq \min\,[B_k]^{\alpha} \right\} \cap$$

$$\cap \left\{ x \geq 0 \;/\; \sum_{l=1}^{n} \min\,[A_{kl}]^{\alpha}\, x_l \leq \max\,[B_k]^{\alpha} \right\}.$$

(iv) $\mathcal{F}_0\left(\bar{A}, \bar{b}\right) = R^n_+$;

(v) $\mathcal{S}_0\left(\bar{A}, \bar{b}, \bar{c}\right) = R^n_+$.

Proof. (i) Let $i \in I$ and $\alpha \in (0,1]$. Using Proposition 5 we have

$$\{x \geq 0 \;/\; \mathrm{Pos}\,(A_i x \leq B_i) \geq \alpha\} =$$
$$= \left\{ x \geq 0 \;/\; \sup \left\{ p \in [0,1] \;/\; \min \left[\sum_{l=1}^{n} A_{il} x_l \right]^{p} \leq \max [B_i]^p \right\} \geq \alpha \right\} =$$
$$= \left\{ x \geq 0 \;/\; \sup \left\{ p \in [0,1] \;/\; \sum_{l=1}^{n} \min\,[A_{il}]^{p}\, x_l \leq \max\,[B_i]^{p} \right\} \geq \alpha \right\}.$$

Because A_{il}, $l = 1, \ldots, n$ are fuzzy numbers, it follows that $\min\,[A_{il}]^p \geq \min\,[A_{il}]^{\alpha}$ and $\max\,[B_i]^p \leq \max\,[B_i]^{\alpha}$, for all $p \geq \alpha$. Therefore

$$\{x \geq 0 \;/\; \mathrm{Pos}\,(A_i x \leq B_i) \geq \alpha\} = \left\{ x \geq 0 \;/\; \sum_{l=1}^{n} \min\,[A_{il}]^{\alpha}\, x_l \leq \max\,[B_i]^{\alpha} \right\}.$$

The following properties will be proved similarly.

Lemma 1. *If A and B are closed convex subsets of R^n, then*

$$AB = \{xy \;/\; x \in A \;\; and \;\; y \in B\}$$

is closed convex subset of R.

Theorem 8. *The sets $\mathcal{F}_{\alpha}\left(\bar{A}, \bar{b}\right)$, $\mathcal{S}_{\alpha}\left(\bar{A}, \bar{b}, \bar{c}\right)$, and P_{α} are convex and closed for any $\alpha \in [0,1]$.*

Proof. If $\alpha = 0$, then it is evident, that $\mathcal{F}_\alpha\left(\bar{A},\bar{b}\right)$ and $\mathcal{S}_\alpha\left(\bar{A},\bar{b},\bar{c}\right)$ are convex and closed. Since P_0 is product of $\mathcal{F}_0\left(\bar{A},\bar{b}\right)$ and $[C]^0 = [C_1]^0 \times [C_2]^0 \times \ldots \times [C_n]^0$, from Lemma 1 it follows that P_0 is convex and closed, too.

If $\alpha \in (0,1]$, then

$$\mathcal{F}_\alpha\left(\bar{A},\bar{b}\right) = \left(\bigcap_{i=1}^{m}\left\{x \geq 0 \;/\; \sum_{l=1}^{n} \min\left[A_{il}\right]^\alpha x_l \leq \max\left[B_i\right]^\alpha\right\}\right) \bigcap$$
$$\bigcap\left(\bigcap_{j\in J}\left\{x \geq 0 \;/\; \sum_{l=1}^{n} \max\left[A_{jl}\right]^\alpha x_l \geq \min\left[B_j\right]^\alpha\right\}\right) \bigcap$$
$$\bigcap\left(\bigcap_{k\in K}\left\{x \geq 0 \;/\; \sum_{l=1}^{n} \max\left[A_{kl}\right]^\alpha x_l \geq \min\left[B_k\right]^\alpha\right\}\right) \bigcap$$
$$\bigcap\left(\bigcap_{k\in K}\left\{x \geq 0 \;/\; \sum_{l=1}^{n} \min\left[A_{kl}\right]^\alpha x_l \leq \max\left[B_k\right]^\alpha\right\}\right)$$

and

$$\mathcal{S}_\alpha\left(\bar{A},\bar{b},\bar{c}\right) = \left\{x \in \mathcal{F}_\alpha\left(\bar{A},\bar{b}\right) \;/\; \mathrm{Pos}\left(cx \geq cy\right) \geq \alpha,\;\; \forall y \in \mathcal{F}_\alpha\left(\bar{A},\bar{b}\right)\right\} =$$
$$= \mathcal{F}_\alpha\left(\bar{A},\bar{b}\right) \cap \left\{x \geq 0 \;/\; \mathrm{Pos}\left(cx \geq cy\right) \geq \alpha,\;\; \forall y \in \mathcal{F}_\alpha\left(\bar{A},\bar{b}\right)\right\} =$$
$$= \mathcal{F}_\alpha\left(\bar{A},\bar{b}\right) \cap$$
$$\cap\left\{x \geq 0 \;/\; \sum_{l=1}^{n} \max\left[C_l\right]^\alpha x_l \geq \sum_{l=1}^{n} \min\left[C_l\right]^\alpha y_l,\;\; \forall y \in \mathcal{F}_\alpha\left(\bar{A},\bar{b}\right)\right\}.$$

Since the sets

$$\left\{x \geq 0 \;/\; \sum_{l=1}^{n} \min\left[A_{il}\right]^\alpha x_l \leq \max\left[B_i\right]^\alpha\right\},$$

$$\left\{x \geq 0 \;/\; \sum_{l=1}^{n} \max\left[A_{jl}\right]^\alpha x_l \geq \min\left[B_j\right]^\alpha\right\},$$

$$\left\{x \geq 0 \;/\; \sum_{l=1}^{n} \max\left[A_{kl}\right]^\alpha x_l \geq \min\left[B_k\right]^\alpha\right\},$$

$$\left\{x \geq 0 \;/\; \sum_{l=1}^{n} \min\left[A_{kl}\right]^\alpha x_l \leq \max\left[B_k\right]^\alpha\right\}$$

are convex and closed for any $i \in I, j \in J, k \in K$, it follows that the intersection of these sets are convex and closed.

Consider the linear programming problem

$$\text{(33)} \qquad \begin{cases} \sum_{l=1}^{n} \min [C_l]^\alpha y_l \to \max, \\ y \in \mathcal{F}_\alpha \left(\bar{A}, \bar{b}\right). \end{cases}$$

Three cases are possible.

1. If $\mathcal{F}_\alpha \left(\bar{A}, \bar{b}\right) = \emptyset$, then $\mathcal{S}_\alpha \left(\bar{A}, \bar{b}, \bar{c}\right) = \emptyset$. Therefore $\mathcal{S}_\alpha \left(\bar{A}, \bar{b}, \bar{c}\right)$ is convex and closed.

2. If $z_\alpha = \max\limits_{y \in \mathcal{F}_\alpha (\bar{A}, \bar{b})} \sum_{l=1}^{n} \min [C_l]^\alpha y_l = +\infty$, then $\mathcal{S}_\alpha \left(\bar{A}, \bar{b}, \bar{c}\right) = \emptyset$. Therefore $\mathcal{S}_\alpha \left(\bar{A}, \bar{b}, \bar{c}\right)$ is convex and closed.

3. If $z_\alpha = \max\limits_{y \in \mathcal{F}_\alpha (\bar{A}, \bar{b})} \sum_{l=1}^{n} \min [C_l]^\alpha y_l =< +\infty$, then $\mathcal{S}_\alpha \left(\bar{A}, \bar{b}, \bar{c}\right) = \mathcal{F}_\alpha \left(\bar{A}, \bar{b}\right) \cap \left\{ x \geq 0 \; / \; \sum_{l=1}^{n} \max [C_l]^\alpha x_l \geq z_\alpha \right\}$. Because these sets are convex and closed, it follows that their intersection is convex and closed, too.

Since P_α is product of $\mathcal{F}_\alpha \left(\bar{A}, \bar{b}\right)$ and $[C]^\alpha = [C_1]^\alpha \times [C_2]^\alpha \times \ldots \times [C_n]^\alpha$, from Lemma 1 it follows that P_α is convex and closed, too.

If we are using Proposition 5 and Proposition 7, then the general problem (24) can be rewritten as

$$\text{(34)} \qquad \begin{cases} Z = \bar{c}x \to \max \quad (\text{or} \quad \min); \\ \bar{A}_i x \leq \bar{b}_i & i \in I, \\ \left(-\bar{A}_j\right) x \leq \left(-\bar{b}_j\right) & j \in J, \\ \bar{A}_k x \leq \bar{b}_k & k \in K, \\ \left(-\bar{A}_k\right) x \leq \left(-\bar{b}_k\right) & k \in K, \\ x \geq 0. \end{cases}$$

These transformations lead to the standard form of fuzzy linear programming problems

$$\text{(35)} \qquad \begin{cases} Z = \bar{c}x \to \max \quad (\text{or} \quad \min); \\ \bar{A}_i x \leq \bar{b}_i, \quad i \in I, \\ x \geq 0. \end{cases}$$

4. Fuzzy optimal value of linear programming problems with quasi-triangular fuzzy numbers in capacity vector

In this section we present elementary properties of fuzzy optimal value, if the capacity vector of linear programming problem contains fuzzy numbers or quasi-triangular fuzzy numbers.

Such a problem can be written as

$$(36)\qquad \begin{cases} Z = cx \to \max; \\ Ax \leq \bar{b}, \\ x \geq 0, \end{cases}$$

where $c = (c_1, c_2, ..., c_n) \in R^n, x = (x_1, x_2, ..., x_n) \in R^n$, $A = (a_{ij})_{\substack{i=1,...m \\ j=1,...,n}}$ is a matrix of real numbers and $\bar{b} = (B_1, B_2, ..., B_m)$ is a vector of fuzzy numbers.

In this case, for any $\alpha \in (0,1]$

$$(37)\qquad \mathcal{F}_\alpha\left(A, \bar{b}\right) = \left\{ x \geq 0 \;/\; \sum_{j=1}^{n} a_{ij} x_j \leq \max [B_i]^\alpha, \quad i = 1, 2, ...m \right\},$$

(38)

$$\mathcal{S}_\alpha\left(A, b, c\right) = \left\{ x \in \mathcal{F}_\alpha\left(A, \bar{b}\right) \;/\; \sum_{j=1}^{n} c_j x_j \geq \sum_{j=1}^{n} c_j x_j', \quad \forall x' \in \mathcal{F}_\alpha\left(A, \bar{b}\right) \right\},$$

$$(39)\qquad \begin{aligned} \mathcal{P}_\alpha &= \left\{ cx \;/\; x \in \mathcal{S}_\alpha\left(A, \bar{b}, c\right)\right\} = \\ &= \begin{cases} \emptyset, & \text{if} \quad \mathcal{S}_\alpha\left(A, \bar{b}, c\right) = \emptyset; \\ \left\{ \max\limits_{x \in \mathcal{F}_\alpha(A, b)} cx \right\}, & \text{if} \quad \mathcal{S}_\alpha\left(A, \bar{b}, c\right) \neq \emptyset. \end{cases} \end{aligned}$$

Proposition 9. *The following properties are true: i) If* $\alpha' \in (0, \alpha]$, *then* $\mathcal{F}_\alpha\left(A, \bar{b}\right) \subseteq \mathcal{F}_{\alpha'}\left(A, \bar{b}\right)$. *ii) If for some* $\alpha \in (0,1]$, $\mathcal{S}_\alpha\left(A, \bar{b}, c\right) \neq \emptyset$ *and* $\mathcal{F}_1\left(A, \bar{b}\right) \neq \emptyset$, *then* $\mathcal{S}_{\alpha'}\left(A, \bar{b}, c\right) \neq \emptyset$, *for any* $\alpha' \in (\alpha, 1]$. *iii) If for some* $\alpha \in (0,1]$, $\mathcal{S}_\alpha\left(A, \bar{b}, c\right) \neq \emptyset$ *and* $\max [B_i]^{\alpha'} < \infty$, *for all* $i = 1, ..., m$, *where* $\alpha' \in (0, \alpha)$, *then* $\mathcal{S}_{\alpha'}\left(A, \bar{b}, c\right) \neq \emptyset$. *iv) If* $\alpha, \beta \in (0,1]$ *where* $\alpha < \beta$ *and* $\mathcal{S}_\alpha\left(A, \bar{b}, c\right), \mathcal{S}_\beta\left(A, \bar{b}, c\right)$ *are not empty, then for any* $\lambda \in [\alpha, \beta]$, $\mathcal{S}_\lambda\left(A, \bar{b}, c\right)$

is not empty. v) Let $\beta \in [0,1]$ *be the biggest number such that* $\mathcal{S}_\beta\left(A,\bar{b},c\right) \neq \emptyset$, *as well as* $\alpha \in [0,1]$ *be the smallest number such that* $\mathcal{S}_\alpha\left(A,\bar{b},c\right) \neq \emptyset$. *If we denote* $z(\alpha') = \max_{x\in\mathcal{F}_{\alpha'}(A,\bar{b})} cx$, *then the function* $z:[\alpha,\beta] \to R$ *is decreasing. If the functions* $f_i:[\alpha,\beta] \to R$, $f_i(\alpha') = \max[B_i]^{\alpha'}$ *are continuous for all* $i = 1,\dots,m$, *then the function* z *is continuous and* $[M]^{\alpha'} = [z(\beta), z(\alpha')]$ *for any* $\alpha' \in [\alpha,\beta]$.

These properties are consequences of Proposition 7 and Proposition 5.

In the following, we assume that all components of the capacity vector $\bar{b}$ are quasi-triangular fuzzy numbers. Let $B_i(b_i,d_i) \in \mathcal{F}_g$, for all $i = 1,2,\dots,m$. In this case $\max[B_i]^\alpha = b_i + d_i g(\alpha)$ for all $i = 1,2,\dots,m$.

Proposition 10. *If* $\mathcal{S}_1\left(A,\bar{b},c\right)$ *is non-empty then the fuzzy optimal value* M *of the objective function in the problem (36) is a fuzzy number, where* $[M]^\alpha = [z(1), z(\alpha)]$ *for any* $\alpha \in [0,1]$.

To solve problem (36) in a fuzzy sense means, that we determine the fuzzy optimal value of the objective function and give at least one element of $\mathcal{S}_\alpha\left(A,\bar{b},c\right)$ for all $\alpha \in [0,1]$.

In this case we will get the fuzzy optimal value, if we solve the following linear programming problem for all $\alpha \in [0,1]$

$$
(40)\qquad \begin{cases} \sum\limits_{j=1}^{n} c_j x_j \to \max \\ \sum\limits_{j=1}^{n} a_{ij}x_j + x_{n+i} = b_i + d_i g(\alpha), \quad i = 1,\dots,m, \\ x_i \geq 0, \qquad i = 1,\dots,n+m. \end{cases}
$$

For every α, the optimal value of this problem will be an element of $\mathcal{P}_\alpha$ and the optimal solution will be an element of $\mathcal{S}_\alpha\left(A,\bar{b},c\right)$.

If $\alpha = \alpha_0$ then let B be a basis associated to the optimal solution of the linear problem. Let I the set of indices $i \in \{1,2,\dots,m+n\}$, if the i-th vector is in a basis B and let $J = \{1,2,\dots,m+n\}\setminus I$.

The solution of (40) for $\alpha \geq \alpha_0$ associated to B is

$$x_B(\alpha) = B^{-1}e(\alpha) = B^{-1}\left(b + dg(\alpha)\right) = B^{-1}b + B^{-1}dg(\alpha).$$

If we denote

$$x_B^0 = \left(x_1^0, x_2^0, \dots, x_n^0\right) = B^{-1}\left(b + g(\alpha_0)d\right) \quad \text{and} \quad x_B^1 = \left(x_1^1, x_2^1, \dots, x_n^1\right) = B^{-1}d,$$

then the solution $x_B(\alpha)$ is

(41) $$x_B(\alpha) = x_B^0 + (g(\alpha) - g(\alpha_0))\, x_B^1.$$

Let us discuss the following problem. Can we find an upper bound of α, such that $x_B(\alpha)$ remains an optimal solution of (40)?

Theorem 11. *If B is optimal basis on $\alpha = \alpha_0$, then B remains optimal for any $\alpha \in [\alpha_0, \alpha_{\max}]$, where*

Case I. If $x_i^1 \leq 0$ for all $i \in I$, then

(42) $$\alpha_{\max} = 1.$$

Case II. If exist some $p \in I$ such that $x_p^1 > 0$, then

(43) $$\alpha_{\max} = \begin{cases} g^{-1}\left(g(\alpha_0) - \min\limits_{i \in I: x_i^1 > 0} \left(\dfrac{x_i^0}{x_i^1}\right)\right), & \text{if } \min\limits_{i \in I: x_i^1 > 0} \left(\dfrac{x_i^0}{x_i^1}\right) < g(\alpha_0), \\ 1, & \text{if } \min\limits_{i \in I: x_i^1 > 0} \left(\dfrac{x_i^0}{x_i^1}\right) \geq g(\alpha_0). \end{cases}$$

Proof. Let $\alpha \geq \alpha_0$. We know, that the basis B is optimal when $x_B(\alpha)$ is a feasible solution and all components c_j' relative on B are not negative, where

(44) $$c_j' = z_j - c_j = c^B B^{-1} a_j - c_j, \qquad j = 1, ..., n + m.$$

Since c_j' does not depend on α, it follows that it is not negative, because B is optimal for α_0.

If the components of $x_B(\alpha)$ denoted with $x_i(\alpha)$, and the components of x_B^0 and x_B^1 with x_i^0 and x_i^1respectively, then $x_B(\alpha)$ is feasible solution, when

(45) $$x_i(\alpha) = x_i^0 + (g(\alpha) - g(\alpha_0))\, x_i^1 \geq 0, \quad \text{for all } \ i \in I.$$

The values of x_i^0 are not negative, because B is optimal on α_0.

Two cases are possible.

Case I. If $x_i^1 \leq 0$ for all $i \in I$, then for any $\alpha \in [\alpha_0, 1]$ we have $g(\alpha) \leq$ $\leq g(\alpha_0)$. Thus inequalities (45) are performed.

Case II. If there exists some $p \in I$ such that $x_p^1 > 0$, then

(46) $$g(\alpha) \geq g(\alpha_0) + \max_{i \in I: x_i^1 > 0} \left\{\frac{-x_i^0}{x_i^1}\right\} = g(\alpha_0) - \min_{i \in I: x_i^1 > 0} \left\{\frac{x_i^0}{x_i^1}\right\}.$$

This condition is true for any $\alpha \in [\alpha_0, \alpha_{\max}]$, where $\alpha_{\max}$ is given by the formula (43).

Thus, when we know the optimal basis B on α_0, then we can determine an interval in which B remains optimal, if α changes in this interval.

Since we supposed, that the value of α is in the interval $[0,1]$, we put the following question. How can we generate a new optimal basis B on $\alpha > \alpha_{\max}$, if $\alpha_{\max} < 1$?

We assume that $\alpha_{\max} < 1$. If $\alpha = \alpha_{\max}$, then two cases are possible.

A) If there exists only one indice l such that $\min\limits_{x_i^1>0,\ i\in I}\left(\frac{x_i^0}{x_i^1}\right) = \frac{x_l^0}{x_l^1}$, then $x_l^0 + (g(\alpha_{\max}) - g(0))\, x_l^1 = 0$.

B) If there exists more indices $s \in S$ such that $\min\limits_{x_i^1>0,\ i\in I}\left(\frac{x_i^0}{x_i^1}\right) = \frac{x_s^0}{x_s^1}$, then $x_s^0 + (g(\alpha_{\max}) - g(0))\, x_s^1 = 0$ for all $s \in S$.

Let $\varepsilon > 0$ such that $\alpha = \alpha_{\max} + \varepsilon \leq 1$. Then at least one component of $x_B(\alpha)$ will be negative. However the solution $x_B(\alpha)$ remains dual-feasible. Consequently, the transformation of basis generates a new basis. We discuss the two cases separately.

Case A) If we take $\alpha = \alpha_{\max} + \varepsilon$ in place of $\alpha_{\max}$, then the component $x_l(\alpha)$ of the vector $x_B(\alpha)$ will become negative. Therefore we are studying sign of the components g_{lj} in row of x_l. Two cases are possible.

a) All $g_{lj} \geq 0$ when $j \in J$. In this case, if $\alpha > \alpha_{\max}$, then there is no optimal solution of (40).

b) If there exists some indices $j \in J_1 \subset J$ such that $g_{lj} < 0$, then we perform a basis transformation. Let B_1 be a new basis. We get this basis from B, where we replaced the vector x_l with x_k. These basis B and B_1 are optimal for $\alpha = \alpha_{\max}$. Thus, a value $\alpha_{\max}$ is critical, because for this value of α two different optimal basis exist. For basis B_1 it is also possible to determine a new interval, in which B_1 remains also an optimal basis. We denote by $[\alpha_{\max}, \alpha_1]$ this interval. We will show, that $\alpha_1 > \alpha_{\max}$.

Proof. The solution of (40) on $\alpha \geq \alpha_{\max}$ associated to B_1 is

$$\bar{x}_{B_1} = B_1^{-1} e(\alpha) = B_1^{-1}\,(\,b + dg(\alpha)) = B_1^{-1} b + g(\alpha)\, B_1^{-1} d.$$

We determine the interval in which B_1 is optimal.

Components of solution x_{B_1} are

$$\bar{x}_i(\alpha_{\max}) = \bar{x}_i^0 + g(\alpha_{\max})\,\bar{x}_i^1 \geq 0,$$

for all $i \in I - \{l\} - \{k\}$ and $\bar{x}_k(\alpha_{\max}) = 0$.

If $\alpha = \alpha_{\max} + \varepsilon$, then

$$\begin{aligned} \bar{x}_i(\alpha_{\max}+\varepsilon) &= \bar{x}_i^0 + g(\alpha_{\max}+\varepsilon)\,\bar{x}_i^1 = \\ &= \bar{x}_i^0 + g(\alpha_{\max})\,\bar{x}_i^1 + \left(g(\alpha_{\max}+\varepsilon) - g(\alpha_{\max})\right)\bar{x}_i^1 = \\ &= \bar{x}_i(\alpha_{\max}) + \left(g(\alpha_{\max}+\varepsilon) - g(\alpha_{\max})\right)\bar{x}_i^1 \end{aligned} \tag{47}$$

for all $i \in I - \{l\} - \{k\}$, and

(48)

$$\begin{aligned} x_k(\alpha_{\max}+\varepsilon) &= -\frac{1}{g_{lk}}\left(\bar{x}_l^0 + g(\alpha_{\max}+\varepsilon)\,\bar{x}_l^1\right) = \\ &= -\frac{1}{g_{lk}}\left(x_l^0 + g(\alpha_{\max})\,x_l^1 + \left(g(\alpha_{\max}+\varepsilon) - g(\alpha_{\max})\right)\bar{x}_l^1\right) = \\ &= -\frac{1}{g_{lk}}\left(g(\alpha_{\max}+\varepsilon) - g(\alpha_{\max})\right)\bar{x}_l^1. \end{aligned}$$

From (47) and (48) it follows that

- if all components of vector $x^1 = B_1^{-1}d$ are not positive, then B_1 is optimal for any $\varepsilon > 0$. Thus, $\alpha_{\max} = 1$.

- If the component x_l^1 of $x^1$is strictly positive, then B_1 is not optimal for $\varepsilon > 0$. Thus, if $\alpha > \alpha_{\max}$, then there is not optimal solution.

- If $x_l^1 \leq 0$, but there is an $i \in I - \{l\} - \{k\}$ such that $x_i^1 > 0$, then the condition $\bar{x}(\alpha_{\max}+\varepsilon) \geq 0$ is satisfied for ε, where

$$\varepsilon \leq \varepsilon_{\max} = g^{-1}\left(\max_{i:x_i^1>0}\left(\frac{-\bar{x}_i(\alpha_{\max})}{x_i^1}\right) + g(\alpha_{\max})\right) - \alpha_{\max}. \tag{49}$$

Since $\bar{x}_i(\alpha_{\max}) > 0$ for any undegenerate solution, it follows that B_1 is optimal for $\alpha_{\max} < \alpha < \alpha_1 = \alpha_{\max} + \varepsilon_{\max}$.

Using this method, we determine a sequence of optimal basis $B_1, B_2, ..., B_p$ associated to the intervals $[\alpha_1, \alpha_2]$, $[\alpha_2, \alpha_3]$, ..., $[\alpha_{p-1}, \alpha_p]$. The limits of these intervals we call critical values. They constitute an increasing sequence. The endpoint element of this sequence may be equal or not equal to one.

Case B) In this case all components $x_s, s \in S$ of $x(\alpha_{\max})$ are equal to zero and will become nonnegative, if we put $\alpha = \alpha_{\max} + \varepsilon$ in place of $\alpha_{\max}$. To determine a new solution of problem (40) for α, it is necessary that we apply in finite times the simplex-dual algorithm. Henceforth, we obtain the conclusion that optimal basis for $\alpha > \alpha_{\max}$ may exist or it may not exist. If optimal basis exists, then $\alpha_{\max}$ is the critical value of α.

4.1. Solution algorithm of linear programming problems with quasi-triangular fuzzy numbers in capacity vector

Step 1. Let $\alpha_0 = 0$ and $k := 0$.

Step 2. Using simplex method, we decide if exists or does not exist an optimal basis. If exists, then go to step 4, else go to step 3.

Step 3. If $\alpha > \alpha_0$, then we put

$$\mathcal{S}_\alpha\left(A, \bar{b}, c\right) = \emptyset \quad \text{and} \quad \mathcal{P}_\alpha = \emptyset, \tag{50}$$

and after that go to step 8.

Step 4. Let $k := k+1$. We denote optimal basis with B_k, optimal solution with x^k and optimal value with z_k.

Step 5. We determine the value of $\alpha_{\max}$ with formulas:
Case I. If $x_i^1 \leq 0$ for all $i \in I$, then

$$\alpha_{\max} = 1. \tag{51}$$

Case II. If there exists some $p \in I$ such that $x_p^1 > 0$, then

$$\alpha_{\max} = \begin{cases} g^{-1}\left(g\left(\alpha_0\right) - \min\limits_{i \in I: x_i^1 > 0}\left(\dfrac{x_i^0}{x_i^1}\right)\right), & \text{if } \min\limits_{i \in I: x_i^1 > 0}\left(\dfrac{x_i^0}{x_i^1}\right) < g\left(\alpha_0\right), \\ 1, & \text{if } \min\limits_{i \in I: x_i^1 > 0}\left(\dfrac{x_i^0}{x_i^1}\right) \geq g\left(\alpha_0\right). \end{cases} \tag{52}$$

Let $\alpha_k := \alpha_{\max}$. We calculate components of vector $y := B_k^{-1}d$ and after that we determine the optimal solution with the formula

$$x_B\left(\alpha\right) = x^k + \left(g\left(\alpha\right) - g\left(\alpha_0\right)\right)y; \tag{53}$$

and optimal value with the formula

$$z\left(\alpha\right) = z_k + \left(g\left(\alpha\right) - g\left(\alpha_0\right)\right)cy, \tag{54}$$

when $\alpha \in \left[\alpha_{k-1}, \alpha_k\right]$.

Step 6. If $\alpha_{\max} = 1$, then go to step 8. Else we determine

$$b_i := x_i^k + \left(g\left(\alpha_{\max}\right) - g\left(\alpha_0\right)\right)y_i, \tag{55}$$

when $i \in I$,

$$z := z(\alpha_{\max}) \tag{56}$$

and

$$\alpha_0 := \alpha_{\max}. \tag{57}$$

We decide which of cases A) or B) is occurred. Using the correspondent method we decide if exist or not a new optimal basis.

Step 7. If the optimal basis exists, then go to step 4, else go to step 3.

Step 8. We determine the membership function of optimal value with formula

(58)

$$\mu_M(t) = \begin{cases} \sup\limits_{\alpha\in[0,1]} \{\alpha \ / \ t = z(\alpha)\}, & \text{if } \exists\alpha \in [0,1] \text{ such that } t = z(\alpha), \\ 0, & \text{else.} \end{cases}$$

5. Example

Consider the function $g : [0,1] \to R_+$, $g(x) = 1 - x^2$. Assume that $B_1 = (6,2)$, $B_2 = (10,3)$, $B_3 = (10,3)$ are quasi-triangular fuzzy numbers. Then we wish to solve the following problem

$$\begin{cases} z = 2x_1 + x_2 + 2x_3 \to \max \\ x_1 + x_2 + x_3 \leq B_1, \\ x_1 + x_2 + 2x_3 = B_2, \\ 2x_1 + x_2 + x_3 = B_3, \\ x_1, x_2, x_3 \geq 0. \end{cases} \tag{59}$$

We solve this problem using the algorithm described in the previous section.

We write the problem in the following form

$$\begin{cases} z = -2x_1 - x_2 - 2x_3 \to \min \\ x_1 + x_2 + x_3 \leq 6 + 2g(\alpha), \\ x_1 + x_2 + 2x_3 \leq 10 + 3g(\alpha), \\ x_1 + x_2 + 2x_3 \geq 10 - 3g(\alpha), \\ 2x_1 + x_2 + x_3 \leq 10 + 3g(\alpha), \\ 2x_1 + x_2 + x_3 \geq 10 - 3g(\alpha), \\ x_1, x_2, x_3 \geq 0. \end{cases} \tag{60}$$

After that we transform all inequalities of the problem in equations

$$(61)\qquad \begin{cases} z = -2x_1 - x_2 - 2x_3 \longrightarrow \min \\ x_1 + x_2 + x_3 + x_4 = 6 + 2g(\alpha), \\ x_1 + x_2 + 2x_3 + x_5 = 10 + 3g(\alpha), \\ -x_1 - x_2 - 2x_3 + x_6 = -10 + 3g(\alpha), \\ 2x_1 + x_2 + x_3 + x_7 = 10 + 3g(\alpha), \\ -2x_1 - x_2 - x_3 + x_8 = -10 + 3g(\alpha), \\ x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \geq 0. \end{cases}$$

Step 1. Let $\alpha_0 := 0$ and $k := 0$.

Step 2. Using dual-primal method we solve the problem (61) if $\alpha_0 = 0$. We insert the components of d into the last column of simplex table.

The first simplex table if $\alpha_0 = 0$ is

X	x_1	x_2	x_3	b	d
x_4	1	1	1	8	2
x_5	1	1	2	13	3
x_6	[−1]	−1	−2	−7	3
x_7	2	1	1	13	3
x_8	−2	−1	−1	−7	3
c	−2	−1	−2	0	

After four iterations we get the following simplex table

X	x_4	x_2	x_7	b	d
x_6	3	1	−1	4	6
x_5	−3	−1	1	2	0
x_1	−1	0	1	5	1
x_3	2	1	−1	3	1
x_8	0	0	1	6	6
c	−2	−1	−0	−16	

This simplex table is optimal.

Step 4. Let $k := 1$. The optimal basis is B_1, the optimal solution is $x^1 = (5, 0, 3, 0, 2, 4, 0, 6)$, and the optimal value is $z_1 = -16$.

Step 5. We find the values of vectors $x^0 = B_1^{-1}(b + g(0)\,d)$ and $y = B_1^{-1}d$ in the columns b and d of the simplex table. Thus $I = \{6, 5, 1, 3, 8\}$ and $\min\limits_{y_i>0, i\in I}\left\{\frac{x_i^0}{y_i}\right\} = \frac{2}{3}$. Consequently,

(62) $$\alpha_{\max} = g^{-1}\left(g(\alpha_0) - \min_{y_i>0, i\in I}\left\{\frac{x_i^0}{y_i}\right\}\right) = \sqrt{\frac{2}{3}} \approx 0.8165.$$

Therefore $\mathcal{S}_\alpha\left(A, \bar{b}, c\right) \neq \emptyset$ for any $\alpha \in \left(0, \sqrt{\frac{2}{3}}\right]$, because

(63) $$x(\alpha) = \left(5 - \alpha^2, \quad 0, \quad 3 - \alpha^2\right) \in \mathcal{S}_\alpha\left(A, \bar{b}, c\right)$$

and

(64) $$z(\alpha) = -16 + 4\alpha^2.$$

Step 6. Since $\alpha_{\max} < 1$, using formula (55) we replace the column b of the simplex table with

(65) $$b_i := x_i^1 + \left(g(\alpha_{\max}) - g(\alpha_0)\right) y_i \quad \forall i \in I.$$

We put

(66) $$z := z(\alpha_{\max})$$

and

(67) $$\alpha_0 := \alpha_{\max} = \sqrt{\frac{2}{3}}.$$

Thus, we get the following simplex table

X	x_4	x_2	x_7	b	d
x_6	3	1	[−1]	0	6
x_5	−3	−1	1	2	0
x_1	−1	0	1	$\frac{13}{3}$	1
x_3	2	1	−1	$\frac{7}{3}$	1
x_8	0	0	1	2	6
c	−2	−1	0	$-\frac{40}{3}$	

Since we have only one index $l = 6$ such that $\min\limits_{x_i^1>0,\ i\in I}\left(\frac{x_i^0}{x_i^1}\right) = \frac{x_l^0}{x_l^1} = \frac{2}{3}$, we will study the simplex table using the method of the case A). We remark that $g_{67} = -1$. Thus the basis transformation is possible and we obtain the following simplex table

X	x_4	x_2	x_6	b	d
x_7	-3	-1	-1	0	-6
x_5	0	0	1	2	6
x_1	2	1	1	$\frac{13}{3}$	7
x_3	-1	0	-1	$\frac{7}{3}$	-5
x_8	3	1	1	2	12
c	-2	-1	0	$-\frac{40}{3}$	

This simplex table is optimal.

Step 4'. Let $k := 2$. The optimal basis is B_2, the optimal solution is $x^2 = \left(\frac{17}{3}, 0, \frac{7}{3}, 0, 2, 0, 0, 2\right)$, and the optimal value is $z_2 = -\frac{40}{3}$.

Step 5'. We find the values of vectors $x^0 = B_2^{-1}(b + g(0)\, d)$ and $y = B_2^{-1} d$ in the columns b and d of the simplex table. Thus $I = \{7, 5, 1, 3, 8\}$ and $\min_{y_i > 0, i \in I} \left\{ \frac{x_i^0}{y_i} \right\} = \frac{1}{6}$. Consequently,

$$\alpha_{\max} = g^{-1}\left(g(\alpha_0) - \min_{y_i > 0, i \in I} \left\{ \frac{x_i^0}{y_i} \right\} \right) = \sqrt{\frac{5}{6}} \approx 0.91287. \tag{68}$$

Therefore $\mathcal{S}_\alpha\left(A, \bar{b}, c\right) \neq \emptyset$ for any $\alpha \in \left(\sqrt{\frac{2}{3}}, \sqrt{\frac{5}{6}}\right]$, because

$$x(\alpha) = \left(9 - 7\alpha^2, \quad 0, \quad -1 + 5\alpha^2\right) \in \mathcal{S}_\alpha\left(A, \bar{b}, c\right), \tag{69}$$

and

$$z(\alpha) = -16 + 4\alpha^2. \tag{70}$$

Step 6'. Since $\alpha_{\max} < 1$, using formula (55) we replace the column of b of the simplex table with

$$b_i := x_i^1 + \left(g(\alpha_{\max}) - g(\alpha_0)\right) y_i, \quad \forall i \in I. \tag{71}$$

We put

$$\alpha_0 := \alpha_{\max} = \sqrt{\frac{5}{6}}. \tag{72}$$

Since we have only one index $l = 8$ such that $\min\limits_{x_i^1>0,\ i\in I}\left(\frac{x_i^0}{x_i^1}\right) = \frac{x_l^0}{x_l^1} = \frac{1}{6}$, then we will study simplex table using the method of the case A). We remark that $g_{lj} > 0$ for all $j = 4, 2, 6$. Thus it is not possible to make a new optimal basis. Consequently, for any $\alpha > \sqrt{\frac{5}{6}}$ the problem (61) has not an optimal solution.

Step 3′. For $\alpha \in \left(\sqrt{\frac{5}{6}}, 1\right]$ we have

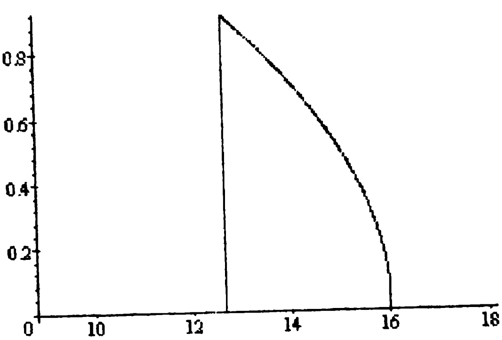
$$S_\alpha\left(A, \bar{b}, c\right) = \emptyset \tag{73}$$

and

$$\Gamma_\alpha = \emptyset. \tag{74}$$

Step 8. The membership function of the fuzzy optimal value is

$$\mu_M(t) = \sup_{\alpha\in[0,1]} \{\alpha \ / \ t = z(\alpha)\} = \begin{cases} \sqrt{\dfrac{16-t}{4}}, & \text{if } \dfrac{38}{3} \le t < 16, \\ 0, & \text{else.} \end{cases} \tag{75}$$



The fuzzy optimal value of the problem (49)

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