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LACUNARY INTERPOLATION WITH ARBITRARY DATA OF HIGH ORDER

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Abstract. In this paper we study at first a lacunary interpolation problem where the conditions given are on the function, on some of its intermediate and its highest derivatives at every node. Standard results such as convergence of optimal order for the solution scheme are established. Then by a suitable reformulation of the analysis the investigation is extended to other types of lacunary interpolation, where the intermediate conditions are relaxed.

1. Introduction

The objective of this investigation is the extension of the analysis done for the case of higher order interpolation [8] namely when information on a function and its highmost derivative is known at a set of nodes. In this study we want to consider the case of an extra condition being known on some intermediate derivatives, the same order at each node. Then we relax somewhat this statement, studying the conditions under which the problem is still meaningful if an arbitrary number of additional intermediate conditions is known. In this case we allow possibly a different number of conditions at each node.

The simplest lacunary interpolation problem under consideration here is the so-called $(0, p, q)$ problem, in which conditions on the function, on some intermediate and on its highest derivatives at every node are given. In the literature many instances of this situation have been considered. For instance,

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[5] considers the two cases $p = 2$ and $p = 3, 4$ respectively, finding a C^1 solution in the class of polynomials of degree 5 or 6. Similar cases are found in [3], [4], where the solution is sought as a G-spline. The solution is also found by means of quartic or quintic splines, see e.g. [1], [11], [7].

The paper is organized as follows. Section 2 contains the statement of the most elementary problem in consideration, the so-called $(0, p, q)$ problem. The structure of the matrix and the algorithm for the solution is presented in the following section. Section 4 contains the error analysis. In Section 5 the analysis of the problem is reformulated, and the changes used to investigate the problem in which the intermediate condition involves a derivative of an order dependent on the node where the condition is given. Finally in the last section the problem is generalized: the conditions are considered, under which an arbitrary number of additional intermediate data are given and lead to a meaningful problem that can be solved by a similar algorithm.

2. The problem

For simplicity, we state the interpolation problem over a set of equispaced nodes $x_k,\ k = 0, \ldots, n$. This statement however is not restrictive, since its elimination would lead to the same algorithm and to the same error estimates we obtain later, with the obvious replacement of h by $H \equiv \max_{1 \le k \le n} h_k$, where $h_k = x_k - x_{k-1}$. Let then be given the interpolation interval $[a, b]$ which we normalize to $[a, b] \equiv [0, 1]$, consider its uniform partition $x_k = kh,\ k = 0, \ldots, n,\ h = 1/n$. We want to determine the spline function $s(x) \in S^{(3)}_{n,q+3}$ where the notation emphasizes the "deficiency", i.e. the fact that it will be possible to satisfy the continuity of the derivatives at the breakpoints only up to three orders below the one of the highest known derivative. We assume that in each subinterval of the partition, $\Delta_k \equiv [x_{k-1},\ x_k]$, the spline is smooth enough, i.e. $s(x) \in C^q[0, 1]$. We denote by $s_k(x)$ the restriction of $s(x)$ to the interval Δ_k, $k = 1, \ldots, n$.

The parameters to be determined are thus $n(q+4)$, while the interpolatory conditions are $3(n + 1)$, explicitly

$$s_k^{(i)}(x_{k-1}) = f_{k-1}^{(i)}, \qquad k = 1, \ldots, n, \qquad i = 0, p, q, \tag{1}$$

$$s_n^{(i)}(x_n) = f_n^{(i)}, \qquad i = 0, p, q, \tag{2}$$

where $f_k^{(j)} = f^{(j)}(x_k)$.

The continuity conditions are instead $(q+1)(n-1)$, explicitly

(3) $$s_k^{(i)}(x_k) = s_{k+1}^{(i)}(x_k), \qquad i = 0, \ldots, q, \qquad k = 1, \ldots, n-1.$$

From this it follows that the free parameters are $n(q+4) - 3(n+1)-$ $-(q+1)(n-1) = q-2$. To have a well-posed problem we need then to choose additional conditions, which we take here in the form of "initial conditions",

(4) $$s_1^{(i)}(x_0) = f_0^{(i)}, \qquad i = 1, \ldots, q-1, \qquad i \neq p.$$

Let us further denote by

$$\begin{aligned} &M_{k-1}^{(i)} = s^{(i)}(x_{k-1}), \qquad i = 0, \ldots, q, \\ &M_{k-1}^{(i)} = s_k^{(i)}(x_{k-1}), \qquad i = q+1,\ q+2,\ q+3, \qquad \text{and} \\ &M_n^{(0)} = s_n(x_n),\ M_n^{(1)} = s_n^{(1)}(x_n), \ldots, M_n^{(q)} = s_n^{(q)}(x_n) \end{aligned}$$

the moments of $s(x)$ and of its successive derivatives at the breakpoints. We then can write

$$s_k(x) = \sum_{j=0}^{q+3} M_{k-1}^{(j)} \frac{(x - x_{k-1})^j}{j!}, \qquad k = 1, \ldots, n, \qquad x \in [x_{k-1}, x_k].$$

Let us recall moreover that

$$\begin{aligned} &M_{k-1}^{(i)} = f_{k-1}^{(i)}, \qquad i = 0, p, q, \qquad k = 2, \ldots, n+1 \ \ \text{and} \\ &M_0^{(i)} = f_0^{(i)}, \qquad i = 0, 1, \ldots, q. \end{aligned}$$

By imposing the interpolation conditions which are not implicitly satisfied, i.e. (2), and the continuity conditions (3) with

$$s_k^{(i)}(x) = \sum_{j=0}^{q+3-i} M_{k-1}^{(i+j)} \frac{(x - x_{k-1})^j}{j!}, \qquad i = 0, \ldots, q,$$

we obtain the square linear algebraic system in $(n-1)(q+1)+3$ equations in the unknowns

$$\begin{aligned} &M_0^{(i)}, \qquad i = q+3, q+2, q+1, \\ &M_s^{(i)}, \qquad s = 1, \ldots, n-1, \qquad i = q+3(-1)1, \qquad i \neq p, q. \end{aligned}$$

The latter are written in inverse order with respect to the derivative order. Several subcases must be distinguished, as it is apparent from the next formulae

$i = q$

$$\sum_{j=1}^{3} M_{k-1}^{(q+j)} \frac{h^j}{j!} = M_k^{(q)} - M_{k-1}^{(q)}, \qquad k = 1, \ldots, n;$$

$i = q - 1(-1)p + 1$

$$\sum_{j=1}^{3} M_0^{(q+j)} \frac{h^{q+j-i}}{(q+j-i)!} - M_1^{(i)} = -\sum_{j=0}^{q-i} M_0^{(j+i)} \frac{h^j}{j!},$$

$$\sum_{\substack{j=0 \\ j \neq q-i}}^{q+3-i} M_{k-1}^{(j+i)} \frac{h^j}{j!} - M_k^{(i)} = -M_{k-1}^{(q)} \frac{h^{q-i}}{(q-i)!}, \qquad k = 2, \ldots, n-1.$$

$i = p$

$$\sum_{j=1}^{3} M_0^{(q+j)} \frac{h^{q+j-p}}{(q+j-p)!} = M_1^{(p)} - \sum_{j=0}^{q-p} M_0^{(j+p)} \frac{h^j}{j!},$$

$$\sum_{\substack{j=1 \\ j \neq q-p}}^{q+3-p} M_{k-1}^{(j+p)} \frac{h^j}{j!} = M_k^{(p)} - M_{k-1}^{(p)} - M_{k-1}^{(q)} \frac{h^{q-p}}{(q-p)!}, \qquad k = 2, \ldots, n.$$

$i = p - 1(-1)1$

$$\sum_{j=1}^{3} M_0^{(q+j)} \frac{h^{q+j-i}}{(q+j-i)!} - M_1^{(i)} = -\sum_{j=0}^{q-i} M_0^{(i+j)} \frac{h^j}{j!},$$

$$\sum_{\substack{j=0 \\ j+i \neq p,q}}^{q+3-i} M_{k-1}^{(i+j)} \frac{h^j}{j!} - M_k^{(i)} = -M_{k-1}^{(p)} \frac{h^{p-i}}{(p-i)!} - M_{k-1}^{(q)} \frac{h^{q-i}}{(q-i)!},$$

$$k = 2, \ldots, n-1.$$

$i = 0$

$$\sum_{j=1}^{3} M_0^{(q+j)} \frac{h^{q+j}}{(q+j)!} = M_1^{(0)} - \sum_{j=0}^{q} M_0^{(j)} \frac{h^j}{j!},$$

$$\sum_{\substack{j=1 \\ j \neq p,q}}^{q+3} M_{k-1}^{(j)} \frac{h^j}{j!} = M_k^{(0)} - M_{k-1}^{(0)} - M_{k-1}^{(p)} \frac{h^p}{p!} - M_{k-1}^{(q)} \frac{h^q}{q!}, \qquad k = 2, \ldots, n.$$

3. The linear system and its solution

We start by rewriting the system in compact form $A\mathbf{m} = \mathbf{b}$ where the matrix A denotes a block Hessemberg matrix, $A \equiv [A_{i,j}]\ , i, j = 1, \ldots, q+1$, with $A_{i,j} \equiv 0$ for $i \leq q-p\ ,\ \ j > i+2$ or $i > q-p\ ,\ \ j > i+1$. Note that not all the blocks of the matrix are of the same dimension. Specifically,

$$\begin{aligned}
&A_{i,j}^{(n,n)}, && i = 1, q-p+1, q+1, && j = 1,2,3,\\
&A_{i,j}^{(n-1,n)}, && i = 2, \ldots, q, \quad i \neq q-p+1, && j = 1,2,3,\\
&A_{i,j}^{(n-1,n-1)}, && i = 2, \ldots, q, \quad i \neq q-p+1, && j = 4, \ldots, q+1,\\
&A_{i,j}^{(n,n-1)}, && i = q-p+1, q+1, && j = 4, \ldots, q+1.
\end{aligned}$$

Let E, F, D be defined as follows

$$\begin{aligned}
E =& (\delta_{i,j})_{n-1,n},\\
F =& (\delta_{i-1,j})_{n-1,n-1},\\
D =& (\delta_{i-1,j})_{n,n-1},
\end{aligned}$$

with $\delta_{i,j}$ denoting the Kronecker symbol.

The nonzero blocks are all either diagonal, subdiagonal or bidiagonal: for $j = 1,2,3$

$$A_{ij} = \begin{cases} \dfrac{h^{i-j+3}}{(i-j+3)!} I_n, & i = 1, q-p+1, q+1, \\ \dfrac{h^{i-j+3}}{(i-j+3)!} E & i = 2, \ldots, q, \quad i \neq q-p+1, \end{cases}$$

for $j = 4, \ldots, q-p+2$, provided that $q-p+2 \geq 4$, ($A_{ij} \neq 0$ if and only if $i \geq j-2$)

$$A_{ij} = \begin{cases} F - I_{n-1}, & i = j-2, \\ \dfrac{h^{i-j+2}}{(i-j+2)!}F, & i = j-1, \ldots, q, \qquad i \neq q-p+1, \\ \dfrac{h^{i-j+2}}{(i-j+2)!}D, & i = q-p+1, q+1, \end{cases}$$

and for $j = q-p+3, \ldots, q+1$, provided that $p \geq 2$, ($A_{ij} \neq 0$ if and only if $i \geq j-1$)

$$A_{ij} = \begin{cases} F - I_{n-1}, & i = j-1, \\ \dfrac{h^{i-j+1}}{(i-j+1)!}F, & i = j, \ldots, q, \\ \dfrac{h^{i-j+1}}{(i-j+1)!}D, & i = q+1. \end{cases}$$

To solve the system, we follow the steps of [8]. "Almost" direct forward substitution can be employed to yield the algorithm. Let us suppose to be at the $k-$th stage of the procedure, $k = 1, \ldots, n-1$:

1) by solving the $k-$th equations of the first, the $(q-p+1)-$th and the $(q+1)-$th row-block, we obtain $M_{k-1}^{(q+3)}, M_{k-1}^{(q+2)}, M_{k-1}^{(q+1)}$; the corresponding columns can then be eliminated;

2) from the $k-$th equations of the $j-$th row-block, $j = 2, \ldots, q$, $j \neq q-p+1$, $M_k^{(i)}$, $i = q-1, (-1), 1$, $i \neq p$ are obtained immediately. These two steps are iterated until the $(n-1)-$st step;

3) at the end of the $(n-1)-$st step, we need to solve the system obtained by the last equations of the first, the $(q-p+1)-$st and $(q+1)-$st row-block, in this way thus determining the remaining unknowns $M_{n-1}^{(q+3)}, M_{n-1}^{(q+2)}, M_{n-1}^{(q+1)}$.

At each stage, the process of elimination of the columns corresponding to the lastly calculated unknowns yields a matrix possessing the same structure of the original one, just of lower dimension. This is a fundamental remark for the validity of the above scheme. Moreover the column elimination corresponds to moving to the right hand side the unknowns just calculated times the corresponding weights given by the respective matrix entries. The latter multiplication when applied to the error equation to be discussed in the next section, gives terms that are of the same order of the corresponding right hand side, as is easily seen by induction.

The algorithm then hinges upon the solution at each stage of a 3 by 3 system for the unknowns $M_j^{(q+3)}, M_j^{(q+2)}, M_j^{(q+1)}, j = 0, \dots, n-1$. Its matrix has the explicit form:

$$C = \begin{pmatrix} \frac{h^3}{3!} & \frac{h^2}{2!} & h \\ \frac{h^{q-p+3}}{(q-p+3)!} & \frac{h^{q-p+2}}{(q-p+2)!} & \frac{h^{q-p+1}}{(q-p+1)!} \\ \frac{h^{q+3}}{(q+3)!} & \frac{h^{q+2}}{(q+2)!} & \frac{h^{q+1}}{(q+1)!} \end{pmatrix},$$

It is easily observed that $\det C \neq 0$, from which the unconditional solvability of the system follows.

Theorem 1. *The algorithm presented above always leads to the solution of the lacunary interpolation problem.*

It is also possible to determine explicitly the inverse of the above matrix:

$$C^{-1} = \begin{pmatrix} \frac{A_1}{h^3} & \frac{B_1}{h^{q-p+3}} & \frac{C_1}{h^{q+3}} \\ \frac{A_2}{h^2} & \frac{B_2}{h^{q-p+2}} & \frac{C_2}{h^{q+2}}, \\ \frac{A_3}{h} & \frac{B_3}{h^{q-p+1}} & \frac{C_3}{h^{q+1}} \end{pmatrix},$$

where $A_i, B_i, C_i \quad i = 1, 2, 3$ are suitable constants. This allows us to obtain immediately a stability result. It turns out however that it is of limited usefulness as will be apparent in the next section.

Theorem 2. *The norm of the inverse matrix of the system satisfies the following estimate*

$$\|C^{-1}\|_2 = O(h^{-q-3}).$$

3.1. An alternative approach

Looking for the solution in the general form, for $k = 1, 2, \dots, n$

$$s_k(x) = f_{k-1} + \sum_{i=1}^{p-1} \frac{a_{k,i}}{i!}(x - x_{k-1})^i + \frac{f_{k-1}^{(p)}}{p!}(x - x_{k-1})^p + \sum_{i=p+1}^{q-1} \frac{a_{k,i}}{i!}(x - x_{k-1})^i$$
$$+ \frac{f_{k-1}^{(q)}}{q!}(x - x_{k-1})^q + \sum_{i=q+1}^{q+3} \frac{a_{k,i}}{i!}(x - x_{k-1})^i\ ,$$

we get the coefficients

$$a_{k,i} = s_{k-1}^{(i)}(x_{k-1}), \quad k = 2, 3, \ldots, n, \quad i = 1, 2, \ldots, q-1, \quad i \neq p.$$

The coefficients $a_{k,q+1}, a_{k,q+2}, a_{k,q+3}$ are found by solving a 3 by 3 linear system, giving

$$\begin{aligned}
a_{k,q+3} &= \frac{3(q+3)(q+2-p)(q+3-p)}{p(q-p)h^2}[R_{k,0} - R_{k,p}] + \\
&\quad + \frac{2 \cdot 3(q+3)(q+3-p)}{q^2(p+3)h^2}[R_{k,0} - R_{k,q}], \\
a_{k,q+2} &= \frac{2(q+2)}{-qh}[R_{k,0} - R_{k,p}] - \frac{q+5}{3(q+3)} h a_{k,q+3}, \\
a_{k,q+1} &= R_{k,q} - \frac{h}{2} a_{k,q+2} - \frac{h^2}{6} a_{k,q+3},
\end{aligned}$$

where

$$\begin{aligned}
R_{k,0} &= \left[f_k - f_{k-1} - \sum_{i=1}^{q-1} \frac{s_{k-1}^{(i)}(x_{k-1})}{(k-1)!} h^{k-1} - \frac{f_{k-1}^{(p)}}{p!} h^p - \frac{f_{k-1}^{(q)}}{q!} h^q \right] \frac{(q+1)!}{h^{q+1}}, \\
R_{k,p} &= \left[f_k^{(p)} - f_{k-1}^{(p)} - \sum_{i=p+1}^{q-1} \frac{s_{k-1}^{(i)}(x_{k-1})}{(i-p)!} h^{i-p} - \frac{f_{k-1}^{(q)}}{(q-p)!} h^{q-p} \right] \frac{(q+1-p)!}{h^{q+1-p}}, \\
R_{k,q} &= \left[f_k^{(q)} - f_{k-1}^{(q)} \right] \frac{1}{h}.
\end{aligned}$$

For $k = 1$ we obtain the coefficients from the Taylor polynomial at x_0

$$a_{1,i} = s_1^{(i)}(x_0) = f_0^{(i)}, \quad i = 1, 2, \ldots, q-1, \quad i \neq p.$$

4. Error analysis

Let us assume that $f(x) \in C^{(q+4)}$ in $[0, 1]$ and let $T(x)$ be the Taylor polynomial in Δ_k,

$$T(x) = \sum_{j=0}^{q+3} f^{(j)}(x_{k-1}) \frac{(x - x_{k-1})^j}{j!}.$$

Then

$$f(x) - T(x) = Kh^{q+4}. \tag{5}$$

Let us define the error vector as

$$\begin{aligned} e^{(m)}(x) &= f^{(m)}(x) - s^{(m)}(x), \quad && m = 0, \dots, q+3, \\ e_k^{(m)} &= f_k^{(m)} - M_k^{(m)}, && m = 1, \dots, q+3, \quad m \neq p, q. \end{aligned}$$

For $x \in \Delta_k$, let us moreover denote the restriction of $e(x)$ on Δ_k by $e_k(x)$. Then

$$e_k(x) = f(x) - T(x) + T(x) - s_k(x), \tag{6}$$

and

$$T(x) - s_k(x) = \sum_{\substack{j=1 \\ j \neq p,q}}^{q+3} e_{k-1}^{(j)} \frac{(x - x_{k-1})^j}{j!}.$$

By imposing the continuity at the points $x_k,\ k = 1, \dots, n-1$ of $e(x)$ and its derivatives and the interpolation conditions at x_n, we obtain a system in the unknowns

$$\begin{aligned} &e_0^{(i)}, \qquad i = q+3, q+2, q+1, \\ &e_s^{(i)}, \qquad s = 1, \dots, n-1, \quad i = q+3(-1)1, \qquad i \neq p, q, \end{aligned}$$

which possesses the same matrix of the system for the calculation of the moments, analyzed in the previous section. To estimate the errors $e_k^{(m)}$, the remarks of the previous section apply. In order to calculate $e_k^{(q+3)}$, $e_k^{(q+2)}$, $e_k^{(q+1)}$ we need to solve the system

$$\begin{pmatrix} \dfrac{h^3}{3!} & \dfrac{h^2}{2!} & h \\ \dfrac{h^{q-p+3}}{(q-p+3)!} & \dfrac{h^{q-p+2}}{(q-p+2)!} & \dfrac{h^{q-p+1}}{(q-p+1)!} \\ \dfrac{h^{q+3}}{(q+3)!} & \dfrac{h^{q+2}}{(q+2)!} & \dfrac{h^{q+1}}{(q+1)!} \end{pmatrix} \begin{pmatrix} e_k^{(q+3)} \\ e_k^{(q+2)} \\ e_k^{(q+1)} \end{pmatrix} = \begin{pmatrix} K_{k,1}h^4 \\ K_{k,2}h^{q-p+4} \\ K_{k,3}h^{q+4} \end{pmatrix} \tag{7}$$

from which, easily,

$$e_k^{(q+i)} = O(h^{4-i}), \qquad i = 3, 2, 1, \qquad k = 0, \dots, n-1.$$

This corresponds to Step 1 of the algorithm in Section 3. Making use of this result, Step 2 yields equations where the only unknowns are $e_k^{(i)}$, $i = q-1\,(-1)\,1$, $i \neq p$, $k = 1\,(1)\,n-1$. Their value is immediately read off and can be used in the solution of the next equations. Note indeed that the coefficient of each such term is -1, the matrix being lower bidiagonal with constant entries given by -1 and 1. Also by differentiating i times (6) and recalling (5), each right hand side is seen to be of order $O\left(h^{q+4-i}\right)$, implying that this is also the order of $e_k^{(i)}$. Hence

$$e_k^{(i)} = O(h^{q+4-i}), \qquad i = 1, \ldots, q+3, \quad i \neq p, q, \quad k = 1, \ldots, n-1.$$

Remembering that by definition $e_k^{(q)} = e_k^{(p)} = 0$, by iterating the procedure alternating at each stage between Step 1 and Step 2, we then explicitly find

- by solving system (7), $e_0^{(q+3)}$, $e_0^{(q+2)}$, $e_0^{(q+1)}$;
- by reading off the values, $e_0^{(q-1)}, ..., e_0^{(1)}$;
- by solving system (7), $e_1^{(q+3)}$, $e_1^{(q+2)}$, $e_1^{(q+1)}$;
- by reading off the values, $e_1^{(q-1)}, ..., e_1^{(1)}$;

...

- by solving system (7), $e_{n-2}^{(q+3)}$, $e_{n-2}^{(q+2)}$, $e_{n-2}^{(q+1)}$;
- by reading off the values, $e_{n-2}^{(q-1)}, ..., e_{n-2}^{(1)}$;

Finally, Step 3 gives

- by solving system (7), $e_{n-1}^{(q+3)}$, $e_{n-1}^{(q+2)}$, $e_{n-1}^{(q+1)}$.

On using these results into (6), together with the triangular inequality, we finally have the convergence result. We summarize the results just obtained in the following

Theorem 3. *For the error of the spline function and its derivatives, the following estimates hold*

$$(8) \quad f^{(m)}(x) - s^{(m)}(x) = O(h^{q+4-m}), \qquad x \in [0,1], \qquad m = 0, \ldots, , q+3.$$

5. Problem reformulation and extension

In this section we would like to consider an extension of the previuos problem, by relaxing the condition that the intermediate interpolatory one be on the same derivative at every breakpoint. In order to quickly establish the former results also for this case, we reinterpret the problem in a more general context. Namely, it is possible to reformulate the problem by enlarging the matrix of the system; this operation is certainly not apt for the implementation, but will enlighten the error analysis for more complicated cases. The matrix of the system is partitioned into two big horizontal blocks, the upper one being similar to the matrix A discussed in the previous sections and in [8], the lower one instead by explicitly stating the known intermediate interpolation conditions. In fact, let us consider the moments $M_k^{(p)}$, $k = 1, \dots, n-1$ as unknowns. The matrix A will have $n-1$ more columns, which can be considered as a block column of structure similar to the ones studied earlier. Now, if we add to our system $n-1$ equations of the type

$$M_k^{(p)} = f_k^{(p)}, \qquad k = 1, \dots, n-1 \tag{9}$$

and if the extra initial conditions are the same as (4), we get back a square matrix.

This operation allows us to consider the same algorithm as given in the previous section (with minor modifications to take into account that some intermediate information is known) and to reformulate the error equation in which the topmost part of the right hand side is once again given by the consistency error as before, but in which the bottom part consists only of zeros, since the interpolation conditions (9) are known exactly. It turns out then that the same scheme for the previous case can be used also for the error analysis and the same convergence rates will apply.

To extend the previous case, we can now assume to substitute the conditions (9) and the intermediate condition on the last node with the following n conditions of the form

$$M_k^{(p_k)} = f_k^{(p_k)}, \qquad k = 1, \dots, n, \qquad 1 \le p_k \le q-1.$$

The matrix of the system will be the same and every consideration made earlier carried on also to this case. The only modification will be in the structure of the $q - p_n + 1$-st row, related to the condition on the last node. It will be quite similar to the $q - p + 1$-st row of the matrix of the previus case.

Theorem 4. *For the extended lacunary interpolation problem with one intermediate condition on each node, placed on a derivative of arbitrary order, possibly differing at each node, the same error estimates (8) hold.*

6. The general case

In this section we turn to examining the following questions:

1) Leaving the conditions $(0, q)$ "clamped", what does it happen if we vary the number of additional conditions?

2) How many such additional conditions can be added on the very same node?

3) Is it possible for the nodes to have a different number of conditions?

In order to answer these questions, let $q + s - 1$ be the degree of the spline and let $(q + s)n$ be the number of coefficents which need to be determined.

Since the numbers of interpolatory and continuity conditions are respectively $2(n + 1)$ and $(n - 1)(q + 1)$, there will be $3n + 1 - q + nq$ equations.

Now let N be the number of additional conditions on the internal nodes $x_1, \dots, x_{n-1}$. Let also X be the number of free parameters which will yield conditions on the extreme nodes. Thus we must have max $X = 2(q - 1)$. It follows then that $nq + ns = 3n + 1 - q + nq + N + X$, i.e.

$$s = 3 + \frac{1 - q + N + X}{n} = 3 + \text{ an integer number.}$$

To ensure that s does not grow, we must have

$$N + X = q - 1.$$

In [8] $X = q - 1$ implies that $N = 0$ and the error is $O(h^{q+3})$.

To answer the above questions, as long as the number N of additional conditions does not exceed $q - 1$, the problem is well posed and the spline does not increase its degree, provided that we decrease the endpoint conditions by the same amount by which N increases, so that $N + X$ remains constant.

It does not matter how many additional conditions are imposed on the same internal node. The block structure of the matrix will of course be preserved, but the strategy for the resolution of the system will change accordingly to the changes imposed by the new conditions.

If instead $N + X = n + q - 1$, we have $s = 4$.

The $(0, p, q)$ case is now the particular case in which $N = n - 1$ conditions are imposed on the intermediate nodes, all of them being imposed on the p-th derivative, and where additionally there are $q - 1$ conditions on the very first node and 1 condition always on the p-th derivative is imposed on the very last node.

Any other distribution of the $N+X$ conditions, as long as the requirement $N+X=n+q-1$ is satisfied, will not alter the order of the error in the solution.

The block structure of the matrix remains unchanged, although the strategy for solving the problem will change. However, it will still be similar to the one examined earlier.

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