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ON A VERY THIN SEQUENCE OF INTEGERS

J.-M. De Koninck and N. Doyon (Québec, Canada)

Abstract. We say that an integer $n > 1$ is an insolite number if it does not contain the digit 0 and if both the sum and the product of the squares of its digits divide n. We first display several infinite families of insolite numbers. Denoting by $I(x)$ the number of insolite numbers $\leq x$, we then show that

$$\exp\left\{\frac{1}{5}(\log\log x)^2 + O(\log\log x \log\log\log x)\right\} \ll I(x) \ll x^{0.462}.$$

We also provide a heuristic argument in favor of $I(x) \gg x^\eta$ for some real $\eta > 0$. Finally we display the list of all 195 insolite numbers less than 10^{18} and show that no three consecutive insolite numbers exist.

1. Introduction

A positive integer n is called a Niven number (or a Harshad number) if it is divisible by the sum of its (decimal) digits. Niven numbers have been extensively studied (see for instance Kennedy & Cooper [3], Grundman [2] or Pickover [4]). Recently [1], we showed that, given any $\varepsilon > 0$, the number $N(x)$ of Niven numbers not exceeding x satisfies

$$x^{1-\varepsilon} \ll N(x) \ll \frac{x \log\log x}{\log x}$$

and conjectured that $N(x) \sim cx/\log x$, as $x \to \infty$, with $c = \frac{14}{27}\log 10 \approx 1.1939$.

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In this paper, we consider a much thinner set of integers, namely those integers $n > 1$ which do not contain the digit 0 and which are divisible by both the sum and the product of the square of their digits. We call these integers insolite numbers and denote their set by I.

The smallest fifteen elements of I are:

111, 11 112, 1 122 112, 111 111 111, 122 121 216, 1 111 112 112, 1 111 211 136, 1 116 122 112, 1 211 162 112, 11 111 113 116, 11 111 121 216, 11 112 122 112, 11 121 114 112, 11 132 111 232 and 11 133 122 112.

The list of all 195 insolite numbers $< 10^{18}$ is given in Section 6.

As we shall see, it turns out that the estimation of the counting function $I(x) := \{n \le x \ : \ n \in I\}$ offers a much greater challenge than that of $N(x)$. Nevertheless, after displaying several infinite families of insolite numbers, we shall establish that

$$\exp\left\{\frac{1}{5}(\log\log x)^2 + O(\log\log x \log\log\log\log x)\right\} \ll I(x) \ll x^{0.462}$$

and provide a heuristic argument in favour of $I(x) \gg x^{\eta}$ for some real $\eta > 0$.

Finally we establish that no three consecutive insolite numbers exist and conjecture that no two consecutive ones exist.

For convenience, throughout this paper, given a positive integer n, we shall denote by $p(n)$ (resp. $s(n)$) the product (resp. the sum) of the squares of its digits.

2. Insolite numbers with certain digit patterns

The following table gives, for each positive integer $k \le 9$, the smallest $n \in I$ containing the digit k:

k	n insolite
1	111
2	11 112
3	1 111 211 136
4	11 121 114 112
5	$\underbrace{11\ldots1}_{11}3111113\underbrace{11\ldots1}_{11}75$ (*)

k	n insolite
6	122 121 216
7	123 412 112 474 112
8	121 111 216 128
9	911 131 213 824

The star (*) in this table indicates that the given number is only the smallest known $n \in I$ containing the digit 5. Since an insolite number containing the digit 5 must be a multiple of 25, it is clear that all its digits must be odd and that it must end with the digits 75, since otherwise it would be a multiple of 10 and thus contain the digit 0.

It is clear that an odd integer n belongs to I only if it has an odd number of digits. Moreover since an odd insolite number cannot have any even digit, it is not so surprising that odd insolite numbers are quite rare. In fact, the only odd insolite numbers $< 10^{22}$ are the 7 numbers 111, 111 111 111, 11 111 731 111 111 113, 11 117 311 111 311 111, 11 131 117 111 113 111, 13 111 131 117 111 111 and 17 111 113 131 111 111.

It is also interesting to observe that the smallest insolite number which contains the maximum possible number of distinct digits, that is whose set of digits is precisely $\{1, 2, 3, 4, 6, 7, 8, 9\}$, is the 18-digit number 711 813 411 914 121 216.

3. Infinite families of insolite numbers

The sequence of insolite numbers is infinite. Indeed, this follows by observing that numbers of the form $\underbrace{11\ldots1}_{k}$, where $k = 3^{\alpha}$ for some positive integer α, are insolite. This is easily shown by induction on α.

Among the insolite numbers $< 10^{15}$, only two contain the digit 9, namely

$$911\,131\,213\,824 \quad \text{and} \quad 691\,112\,321\,114\,112.$$

Nevertheless, we can show that there exist infinitely many insolite numbers containing the digit 9.

In fact, we will show that, for each positive integer m, there exists $n \in I$ containing m times the digit 9 and $(3^{4m+1} - 81m)$ times the digit 1.

We first observe that for such an integer n, we have $p(n) = 3^{4m}$ and $s(n) = 3^{4m+1}$. Hence if we can show that, by placing the 9's in the appropriate positions in the chosen integer, then $3^{4m+1}|n$, from which it will follow that $n \in I$.

Moreover, regardless of the positions of the $9's$, it is clear that such a number is necessarily divisible by 9, since the sum of its digits is a multiple of 9.

We first consider the case $m = 1$. In this case, we examine the numbers containing one 9 and 162 times the digit 1. For each non-negative integer $k < 162$, set

$$n_k := \underbrace{11\ldots1}_{k}9\underbrace{11\ldots1}_{162-k}$$

and for any two integers k, ℓ, with $0 \le k < \ell < 162$, consider the difference

$$\begin{aligned} n_k - n_\ell &= \underbrace{11\ldots1}_{k}9\underbrace{11\ldots1}_{162-k} - \underbrace{11\ldots1}_{\ell}9\underbrace{11\ldots1}_{162-\ell} = 7\underbrace{99\ldots9}_{\ell-k-1}2\underbrace{00\ldots0}_{162-\ell} = \\ &= 72 \times \underbrace{11\ldots1}_{\ell-k}\underbrace{00\ldots0}_{162-\ell}. \end{aligned}$$

It is clear that $9|(n_k - n_\ell)$ and that

$$(1) \qquad 243|(n_k - n_\ell) \Longleftrightarrow 27|\underbrace{11\ldots1}_{\ell-k} \Longleftrightarrow 27|(\ell - k).$$

Our goal will be to show that there exists k_0, $0 \le k_0 \le 26$, such that $n_{k_0} \equiv \equiv 0 \pmod{243}$, thereby allowing us to conclude that $n_{k_0} \in I$. We shall proceed by contradiction in assuming that such a number k_0 does not exist, that is that

$$(2) \qquad n_k \not\equiv 0 \pmod{243} \quad \text{for } k = 0, 1, 2, \ldots, 26.$$

But for $0 \le k < \ell \le 26$, we have that 27 does not divide $\ell - k$, in which case because of (1) it follows that

$$(3) \qquad n_k \not\equiv n_\ell \pmod{243}.$$

But the 27 numbers n_k each belong to one of the 27 congruence classes $0, 9, 18, 27, \ldots, 234$ modulo 243 (because $9|n_k$ for each k). Because of hypothesis (2), the first congruence class must be excluded and we conclude that the 27 numbers n_k each belong to one of the 26 remaining congruence classes 9,18,27,...,234 modulo 243. By Dirichlet's principle, it follows that two of the n_k's must belong to the same congruence class, say n_k and n_ℓ for certain k, ℓ with $0 \le k < \ell \le 26$, in which case $n_k \equiv n_\ell \pmod{243}$, which contradicts (3). We may therefore conclude that there exists an integer $k_0 \in [0, 26]$ such that $n_{k_0} \equiv 0 \pmod{243}$ and therefore that $n_{k_0} \in I$.

In fact, using a computer, we find that $k_0 = 26$ is the appropriate choice and moreover that the six 163-digit numbers

$$\underbrace{11\ldots1}_{161}91, \quad \underbrace{11\ldots1}_{134}9\underbrace{11\ldots1}_{28}, \quad \underbrace{11\ldots1}_{107}9\underbrace{11\ldots1}_{55}, \quad \underbrace{11\ldots1}_{80}9\underbrace{11\ldots1}_{82},$$
$$\underbrace{11\ldots1}_{53}9\underbrace{11\ldots1}_{109} \text{ and } \underbrace{11\ldots1}_{26}9\underbrace{11\ldots1}_{136}$$

are all insolite numbers.

We now move to the case $m = 2$ and follow essentially the same reasoning. Indeed, since $3^{4m+1} - 81m = 19683 - 162 = 19521$, for each non-negative integer $k < 19521$, we define

$$n_k := \underbrace{11\ldots1}_{k}9\underbrace{11\ldots1}_{19521-k}9$$

and, for $0 \leq k < \ell \leq 2186 = 3^7 - 1$, we consider the difference

$$n_k - n_\ell = \underbrace{11\ldots1_k}_{}9\underbrace{11\ldots1}_{19521-k}9 - \underbrace{11\ldots1}_{\ell}9\underbrace{11\ldots1}_{19521-\ell}9 = 72 \times \underbrace{11\ldots1}_{\ell-k}\underbrace{00\ldots0}_{19522-\ell}.$$

As in the case $m = 1$, it is clear that $9|(n_k - n_\ell)$ and moreover that $19683|(n_k - n_\ell)$ if and only if $2187|(\ell - k)$. But, for $0 \leq k < \ell \leq 2186$, we have that 2187 does not divide $\ell - k$, in which case $n_k \not\equiv n_\ell \pmod{19683}$. Following the same reasoning, we conclude that there exists an integer $k_0 \in [0, 2186]$ such that 19683 divides n_{k_0}, which identifies n_{k_0} as an insolite number. A computer search reveals that $k_0 = 1879$.

For the general case, given a positive integer m, we define

$$n_k := \underbrace{11\ldots1}_{k}9 \quad \underbrace{11\ldots1}_{3^{4m+1}-81m-k} \quad \underbrace{99\ldots9}_{m-1} \tag{4}$$

and consider the difference

$$n_k - n_\ell = 72 \times \underbrace{11\ldots1}_{\ell-k} \quad \underbrace{00\ldots0}_{3^{4m+1}-80m-1-\ell} \tag{5}$$

Again we have that $9|(n_k - n_\ell)$ and moreover that $3^{4m+1}|(n_k - n_\ell)$ if and only if $3^{4m-1}|(\ell - k)$. But, for $0 \leq k < \ell \leq 3^{4m-1} - 1$, we have that 3^{4m-1} does not divide $\ell - k$, so that

$$n_k \not\equiv n_\ell \pmod{3^{4m+1}}.$$

Following the same reasoning, we conclude that there exists an integer $k_0 \in$ $\in [0, 3^{4m-1} - 1]$ such that $3^{4m+1}|n_{k_0}$, which identifies n_{k_0} as an insolite number.

Hence to each positive integer m, there corresponds an insolite number containing m 9's. It follows from this that there exist an infinite number of insolite numbers containing the digit 9.

Remark 1. Using essentially the same argument, it is possible to show that there exist infinitely many insolite numbers containing the digit 3. Indeed given a positive integer m, one can construct $n \in I$ made up of $3m$ times the digit 3 and $3^{6m-1} - 27m$ times the digit 1. For such an integer $n \in I$, we have

$p(n) = 3^{6m}$, while $s(n) = 3^{6m-1}$. Hence proving that $3^{6m}|n$ will establish that $n \in I$. We shall only examine the case $m = 1$, the general case being generally similar to the above "digit 9" proof.

We want to construct an integer n made up of 3 times the digit 3 and $3^5 - 27 = 216$ times the digit 1 in such a way that if the 3's and the 1's are placed in the appropriate positions, we have that $n \in I$. To do so, for each non-negative integer $k < 216$, we define

$$n_k = \underbrace{11\ldots1}_{k}3\underbrace{11\ldots1}_{216-k}33.$$

It is clear that $9|n_k$ since the sum of its digits is a multiple of 9, namely 225.

Now for $0 \le k < \ell \le 3^4 - 1 = 80$, consider the difference

$$\begin{aligned} n_k - n_\ell &= \underbrace{11\ldots1}_{k}3\underbrace{11\ldots1}_{216-k}33 - \underbrace{11\ldots1}_{\ell}3\underbrace{11\ldots1}_{216-\ell}33 = \\ &= 1\underbrace{99\ldots9}_{\ell-k-1}8\underbrace{00\ldots0}_{216-\ell+2} = 9 \times \underbrace{22\ldots2}_{\ell-k}\underbrace{00\ldots0}_{216-\ell+2}. \end{aligned} \tag{6}$$

We want to show that there exists an integer $k_0 \in [0, 3^4 - 1]$ such that $n_{k_0} \equiv \equiv 0 \pmod{3^6}$. Assume the contrary, that is that

$$n_k \not\equiv 0 \pmod{3^6} \quad \text{for } k = 0, 1, 2, \ldots, 80. \tag{7}$$

It is clear, from (6), that $81 \not| (n_k - n_\ell)$ for $0 \le k < \ell \le 80$, which certainly implies that

$$n_k \not\equiv n_\ell \pmod{3^6}. \tag{8}$$

But the 81 numbers $n_0, n_1, \ldots, n_{80}$ each belong to one of the 81 congruence classes $0, 9, 18, \ldots, 720$ modulo 729 (since $9|n_k$ for all k). Because of hypothesis (7), the first congruence class must be excluded and therefore the 81 numbers n_k must belong to the remaining 80 classes $9, 18, \ldots, 720$ modulo 729. Hence by Dirichlet's principle, two of the n_k's must belong to the same congruence class, say n_k and n_ℓ for certain k, ℓ with $0 \le k < \ell \le 80$, in which case we have $n_k \equiv n_\ell \pmod{3^6}$, which contradicts (8). This proves that there exists an integer $k_0 \in [0, 80]$ such that $n_{k_0} \equiv 0 \pmod{3^6}$ and therefore that $n_{k_0} \in I$. A computer search reveals that $k_0 = 8$ is the appropriate choice.

Given an arbitrary positive integer m, we consider the integers

$$n_k = \underbrace{11\ldots1}_{k}3 \quad \underbrace{11\ldots1}_{3^{6m-1}-27m-k} \quad \underbrace{33\ldots3}_{3m-1},$$

and observe that $p(n_k) = 3^{6m}$, $s(n_k) = 3^{6m-1}$ and that the sum of the digits of n_k is a multiple of 9, namely $3^{6m-1} - 18m$. Then following the same argument, we find an appropriate integer $k_0 \in [0, 3^{6m-2} - 1]$ such that $n_{k_0} \in I$.

Remark 2. A quick look at the list of insolite numbers given in Section 6 suggests that most insolite numbers are even. However, so far, we have only displayed infinite families of odd insolite numbers. By slightly modifying the definition of the n_k's given in Remark 1, we can construct an infinite family of even insolite numbers. Indeed given a positive integer m, we first consider the integers

$$n_k = \underbrace{11\ldots1}_{k}3\quad\underbrace{11\ldots1}_{3^{6m+4}-27m-33-k}\quad\underbrace{33\ldots3}_{3m+1}122112,$$

where $0 \leq k < 3^{6m-1} - 27m$. For such an integer n_k, we have that

$$p(n_k) = 3^{6m+4} \cdot 2^6 \quad \text{and} \quad s(n_k) = 3^{6m+4}.$$

Clearly $2^6 | n_k$ for all k. Hence we only need to show that $3^{6m+4} | n_k$ for some positive integer $k = k_0$.

But observing that $9 | n_k$ for each k and then considering the difference $n_k - n_\ell$ for $0 \leq k < \ell \leq 3^{6m+2} - 1$, we obtain, following the same argument as in Remark 1, the existence of an integer $k_0 \in [0, 3^{6m+2} - 1]$ such that $n_{k_0} \in I$, thereby putting together an infinite family of even insolite numbers.

4. An upper bound for the number of insolite numbers smaller than a given quantity

It is clear that the set of insolite numbers is of density zero. Indeed, let $I(x)$ be the number of insolite numbers $\leq x$ and set $\mu := [\log x / \log 10]$. Since by definition an insolite number does not contain the digit 0, it follows that the number of k-digit insolite numbers $\leq x$ is smaller than

$$9^k \leq 9^\mu \leq 9^{\log x / \log 10} = x^{\log 9 / \log 10} < x^{0.955},$$

which implies that

$$I(x) \leq \sum_{k=1}^{\mu} 9^k \leq \mu \cdot 9^\mu \ll \log x \cdot x^{0.955} \ll x^{0.96},$$

which proves our claim.

This upper bound for $I(x)$ can easily be improved. First write

$$I(x) = I^{(o)}(x) + I^{(e)}(x), \tag{9}$$

where $I^{(o)}(x)$ (resp. $I^{(e)}(x)$) stand for the number of odd (resp. even) insolite numbers $\leq x$. Since the only possible digits of an odd insolite number are $1, 3, 5, 7$ and 9, we have that

$$I^{(o)}(x) \leq \sum_{k=1}^{\mu} 5^k \leq \mu \cdot 5^\mu \ll \log x \cdot 5^{\log x/\log 10} = \log x \cdot x^{\log 5/\log 10} \ll x^{0.7}. \tag{10}$$

On the other hand, since the only possible digits of an even insolite number are $1, 2, 3, 4, 6, 7, 8$ and 9, we have that

$$I^{(e)}(x) \leq \sum_{k=1}^{\mu} 8^k \leq \mu \cdot 8^\mu \ll \log x \cdot 8^{\log x/\log 10} = \log x \cdot x^{\log 8/\log 10} \ll x^{0.9031}. \tag{11}$$

Combining (10) and (11) in (9), we get that

$$I(x) \ll x^{0.904}.$$

Now using a more delicate argument, we can show that

$$I(x) \ll x^c \quad \text{with } c \approx 0.523. \tag{12}$$

To prove this, first write J_k for the number of k-digit even insolite numbers. Fix x large and let μ be as above so that $I^{(e)}(x) = \sum_{k=1}^{\mu} J_k$. By writing $n = \langle d_1, d_2 \ldots, d_k \rangle$, we mean that $d_1, d_2, \ldots, d_k$ are the k decimal digits of $n = \sum_{i=1}^{k} d_i 10^{k-i}$.

For a given $k \leq \mu$, write $k = r + s$, where $r \geq s > 0$ are to be chosen later, and for each non-negative integer $j \leq r$, denote by I_j the number of k-digit even insolite numbers $n = \langle d_1, d_2, \ldots, d_r, d_{r+1}, \ldots, d_{r+s} \rangle$ for which j of the first r digits are different from 1.

Now fixing the number $R := \langle d_1, d_2, \ldots, d_r \rangle$ and using the fact that $p := \prod_{i=1}^{r} d_i^2$ divides n and that the last digit of n is even, we want to count those integers m such that

$$R \cdot 10^s + \underbrace{11\ldots1}_{s-1}2 \leq n = 4 \cdot p \cdot m \leq R \cdot 10^s + \underbrace{99\ldots9}_{s-1}6,$$

that is
$$\frac{R\cdot 10^s+\underbrace{11\dots1}_{s-1}2}{4p}\le m\le\frac{R\cdot 10^s+\underbrace{99\dots9}_{s-1}6}{4p}.$$

Since the number of positive integers in an interval $[a,b]$ does not exceed $b-a+1$, it follows that the number of such integers m is less than

$$\frac{\underbrace{99\dots9}_{s-1}6}{4p}-\frac{\underbrace{11\dots1}_{s-1}2}{4p}+1<\frac{\underbrace{88\dots8}_{s}}{4p}+1=\frac{8}{4p}\cdot\frac{10^s-1}{9}+1<\frac{8}{36}\cdot\frac{10^s}{p}+1\le$$
$$\le\frac{2}{9}\cdot\frac{10^s}{4^j}+1,$$

since $p\ge 4^j$. Note further that among the first r digits of such an integer n, there are $\binom{r}{j}$ possible positions for these j digits, and in each of these $\binom{r}{j}$ positions we may place either one of the 7 digits $2,3,4,6,7,8,9$; here we omitted the digit 5 because n is even.

Combining these estimates, we obtain that

$$J_k=\sum_{j=0}^{r}I_j< \tag{13}$$
$$<\sum_{j=0}^{r}\binom{r}{j}7^j\left(\frac{8}{36}\frac{10^s}{4^j}+1\right)=\frac{2}{9}\cdot 10^s\sum_{j=0}^{r}\binom{r}{j}\left(\frac{7}{4}\right)^j+\sum_{j=0}^{r}\binom{r}{j}7^j=$$
$$=\frac{2}{9}\cdot 10^s\left(1+\frac{7}{4}\right)^r+(1+7)^r=\frac{2}{9}\cdot 10^s\left(\frac{11}{4}\right)^r+8^r<10^s\left(\frac{11}{4}\right)^r+8^r.$$

In the above, we only used the fact that $p\ge 4^j$, where j is the number of digits among $d_1,\dots,d_r$ which are greater than 1. We can do better if, among these j digits, we consider those j_2 digits which are greater than 2; for these we shall get $p\ge 9^{j_2}\cdot 4^{j-j_2}$, in which case we have

$$J_k<\sum_{j=0}^{r}\binom{r}{j}\sum_{j_2=0}^{j}\binom{j}{j_2}6^{j_2}\left(\frac{10^s}{9^{j_2}\cdot 4^{j-j_2}}+1\right)=$$
$$=10^s\sum_{j=0}^{r}\binom{r}{j}\frac{1}{4^j}\sum_{j_2=0}^{j}\binom{j}{j_2}\left(\frac{6\cdot 4}{9}\right)^{j_2}+\sum_{j=0}^{r}\binom{r}{j}\sum_{j_2=0}^{j}\binom{j}{j_2}6^{j_2}=$$

$$= 10^s \sum_{j=0}^{r} \binom{r}{j} \left(\frac{33}{36}\right)^j + \sum_{j=0}^{r} \binom{r}{j} 7^j =$$

$$= 10^s \left(\frac{23}{12}\right)^r + 8^r,$$

which improves (13) since $\frac{23}{12} < \frac{11}{4}$.

We now take this procedure a step further, by considering, among the j_2 digits ≥ 3, those j_3 digits which are ≥ 4, thus providing $p \geq 16^{j_3} 9^{j_2-j_3} 4^{j-j_2}$. Repeating this until we reach the digit 9 (again omitting the digit 5), we get in the end

$$J_k < \sum_{j=0}^{r} \binom{r}{j} \sum_{j_2=0}^{j} \binom{j}{j_2} \sum_{j_3=0}^{j_2} \binom{j_2}{j_3} \sum_{j_4=0}^{j_3} \binom{j_3}{j_4} \sum_{j_5=0}^{j_4} \binom{j_4}{j_5} \times$$
$$\times \sum_{j_6=0}^{j_5} \binom{j_5}{j_6} \sum_{j_7=0}^{j_6} \binom{j_6}{j_7} \left(\frac{10^s}{p^*} + 1\right),$$

with

$$p^* = 81^{j_7} \cdot 64^{j_6-j_7} \cdot 49^{j_5-j_6} \cdot 36^{j_4-j_5} \cdot 16^{j_3-j_4} \cdot 9^{j_2-j_3} \cdot 4^{j-j_2}.$$

It then follows that

$$(14) \qquad J_k < 10^s \cdot c_1^r + 8^r, \quad \text{with } c_1 = \frac{380965}{254016} \approx 1.49977.$$

It remains to choose r and thereby $s = k - r$ so that the right hand side of the above inequality is minimal. We therefore need to find the minimum of

$$f(r) := 10^k \left(\frac{c_1}{10}\right)^r + 8^r,$$

k being fixed. For this purpose, assume that r is real and look for r_0 such that $f'(r_0) = 0$. Doing so, we find

$$r_0 = \frac{k \log 10 + \log(\log(10/c_1)/\log 8)}{\log(80/c_1)} = c_2 k + \frac{\log(\log(10/c_1)/\log 8)}{\log(80/c_1)},$$

where

$$c_2 = \frac{\log 10}{\log(80/c_1)} \approx 0.579.$$

Hence substituting $r = [c_2 k]$ back into (14) and using the fact that $k \le \eta = [\log x/\log 10]$, we finally get

$$J_k \ll x^{c_3}, \quad \text{with } c_3 \approx 0.5228,$$

which proves that

$$I^{(e)}(x) = \sum_{k=1}^{\mu} J_k \le \frac{\log x}{\log 10} J_\mu \ll (\log x) x^{c_3} \ll x^{0.523}. \tag{15}$$

In order to estimate $I^{(o)}(x)$, we use the same method. Since all the digits of an odd insolite number are odd, we get

$$\begin{aligned} I^{(o)}(x) &< \frac{8}{9} \sum_{j=0}^{r} \binom{r}{j} \sum_{j_2=0}^{j} \binom{j}{j_2} \sum_{j_3=0}^{j_2} \binom{j_2}{j_3} \sum_{j_4=0}^{j_3} \binom{j_3}{j_4} \times \\ &\quad \times \left(\frac{10^s}{81^{j_4} 49^{j_3-j_4} 25^{j_2-j_3} 9^{j-j_2}} + 1 \right), \\ &= \frac{8}{9} \cdot 10^s \cdot c_4^r + 5^r < 10^s \cdot c_4^r + 5^r, \end{aligned}$$

where $c_4 \approx 1.184$, which leads to

$$I^{(o)}(x) \ll x^{0.431}. \tag{16}$$

Combining (15) and (16), and recalling that $I(x) = I^{(o)}(x) + I^{(e)}(x)$, we obtain (12).

We shall now refine this argument in order to obtain $I(x) \ll x^{0.462}$; to do this, because of (16), we only need to show that

$$I^{(e)}(x) \ll x^c \quad \text{with } c = 0.462. \tag{17}$$

Let $1 < s < k$ be fixed integers. Let n be a k-digit even insolite number and write it as follows:

$$n = \langle d_1, d_2, \ldots, d_s, d_{s+1}, \ldots, d_{s+t}, d_{s+t+1}, \ldots, d_k \rangle, \tag{18}$$

where t is to be determined later. For convenience, define u implicity by

$$s + t + u = k.$$

For the moment, let the digits $d_1, d_2, \ldots, d_s$ be fixed, denoting this part of n by S, and set $p_S = \prod_{i=1}^{s} d_i^2$ and denote by H_S the number of such even insolite numbers n. Consider the digits $d_{s+1}, d_{s+2}, \ldots, d_{s+t}$, denoting this part of n by T. Proceeding as above, but this time letting j be the number of digits among those in part T which are greater than 1, we get that

$$H_S \leq \sum_{j=0}^{t} \binom{t}{j} \sum_{j_2=0}^{j} \binom{j}{j_2} \sum_{j_3=0}^{j_2} \binom{j_2}{j_3} \sum_{j_4=0}^{j_3} \binom{j_3}{j_4} \sum_{j_5=0}^{j_4} \binom{j_4}{j_5} \sum_{j_6=0}^{j_5} \binom{j_5}{j_6} \times$$
$$\times \sum_{j_7=0}^{j_6} \binom{j_6}{j_7} \left(\frac{8}{36} \frac{10^u}{p_S \cdot p^*(T)} + 1 \right),$$

where

$$p^*(T) = 81^{j_7} \cdot 64^{j_6 - j_7} \cdot 49^{j_5 - j_6} \cdot 36^{j_4 - j_5} \cdot 16^{j_3 - j_4} \cdot 9^{j_2 - j_3} \cdot 4^{j - j_2}.$$

It follows that

$$H_S < \frac{10^u \cdot c_1^t}{p_S} + 8^t, \tag{19}$$

where c_1 is the constant defined in (14).

We now fix the sum $t + u = r$ and look for the minimum value of the right hand side of (19) as t varies. First we write

$$f(t) := \frac{10^{r-t} \cdot c_1^t}{p_S} + 8^t = \frac{e^{r \log 10} \cdot e^{t(\log c_1 - \log 10)}}{p_S} + e^{t \log 8}.$$

Solving

$$f'(t) = \frac{e^{t(\log c_1 - \log 10)}(\log c_1 - \log 10) e^{r \log 10}}{p_S} + \log 8 \cdot e^{t \log 8} = 0,$$

we find

$$t = \frac{r \log 10 - \log p_S}{\log(80/c_1)} - c_5, \tag{20}$$

where $c_5 = \dfrac{\log\log 8 - \log\log(10/c_1)}{\log(80/c_1)} \approx 0.0368$.

In order to keep t non-negative, we shall set $t = 0$ if $p_S > 10^r$. Substituting $t = t_0 := \left[\frac{r\log 10 - \log p_S}{\log(80/c_1)}\right]$ in the right hand side of (19), we conclude that

$$H_S < \frac{10^{r-t_0+1}c_1^{t_0}}{p_S} + 8^{t_0} < \frac{20 \cdot c_6^r}{p_S^{c_7}},$$

where

$$c_6 \approx 3.33, \quad c_7 \approx 0.523.$$

So far, we have kept the first s digits fixed. We shall now allow these digits to vary. Recalling that J_k stands for the number of k-digit even insolite numbers and letting m denote the number of digits (among $d_1, \ldots, d_s$) which are larger than 1, while $m_2, \ldots, m_7$ are defined as the j_i's above, we have if $p_S < 10^r$, that

$$\begin{aligned} J_k < &\sum_{m=0}^{s}\binom{s}{m}\sum_{m_2=0}^{m}\binom{m}{m_2}\sum_{m_3=0}^{m_2}\binom{m_2}{m_3}\sum_{m_4=0}^{m_3}\binom{m_3}{m_4}\sum_{m_5=0}^{m_4}\binom{m_4}{m_5}\times \\ &\times\sum_{m_6=0}^{m_5}\binom{m_5}{m_6}\sum_{m_7=0}^{m_6}\binom{m_6}{m_7}\frac{20 \cdot c_6^r}{p_S^{c_7}}, \end{aligned} \tag{21}$$

where

$$p_S = 81^{m_7} \cdot 64^{m_6-m_7} \cdot 49^{m_5-m_6} \cdot 36^{m_4-m_5} \cdot 16^{m_3-m_4} \cdot 9^{m_2-m_3} \cdot 4^{m-m_2}.$$

Proceeding as we did earlier, we find that

$$J_k < 20_{c_6}^r \cdot c_8^s, \tag{22}$$

where

$$c_8 = 1 + \frac{1}{81^{c_7}} + \frac{1}{64^{c_7}} + \frac{1}{49^{c_7}} + \frac{1}{36^{c_7}} + \frac{1}{16^{c_7}} + \frac{1}{9^{c_7}} + \frac{1}{4^{c_7}} \approx 2.5339.$$

It remains to consider the case $p_S > 10^r$, for which we get

$$\begin{aligned} J_k < &\sum_{m=0}^{s}\binom{s}{m}\sum_{m_2=0}^{m}\binom{m}{m_2}\sum_{m_3=0}^{m_2}\binom{m_2}{m_3}\sum_{m_4=0}^{m_3}\binom{m_3}{m_4}\sum_{m_5=0}^{m_4}\binom{m_4}{m_5}\times \\ &\times\sum_{m_6=0}^{m_5}\binom{m_5}{m_6}\sum_{m_7=0}^{m_6}\binom{m_6}{m_7} 1 = 8^s. \end{aligned} \tag{23}$$

Thus combining (22) and (23) to cover both cases, we get

$$(24) \qquad J_k < 20 \cdot c_6^r \cdot c_8^s + 8^s = 20 \cdot c_6^{k-s} \cdot c_8^s + 8^s = g(s),$$

say. We shall now look for the minimum of $g(s)$ as s varies. Hence solving $g'(s) = 0$, we find

$$(25) \qquad s = \frac{k \log c_6}{\log(8c_6/c_8)} - \frac{\log\log 8 - \log\log(c_6/c_9)}{\log(8c_6/c_8)} = c_{10}k - c_{11},$$

where $c_{10} \approx 0.5116$ and $c_{11} \approx 0.8603$. Substituting $s = [c_{10}k]$ in (24), we obtain, for each $k \leq \mu$,

$$(26) \qquad J_k \ll c_6^{k-c_{10}k+1} c_8^{c_{10}k} + 8^{c_{10}k} \ll x^{c_{10}\log 8/\log 10} = x^{c_{12}},$$

say, where $c_{12} < 0.462$, which in turn yields

$$I^{(e)}(x) = \sum_{k=1}^{\mu} J_k \leq \frac{\log x}{\log 10} J_\mu \ll (\log x) x^{c_{12}} \ll x^{0.462},$$

which establishes (17) and thus proves that

$$I(x) \ll x^c \quad \text{with } c = 0.462.$$

5. A lower bound for $I(x)$

From the fact that the numbers $\underbrace{11\ldots1}_{3^\alpha}$, where $\alpha = 1, 2, \ldots$ are all insolite, it follows that $I(x) \gg \log\log x$.

We shall now exploit the argument developped in Section 3 to show that

$$(27) \qquad I(x) \gg e^{\frac{1}{5}(\log\log x)^2 + O(\log\log x \log\log\log x)}.$$

First we observe that while still maintaining $3^{4m+1} - 80m - 1 - \ell$ zeros in the tail end of the difference $n_k - n_\ell$ given by (5), we can vary slightly the definition of n_k given in (4), by moving to the right some or all of the $m - 1$ digits 9 which now all appear in the tail of n_k. In other words, our proof would have followed the same path from relation (5) on, had we defined n_k as

$$n_k := \underbrace{11\ldots1}_{k}9\ \underbrace{11\ldots1}_{3^{4m-1}-k-1}\ \underbrace{11\ldots191\ldots11\ldots191\ldots1}_{8\cdot3^{4m-1}-81m+1+(m-1)\text{ digits}},$$

where this last part of n_k is made of $8\cdot3^{4m-1}-80m+1$ times the digit 1 and $m-1$ times the digit 9. This means that the number of acceptable choices for the definition of n_k is equal to $Y := \binom{8\cdot3^{4m-1}-80m}{m-1}$.

Noting that for a fixed positive integer m, the corresponding number n_k has $3^{4m+1}-80m$ digits, it follows that, given a large number x and setting $\mu = [\log x/\log 10]$, one can easily see that for the largest integer $n_k \le x$, we have that

$$3^{4m+1}-80m \le \mu < 3^{4(m+1)+1}-80(m+1) = 9\cdot3^{4m+3}-80m-80. \tag{28}$$

Hence, if m is a large enough, we have

$$8\cdot3^{4m-1}-80m > \frac{4}{9^3}\left(9\cdot3^{4m+3}-80m-80\right) > \frac{4}{9^3}\mu > \frac{1}{729}\cdot\log x. \tag{29}$$

It also follows from (28) that, if m is large enough,

$$2\cdot3^5\cdot3^{4m} > 3^4\cdot3^{4m+1}-80m-80 > \mu,$$

so that

$$3^{4m} > \frac{\mu}{2\cdot3^5}$$

and therefore

$$m > \frac{\log\mu-\log(2\cdot3^5)}{4\log3} > \frac{1}{5}\log\log x. \tag{30}$$

It follows from (29) and (30) that

$$Y > \binom{\left[\frac{1}{729}\log x\right]}{\left[\frac{1}{5}\log\log x\right]} = \binom{a}{b}, \tag{31}$$

say.

Using Stirling's formula in the form

$$n^n e^{-n}\sqrt{2\pi n} < n! < \left(1+\frac{1}{n}\right) n^n e^{-n}\sqrt{2\pi n}, \tag{32}$$

we obtain

$$\binom{a}{b} = \frac{a!}{b!(a-b)!} > \frac{a^{a+1/2}}{b^{b+1/2}(a-b)^{a-b+1/2}(1+1/b)(1+1/(a-b))\sqrt{2\pi}}. \tag{33}$$

Using the fact that $\log(1-\theta) < -\theta$ for $0<\theta<1$ and that $(1+1/b)(1+1/(a-$ $-b))\sqrt{2\pi} < 3$ if x is large enough, we get from (33) that

$$\log\binom{a}{b} >$$

$$\begin{aligned}
&> \left(a+\frac{1}{2}\right)\log a - \left(b+\frac{1}{2}\right)\log b - \left(a-b+\frac{1}{2}\right)\left(\log a - \frac{b}{a}\right) - \log 3 = \\
&= b\log a - b\log b + b - \frac{1}{2}\log b - \frac{b^2}{a} + \frac{b}{2a} - \log 3 = \\
&= b\log a - b\log b + b + O(\log b) = \\
&= \frac{1}{5}(\log\log x)^2 + O(\log\log x \log\log\log x),
\end{aligned}$$

which, recalling (31), proves (27).

Remark 3. By a heuristic argument, we can show that $I(x) > x^{\eta}$ for some real number $\eta > 0$. The argument goes as follows.

Let $n = \langle d_1, d_2, \ldots, d_k\rangle$ be a k-digit number made up of m times the digit 3 and $k-m$ times the digit 1, so that

$$p = p(n) := \prod_{i=1}^{k} d_i^2 = 9^m \quad \text{and} \quad s = s(n) := \sum_{i=1}^{k} d_i^2 = 9m + (k-m) = 8m + k.$$

In order to have that $s|p$, we let $k = 19m$ (so that $s = 27m$) and $m = 3^{\beta}$, $\beta \geq 1$, so that

$$s = 27m = 3^{\beta+3} | 3^{2m} = p,$$

since $\beta + 3 \leq 2m$. We therefore have $\binom{k}{m} = \binom{19m}{m}$ such integers n.

Noting that $\sum_{i=1}^{k} d_i = 3m + (k - m) = 2m + k = 21m = 21 \cdot 3^\beta$, it is clear that $9|n$. Therefore it follows that the expected number of insolite numbers among these numbers n, is equal to

$$Q(m) := \frac{1}{3^{2m-2}} \binom{k}{m} = \frac{1}{3^{2m-2}} \binom{19m}{m}.$$

It remains to find a lower bound for $Q(m)$. Again using (32), we get that, for each $m \geq 1$,

$$\begin{aligned} Q(m) = 9\frac{1}{9^m}\frac{(19m)!}{(18m)!m!} &> \frac{9}{2\sqrt{2\pi}}\frac{1}{\sqrt{m}}\frac{1}{9^m}\frac{(19m)^{19m}}{(18m)^{18m}m^m} = \\ &< \frac{1}{\sqrt{m}}\frac{1}{9m}\frac{19^{19m}}{18^{18m}} = \frac{1}{\sqrt{m}}\left(\frac{19^{19}}{9 \cdot 18^{18}}\right)^m > (5.5)^m. \end{aligned}$$

Hence, given a large number x, let $\mu = [\log x/\log 10]$. Then let k be the largest integer of the form $k = 19 \cdot 3^\beta \leq \mu$, so that

$$\begin{aligned} k = 19 \cdot 3^\beta &\leq \mu < 19 \cdot 3^{\beta+1}, \\ 19m &\leq \mu < 19 \cdot 3 \cdot m. \end{aligned}$$

We then have

$$57m > \mu > \frac{\log x}{\log 10} - 1,$$

so that

$$m \geq \frac{\log x}{57 \cdot \log 10}.$$

It follows that

$$I(x) > 9(5.5)^m > 9(5.5)^{\log x/(57 \cdot \log 10)} = x^\eta,$$

where

$$\eta = \frac{\log 5.5}{57 \cdot \log 10} \approx 0.0129,$$

as required.

6. The list of insolite numbers $< 10^{18}$

To identify the insolite numbers smaller than 10^{18}, we proceed as follows. First, using a computer, one easily finds the only two insolite numbers of no more than 6 digits, namely 111 and 11 112. To find all insolite numbers of k digits, for each integer $k \in [7, 18]$, we use the procedure developped in Section 4, treating separately the search for even insolite numbers from that of the odd ones. We thus obtain the 195 elements of I given below.

111	11 111 121 216	32 111 111 232	131 111 132 112
11 112	11 112 122 112	111 122 111 232	131 112 122 112
1 122 112	11 121 114 112	111 132 122 112	211 111 322 112
111 111 111	11 132 111 232	111 211 322 112	211 121 114 112
122 121 216	11 133 122 112	111 312 122 112	311 112 122 112
1 111 112 112	11 213 111 232	112 111 322 112	911 131 213 824
1 111 211 136	11 311 322 112	113 112 122 112	1 111 111 113 312
1 116 122 112	12 111 213 312	121 111 216 128	1 111 121 114 112
1 211 162 112	21 111 311 232	121 111 322 112	1 121 313 321 216
11 111 113 116	31 111 221 312	121 121 114 112	1 331 611 322 112

11 111 111 114 112	12 113 411 162 112	111 111 311 111 112
11 111 111 211 312	12 121 141 211 136	112 221 411 213 312
11 111 111 312 112	12 142 111 113 216	123 111 311 118 336
11 126 112 141 312	21 111 212 122 112	123 412 112 474 112
11 211 111 111 312	21 132 161 114 112	211 211 261 116 416
11 221 121 114 112	21 214 111 113 216	211 912 113 131 712
11 311 111 112 112	21 311 114 121 216	311 111 111 111 112
11 611 142 111 232	41 111 131 226 112	311 111 111 411 136
12 111 212 122 112	111 111 111 111 312	323 113 121 114 112
12 111 213 146 112	111 111 111 122 112	691 112 321 114 112

1 111 111 111 112 112	1 221 114 113 114 112	11 111 111 111 642 112
1 111 111 137 122 112	1 311 213 111 386 112	11 111 111 212 122 112
1 111 112 121 434 112	1 312 414 721 114 112	11 111 131 121 421 312
1 111 311 612 122 112	1 711 122 111 111 168	11 111 713 119 122 112
1 112 111 412 314 112	2 112 121 221 414 912	11 111 731 111 111 113
1 112 112 142 221 312	2 121 114 412 122 112	11 114 216 111 112 192

1 118 123 112 333 312
1 133 111 113 221 312
1 143 212 114 313 216
1 221 111 131 111 424
2 131 214 171 111 424
2 311 141 121 114 112
2 911 112 172 122 112
8 111 111 182 221 312
11 117 311 111 311 111
11 121 111 121 416 192
11 131 111 113 818 112
11 131 117 111 113 111

11 161 121 122 271 232
11 311 181 111 181 312
11 322 114 111 111 168
11 672 111 311 322 112
11 914 213 121 114 112
12 111 141 121 114 112
12 231 111 621 722 112
13 111 182 132 314 112
13 111 222 111 322 112
13 111 131 117 111 111
13 311 611 232 141 312
14 113 111 811 162 112
14 312 122 311 131 136
16 111 117 116 122 112
16 211 411 111 411 712
17 111 113 131 111 111
18 111 112 112 111 616
18 111 132 422 111 232
18 111 211 411 341 312
18 211 112 111 112 192
21 111 111 213 311 232
21 113 121 132 122 112
22 111 111 212 122 112
22 214 111 618 138 112
22 413 113 311 113 216
27 121 123 116 122 112
31 211 413 313 421 312
32 111 132 114 313 216
32 411 112 111 194 112
61 161 114 113 114 112

111 111 113 111 121 216
111 112 312 223 711 232
111 117 113 126 111 232
111 117 322 241 114 112
111 122 317 121 421 312
111 141 121 123 418 112
111 211 113 112 141 824
111 211 114 612 211 712
111 211 141 121 114 112
111 272 111 211 131 136
111 281 413 121 114 112
111 311 221 121 114 112
111 311 322 211 418 112
111 711 111 242 121 216
111 711 312 211 113 216
111 721 311 311 162 112
112 111 111 121 114 112
112 172 113 111 113 216
112 211 418 113 114 112
112 217 111 114 121 216
112 433 111 114 121 216
112 711 113 121 211 136
113 111 221 344 141 312
113 133 111 111 122 112
113 171 111 111 181 312
113 211 111 122 141 184
113 311 121 411 211 264
113 341 142 112 141 312
114 217 113 132 122 112
116 113 171 111 322 112

119 612 124 161 114 112
121 111 313 111 111 232
121 126 117 121 114 112
121 131 111 132 122 112
122 291 211 114 381 312
131 121 114 211 418 112
133 243 141 121 114 112
141 111 212 114 313 216
142 121 211 121 631 232
161 212 211 111 411 712
164 116 129 218 822 144
211 111 111 261 274 112
211 113 121 713 242 112
211 117 212 161 114 112
211 131 112 223 711 232
211 216 141 313 114 112
211 231 237 121 114 112
211 311 611 111 123 712
211 431 132 161 114 112
212 133 113 111 642 112
213 131 211 611 111 424
217 311 112 113 242 112
221 371 111 313 114 112
224 111 222 411 231 232
311 121 111 114 842 112
313 141 212 161 114 112
331 111 112 211 431 424
331 119 133 111 112 112
331 122 111 127 911 936
371 161 111 111 322 112

411 181 112 321 114 112
421 141 121 311 113 216
611 131 121 414 111 232
623 131 141 121 114 112
711 813 411 914 121 216

7. Consecutive insolite numbers

It has been established by Kennedy & Cooper [3] that there exist infinitely many 20-tuples $(n, n+1, \ldots, n+19)$ of Niven numbers and furthermore that there was no such thing as 21 consecutive Niven numbers.

We can show that no three consecutive insolite numbers exist. Indeed assume first that both $n = \langle d_1, d_2, \ldots, d_k \rangle$ and $n+1 = \langle e_1, e_1, \ldots, e_k \rangle$ belong to I (clearly they must have the same number of digits). It is obvious that $1 \le d_k \le 8$ and that $d_i = e_i$ for $1 \le i \le k-1$. Hence $d_i | n$ and $d_i | n+1$ for $1 \le i \le k-1$, which implies that $d_i = e_i = 1$ for $1 \le i \le k-1$ and $e_k = d_k + 1$. Assuming that both $n = \langle 1, 1, \ldots, 1, d_k \rangle$ and $n+1 = \langle 1, 1, \ldots, 1, d_k + 1 \rangle$ belong to I, we consider separately the cases $d_k = 1, 2, \ldots, 8$. First observe that no insolite number ends with the digits 14 or 18. We may also exclude $d_k = 5$ because the last two digits of n would then be 15, instead of 75 as we have seen in Section 2. Similarly, we can exclude $d_k = 3, 7$ because then $e_k = 4, 8$ respectively, also a nonsense. Therefore the only three remaining possibilities are $d_k = 1, 2$ and 6.

Consider the first of these three cases. Let $n = 11 \ldots 1$ be a k-digit number such that $n, n+1 \in I$. Since $s(n+1) = k - 1 + 4 = k + 3$ must divide $n+1$, it follows that if we can show that k is divisible by 3, we shall have reached a contradiction (because then both n and $n+1$ would be multiples of 3). Since $s(n) = k$, we have that $k \left| \frac{10^k - 1}{9} \right.$, which means in a particular that $10^k \equiv 1 \pmod{k}$ and therefore that

$$10^k \equiv 1 \pmod{p} \tag{34}$$

where p is the smallest prime factor of k. On the other hand if we let a be the smallest positive integer such that $10^a \equiv 1 \pmod{p}$, then it follows from Fermat's Little Theorem that $a | (p-1)$ $(< p)$ and, because of (34), that $a | k$. But p being the smallest prime factor of k, this can only occur if $a = 1$. We have thus established that $10 \equiv 1 \pmod{p}$ and therefore that $p = 3$, thus proving that k is a multiple of 3.

We have thus shown that in order that $n, n+1 \in I$, we must have $n = \langle 1, 1, \ldots, 1, d_k \rangle$ and $n+1 = \langle 1, 1, \ldots, 1, d_k + 1 \rangle$ with $d_k = 2$ or $d_k = 6$. But in the first of these cases, the number $n+2$ ends with the digit 4, while in the second case it ends with the digit 8, and therefore in both cases $n+2 \notin I$, thus proving that no three consecutive insolite numbers exist.

As for the possibility of having two consecutive insolite numbers, it is most unlikely. Indeed, using a computer, one obtains that the only k-digit insolite numbers of the form $11\ldots12$, with $k < 10^5$ are those with $k = 5$, 86, 3701, 7766 and 63769, while those of the form $11\ldots13$ are those with $k = 2383$ and 25891. Finally, the only k-digit insolite numbers of the form $11\ldots16$ are those with $k = 193$, 769 and 2281, while we did not find any insolite number of the form $11\ldots17$ with less than one million digits.

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J.-M. De Koninck and N. Doyon
Département de mathématiques et de statistique
Université Laval
Québec G1K 7P4, Canada
jmdk@mat.ulaval.ca
ndoyon@mat.ulaval.ca