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SIMULTANEOUS APPROXIMATION TO A DIFFERENTIABLE FUNCTION AND ITS DERIVATIVE BY PÁL–TYPE INTERPOLATION ON THE ROOTS OF JACOBI POLYNOMIALS

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Abstract. The Pál-type interpolation on the roots of Jacobi polynomials with boundary (Hermite) conditions at the endpoints of the interval $[-1,1]$ gives a simultaneous approximation to a differentiable function and its derivative. The order of convergence depends only on the smoothness of the function.

1. Introduction

In Pál-type interpolation [6] the function values are interpolated at the zeros of the polynomial $w(x)$, while the first derivative values are interpolated at the roots of $w'(x)$. In inverse Pál-type interpolation only the role of $w(x)$ and $w'(x)$ is interchanged, namely, the derivative values are interpolated at the roots of $w(x)$ and the function values at the roots of $w'(x)$. That is, the derivative of the interpolational polynomial interpolates the derivative of the function at the roots of $w'(x)$ or of $w(x)$. This feature of the interpolation indicated the question under what conditions gives the Pál-type interpolation a simultaneous approximation to a differentiable function and its derivative.

In this paper we will study under what conditions gives the following interpolational procedure a simultaneous approximation to a differentiable function and its derivative.

Let the set of the knots be given by

$$(1.1) \qquad -1 = x_n < x_n^* < x_{n-1} < x_{n-1}^* < \ldots < x_1 < x_1^* < x_0 = 1 \quad (n \geq 1),$$

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where $\{x_i\}_{i=1}^{n-1}$ and $\{x_i^*\}_{i=1}^{n}$ are the roots of the Jacobi (ultraspherical) polynomials $P_{n-1}^{(k+1)}(x)$ and $P_n^{(k)}(x)$, respectively. On the knots (1.1) there exists a unique polynomial $Q_m(x)$ of degree at most $m = 2n + 2k + 1$ satisfying the $(0;1)$ interpolation conditions

$$\begin{aligned} Q_m(x_i) &= y_i \quad (i = 1, \ldots, n-1), \\ Q'_m(x_i^*) &= y'_i \quad (i = 1, \ldots, n), \end{aligned} \tag{1.2}$$

with the boundary (Hermite) conditions

$$\begin{aligned} Q_m^{(j)}(x_0) &= Q_m^{(j)}(1) = \alpha_j \quad (j = 0, \ldots, k), \\ Q_m^{(j)}(x_n) &= Q_m^{(j)}(-1) = \beta_j \quad (j = 0, \ldots, k+1), \end{aligned} \tag{1.3}$$

where $y_i, y'_i, \alpha_j, \beta_j$ are given real numbers, and k is a fixed non-negative integer.

The explicit formulae for the fundamental polynomials are given and the uniqueness of the interpolation with convergence theorems are proved by the author in [4]. The knots $x_0, x_1, \ldots, x_n$ are the roots of $w(x) = (1-x^2)^{k+1} P_{n-1}^{(k+1)}(x)$, and the knots $x_0^*, x_1^*, \ldots, x_n^*, x_{n+1}^*$, $(x_0^* = -1,\ x_{n+1}^* = 1)$ are the roots of

$$w'(x) = -n(n+2k-1)(1-x^2)^k P_n^{(k)}(x),$$

hence this interpolation problem is a modified Pál-type problem with boundary conditions. In inverse Pál-type interpolation the function values are interpolated on the roots of $w'(x)$, while the derivative values are interpolated on the roots of $w(x)$. As

$$P_n^{(k)\prime}(x) = \frac{n+2k-1}{2}\, P_{n-1}^{(k+1)}(x),$$

this problem could be considered also as an inverse Pál-type problem on the roots of the Jacobi polynomial $P_n^{(k)}(x)$ with boundary (Hermite) conditions at the endpoints of the interval $[-1,1]$.

The convergence of this interpolational process was first studied by Eneduanya [2] for $k = 0$. Later Xie [9] proved that if $f \in C^r[-1,1]$ for $x \in [-1,1]$

$$|f(x) - Q_{2n+1}(x;f)| = O\left(n^{-r+1}\right) w\left(f^{(r)}, \frac{1}{n}\right). \tag{1.4}$$

For $k \geq 1$ the author [4] proved that if $f \in C^r[-1,1]$ for $x \in [-1,1]$

$$|f(x) - Q_m(x;f)| = O\left(n^{k-r+\frac{1}{2}}\right) w\left(f^{(r)}, \frac{1}{n}\right), \tag{1.5}$$

where $w(f^{(r)},.)$ denotes the modulus of continuity of the r-th derivative of the function $f(x)$. Hence $Q_m(x;f)$ converges uniformly to $f(x)$ on $[-1,1]$, if $f \in$ $\in C^{k+1}[-1,1]$. Furthermore, while $Q_m(x;f)$ interpolates $f(x)$ at the roots of $w(x)$, its derivative $Q'_m(x;f)$ interpolates $f'(x)$ at the roots of $w'(x)$. This feature of the Pál-type interpolation indicated the question under what conditions would $Q'_m(x;f)$ converge to $f'(x)$ uniformly on $[-1,1]$. Xie and Zhou [10] proved for $k=0$ that if $f \in C^r[-1,1]$ $(r \geq 2)$ then

$$|f'(x) - Q'_{2n+1}(x;f)| = O\left(n^{-r+\frac{5}{2}}\right) w\left(f^{(r)}, \frac{1}{n}\right), \tag{1.6}$$

that is, if $f \in C^2[-1,1]$, $f'' \in \text{Lip}\ \alpha$, $\alpha > \dfrac{1}{2}$, then $Q'_{2n+1}(x;f)$ converges uniformly to $f'(x)$ on $[-1,1]$.

In §3 we will prove, that if $k \geq 1$, $f \in C^{k+2}[-1,1]$, $f^{(k+2)} \in \text{Lip}\ \alpha$, $\alpha > \dfrac{1}{2}$, then $Q'_m(x;f)$ converges uniformly to $f'(x)$ on $[-1,1]$. In §2 we recall the properties of the ultraspherical polynomials and the explicit formulae of the fundamental polynomials.

The simultaneous approximation to a differentiable function and its derivative by inverse Pál-type interpolation polynomials was studied by Bao [1] on the roots of the integrated Legendre polynomials, introduced by Szili [8]. The same properties of the Pál-type interpolation was studied by Lu [5] on the zeros of Chebyshev polynomial of second kind.

2. Preliminaries

Let $P_n^{(\alpha)}(x) = P_n^{(\alpha,\alpha)}(x)$ $(\alpha > -1,\ n \geq 0)$ denote the ultraspherical polynomial of degree n with the normalization $P_n^{(\alpha)}(1) = \binom{n+\alpha}{n}$. Now we list those properties of the ultraspherical polynomials which will be needed to prove the theorems in §3 (we refer to [7] (4.2.1), (4.21.7), (7.32.6)):

$$(1-x^2)P_n^{(\alpha)''}(x) - 2(\alpha+1)xP_n^{(\alpha)'}(x) + n(n+2\alpha+1)P_n^{(\alpha)}(x) = 0, \tag{2.1}$$

$$P_n^{(\alpha)'}(x) = \frac{n+2\alpha+1}{2} P_{n-1}^{(\alpha+1)}(x), \tag{2.2}$$

and

$$|P_n^{(\alpha)}(x)| = O(n^\alpha) \qquad x \in [-1,1], \tag{2.3}$$

$$(1-x^2)^{\frac{\alpha}{2}+\frac{1}{4}}|P_n^{(\alpha)}(x)| = O\left(\frac{1}{\sqrt{n}}\right) \quad x \in [-1,1], \tag{2.4}$$

where $O(n)$ is independent of x.

If $x_1 > \ldots > x_n$ are the roots of $P_n^{(\alpha)}(x)$, then the following asymptotical relations hold (cf. [7] (8.9.1), (8.9.2))

$$1 - x_j^2 \sim \begin{cases} \dfrac{j^2}{n^2} & (x_j \geq 0), \\[2mm] \dfrac{(n-j)^2}{n^2} & (x_j < 0), \end{cases} \tag{2.5}$$

$$|P_n^{(\alpha)'}(x_j)| \sim \begin{cases} \dfrac{n^{\alpha+2}}{j^{\alpha+\frac{3}{2}}} & (x_j \geq 0), \\[2mm] \dfrac{n^{\alpha+2}}{(n-j)^{\alpha+\frac{3}{2}}} & (x_j < 0), \end{cases} \tag{2.6}$$

where $a_n \sim b_n$ means that $a_n = O(b_n)$ and $b_n = O(a_n)$. If $\ell_j(x)$ denotes the fundamental polynomial of Lagrange interpolation on the knots $x_1, \ldots, x_n$ which corresponds to the knot x_j, then (cf. [7] (4.5.2), (4.3.3))
(2.7)

$$\ell_j(x) = \frac{P_n^{(\alpha)}(x)}{P_n^{(\alpha)'}(x_j)(x-x_j)} = \frac{\tilde{h}_n^{(\alpha)}}{(1-x_j^2)\left[P_n^{(\alpha)'}(x_j)\right]^2} \sum_{\nu=0}^{n-1} \frac{1}{h_\nu^{(\alpha)}} P_\nu^{(\alpha)}(x_j) P_\nu^{(\alpha)}(x),$$

where

$$\tilde{h}_n^{(\alpha)} = 2^{2\alpha} \frac{\Gamma^2(n+\alpha+1)}{\Gamma(n+1)\Gamma(n+2\alpha+1)} \sim C_1, \tag{2.8}$$

$$h_\nu^{(\alpha)} = \frac{2^{2\alpha+1}}{2\nu+2\alpha+1} \frac{\Gamma^2(\nu+\alpha+1)}{\Gamma(\nu+1)\Gamma(\nu+2\alpha+1)} \begin{cases} \sim \dfrac{1}{\nu} & (\nu > 0), \\[2mm] = C_2 & (\nu = 0), \end{cases} \tag{2.9}$$

where the constants C_1, C_2 depend only on α.

We will denote by $l_j(x)$ and $l_j^*(x)$ the fundamental polynomials of Lagrange interpolation on the knots $\{x_i\}_{i=1}^{n-1}$ and $\{x_i^*\}_{i=1}^{n}$, defined in (1.1), that is

$$l_j(x) = \frac{P_{n-1}^{(k+1)}(x)}{P_{n-1}^{(k+1)'}(x_j)(x-x_j)} \quad \text{and} \quad l_j^*(x) = \frac{P_n^{(k)}(x)}{P_n^{(k)'}(x_j^*)(x-x_j^*)}. \tag{2.10}$$

The polynomial $Q_m(x)$ satisfying the interpolation conditions (1.2) and (1.3) can be written in the form (cf. [4])

$$Q_m(x) = \sum_{j=1}^{n-1} y_j A_j(x) + \sum_{j=1}^{n} y_j' B_j(x) + \sum_{j=0}^{k} \alpha_j C_j(x) + \sum_{j=0}^{k+1} \beta_j D_j(x), \tag{2.11}$$

where the fundamental polynomials of first kind are

$$\begin{aligned} A_j(x) = {} & \frac{(1-x^2)^{k+1}}{(1-x_j^2)^{k+1}(1+x_j)P_n^{(k)}(x_j)} \times \\ & \times \left\{ (1+x)P_n^{(k)}(x)l_j(x) - P_n^{(k)'}(x)\int_{-1}^{x}(1+t)l_j(t)dt \right\} \end{aligned} \tag{2.12}$$

$(j = 1, \dots, n-1)$; the fundamental polynomials of second kind are

$$B_j(x) = \frac{(1-x^2)^{k+1}P_n^{(k)'}(x)}{(1-x_j^{*2})^{k+1}P_n^{(k)'}(x_j^*)} \int_{-1}^{x} l_j^*(t)dt \quad (j = 1, \dots, n); \tag{2.13}$$

and the fundamental polynomials which correspond to the boundary conditions are

$$\begin{aligned} C_j(x) = {} & (1-x)^j(1+x)^{k+2}P_n^{(k)'}(x)P_n^{(k)}(x)p_j(x) + \\ & + (1-x^2)^{k+1}P_n^{(k)'}(x)\int_{-1}^{x} \frac{q_j(t)P_n^{(k)}(t) - (1+t)p_j(t)P_n^{(k)'}(t)}{(1-t)^{k+1-j}}dt \end{aligned} \tag{2.14}$$

$(j = 0, \dots, k)$, where $p_j(x)$ and $q_j(x)$ are uniquely determined polynomials of degree $\leq k-j$;

$$\begin{aligned} D_j(x) = {} & (1-x)^{k+1}(1+x)^j P_n^{(k)'}(x)P_n^{(k)}(x)\tilde{p}_j(x) + \\ & + (1-x^2)^{k+1}P_n^{(k)'}(x)\int_{-1}^{x} \frac{\tilde{q}_j(t)P_n^{(k)}(t) - \tilde{p}_j(t)P_n^{(k)'}(t)}{(1+t)^{k+1-j}}dt \end{aligned} \tag{2.15}$$

($j = 0, \dots, k$), where $\tilde{p}_j(x)$ and $\tilde{q}_j(x)$ are uniquely determined polynomials of degree $\le k+1-j$ and $k-j$, respectively; and

$$D_{k+1}(x) = \frac{(1-x^2)^{k+1} P_n^{(k)'}(x)}{2^{k+1} P_n^{(k)'}}. \tag{2.16}$$

3. The convergence

Lemma 1. *If $k > 0$, $n \ge 2$, for the first derivative of the first kind fundamental polynomials on $[-1,1]$ holds*

$$\sum_{j=1}^{n-1} (1-x_j^2)|A_j'(x)| = O\left(n^{k+\frac{5}{2}}\right). \tag{3.1}$$

Proof. Substituting x_j into the differential equation (2.1)

$$(1-x_j^2) P_n^{(k)''}(x_j) + n(n+2k+1) P_n^{(k)}(x_j) = 0,$$

hence by (2.2)

$$P_n^{(k)}(x_j) = -\frac{1-x_j^2}{2n} P_{n-1}^{(k+1)'}(x_j). \tag{3.2}$$

Differentiating (2.12) and applying (2.1) and (3.2) we get

$$A_j'(x) = \frac{-2n(1-x^2)^k P_n^{(k)}(x)}{(1-x_j^2)^{k+2}(1+x_j) P_{n-1}^{(k+1)'}(x_j)} \left\{ n(n+2k+1) \int_{-1}^{x} (1+t) l_j(t) dt + \right.$$

$$\left. + g(x)(1+x) l_j(x) + (1+x)(1-x^2) l_j'(x) \right\}$$

with $g(x) = 1 - x - 2x(k+1)$. In according to the form of $A_j'(x)$ we can write

$$\sum_{j=1}^{n-1} (1-x_j^2)|A_j'(x)| = \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3. \tag{3.3}$$

Using the decomposition (2.7) for $l_j(x)$ (with $\alpha = k+1$)

$$\sum\nolimits_1 \le \sum_{j=1}^{n-1} \frac{2n^2(n+2k+1)(1-x_j)(1-x^2)^k \left|P_n^{(k)}(x)\right| \tilde{h}_{n-1}^{k+1}}{(1-x_j^2)^{\frac{3k}{2}+\frac{15}{4}} |P_{n-1}^{(k+1)'}(x_j)|^3} \times$$

$$\times \left\{ C_3 + \sum_{\nu=1}^{n-2} \frac{1}{h_\nu^{(k+1)}} (1-x_j^2)^{\frac{k}{2}+\frac{3}{4}} \left|P_\nu^{(k+1)}(x_j)\right| \left| \int_{-1}^{x} (1+t) P_\nu^{(k+1)}(t) dt \right| \right\},$$

where C_3 is a constant, independent of n. Using (2.2) and integrating by parts

$$\int_{-1}^{x} (1+t) P_\nu^{(k+1)}(t) dt = \frac{2}{\nu+2k+2} \int_{-1}^{x} (1+t) P_{\nu+1}^{(k)'}(t) dt =$$

$$= \frac{2}{\nu+2k+2} \left((1+x) P_{\nu+1}^{(k)}(x) - \int_{-1}^{x} P_{\nu+1}^{(k)}(t) dt \right),$$

hence by (2.3) for $\nu \le 1$ we have

$$\left| \int_{-1}^{x} (1+t) P_\nu^{(k+1)}(t) dt \right| = O(\nu^{k-1}).$$

Furthermore, by (2.5) and (2.6) it holds

$$\frac{1}{(1-x_j^2)^{\frac{3k}{2}+\frac{15}{4}} \left|P_{n-1}^{(k+1)'}(x_j)\right|^3} = O\left((n-1)^{-\frac{3}{2}}\right).$$

Now applying these estimates and (2.3), (2.4) we get

$$\sum\nolimits_1 = O(1) n^3 \frac{1}{\sqrt{n-1}} (n-1)^{-\frac{3}{2}} \sum_{j=1}^{n-1} \left(1 + \sum_{\nu=1}^{n-2} \nu \frac{1}{\sqrt{\nu}} \nu^{k-1} \right) = O\left(n^{k+\frac{5}{2}}\right).$$

For the second and third terms of (3.3) we have in a similar way

$$\sum\nolimits_2 \le \sum_{j=1}^{n-1} \frac{2n|g(x)|(1-x_j)(1-x^2)^k |P_n^{(k)}(x)| \tilde{h}_{n-1}^{(k+1)}}{(1-x_j^2)^{\frac{3k}{2}+\frac{15}{4}} \left|P_{n-1}^{(k+1)'}(x_j)\right|^3} \left\{ C_4 + \right.$$

$$\left. + \sum_{\nu=1}^{n-2} \frac{1}{h_\nu^{(k+1)}} (1-x_j^2)^{\frac{k}{2}+\frac{3}{4}} \left|P_\nu^{(k+1)}(x_j)\right| \left|P_\nu^{(k+1)}(x)\right| \right\}$$

and

$$\sum\nolimits_3 \leq \sum_{j=1}^{n-1} \frac{2n(1-x_j)(1+x)|P_n^{(k)}(x)|\tilde{h}_{n-1}^{(k+1)}}{(1-x_j^2)^{\frac{3k}{2}+\frac{15}{4}} \left|P_{n-1}^{(k+1)'}(x_j)\right|^3} \Bigg\{ C_5 +$$

$$+ \sum_{\nu=1}^{n-2} \frac{1}{h_\nu^{(k+1)}} (1-x_j^2)^{\frac{k}{2}+\frac{3}{4}} \left|P_\nu^{(k+1)}(x_j)\right| (1-x^2)^{k+1} \left(\frac{\nu+2k+3}{2}\right) \left|P_{\nu-1}^{(k+2)}(x)\right| \Bigg\},$$

where C_4 and C_5 are constants, independent of n. Hence

$$\sum\nolimits_2 = O(1) n \frac{1}{\sqrt{n}} (n-1)^{-\frac{3}{2}} \sum_{j=1}^{n-1} \left(1 + \sum_{\nu=1}^{n-2} \nu \frac{1}{\sqrt{\nu}} \nu^{k+1}\right) = O\left(n^{k+\frac{5}{2}}\right)$$

and

$$\sum\nolimits_3 = O(1) n n^k (n-1)^{-\frac{3}{2}} \sum_{j=1}^{n-1} \left(1 + \sum_{\nu=1}^{n-2} \nu \frac{1}{\sqrt{\nu}} \nu \frac{1}{\sqrt{\nu}}\right) = O\left(n^{k+\frac{5}{2}}\right),$$

which completes the proof.

Lemma 2. *If $k > 0$, $n \geq 2$, for the first derivative of the second kind fundamental polynomials on $[-1,1]$ holds*

$$\sum_{j=1}^{n} |B_j'(x)| = O\left(n^{k+\frac{3}{2}}\right). \tag{3.4}$$

Proof. Differentiating (2.13) and applying (2.1) and (2.2) we get

$$\sum_{j=1}^{n} |B_j'(x)| = S_1 + S_2, \tag{3.5}$$

where

$$S_1 = \sum_{j=1}^{n} \frac{n(n+2k+1)(1-x^2)^k |P_n^{(k)}(x)|}{(1-x_j^{*2})^{k+1} |P_n^{(k)'}(x_j^*)|} \left| \int_{-1}^{x} l_j^*(t) dt \right|,$$

$$S_2 = \sum_{j=1}^{n} \frac{(n+2k+1)(1-x^2)^{k+1} \left|P_{n-1}^{(k+1)}(x)\right|}{2(1-x_j^{*2})^{k+1} \left|P_n^{(k)'}(x_j^*)\right|} |l_j^*(x)|.$$

As in the previous lemma, we use the decomposition (2.7) for $l_j^*(x)$ (with $\alpha = k$) and we have

$$S_1 \le \sum_{j=1}^{n} \frac{n(n+2k+1)(1-x^2)^k \left|P_n^{(k)}(x)\right| \tilde{h}_n^{(k)}}{(1-x_j^{*2})^{\frac{3k}{2}+\frac{9}{4}} \left|P_n^{(k)\prime}(x_j^*)\right|^3} \Bigg\{ C_6 + \\ + \sum_{\nu=1}^{n-1} \frac{1}{h_\nu^{(k)}} (1-x_j^{*2})^{\frac{k}{2}+\frac{1}{4}} \left|P_\nu^{(k)}(x_j^*)\right| \left| \int_{-1}^{x} P_\nu^{(k)}(t)dt \right| \Bigg\},$$

where the constant C_6 is independent of n. For $\nu \ge 1$ by (2.2) and (2.3)

$$\left| \int_{-1}^{t} P_\nu^{(k)}(t)dt \right| = \frac{2}{\nu+2k} \left| \int_{-1}^{x} P_{\nu+1}^{(k-1)\prime}(t)dt \right| = O(\nu^{k-2}),$$

and by (2.5) and (2.6) holds

$$\frac{1}{(1-x_j^{*2})^{\frac{3k}{2}+\frac{9}{4}} \left|P_n^{(k)\prime}(x_j^*)\right|^3} = O(n^{-\frac{3}{2}}),$$

hence

$$S_1 = O(1) n^2 \frac{1}{\sqrt{n}} n^{-\frac{3}{2}} \sum_{j=1}^{n} \left(1 + \sum_{\nu=1}^{n-1} \nu \frac{1}{\sqrt{\nu}} \nu^{k-2} \right) = O\left(n^{k+\frac{1}{2}}\right).$$

For the second term of (3.5) we get in a similar way

$$S_2 \le \sum_{j=1}^{n} \frac{(n+2k+1)(1-x^2)^{k+1} \left|P_{n-1}^{(k+1)}(x)\right| \tilde{h}_n^{(k)}}{2(1-x_j^{*2})^{\frac{3k}{2}+\frac{9}{4}} \left|P_n^{(k)\prime}(x_j^*)\right|^3} \times \\ \times \left\{ C_7 + \sum_{\nu=1}^{n-1} \frac{1}{h_\nu^{(k)}} (1-x_j^{*2})^{\frac{k}{2}+\frac{1}{4}} \left|P_\nu^{(k)}(x_j^*)\right| \left|P_\nu^{(k)}(x)\right| \right\},$$

where C_7 is constant, independent of n, and

$$S_2 = O(1) n \frac{1}{\sqrt{n}} n^{-\frac{3}{2}} \sum_{j=1}^{n} \left(1 + \sum_{\nu=1}^{n-1} \nu \frac{1}{\sqrt{\nu}} \nu^{k} \right) = O\left(n^{k+\frac{3}{2}}\right),$$

which completes the proof.

Theorem 1. *Let $k \geq 0$ be a fixed integer, $m = 2n+2k+1$ and let the knots $\{x_i\}_{i=1}^{n-1}$ and $\{x_i^*\}_{i=1}^{n}$ be the roots of the ultraspherical polynomials $P_{n-1}^{(k+1)}(x)$ and $P_n^{(k)}(x)$, respectively. If $f \in C^r[-1,1]$ $(r \geq k+1,\ n \geq 2r-k+2)$, then the interpolational polynomial*

$$Q_m(x;f) =$$

$$= \sum_{i=1}^{n-1} f(x_i)A_i(x) + \sum_{i=1}^{n} f'(x_i^*)B_i(x) + \sum_{j=0}^{k} f^{(j)}(1)C_j(x) + \sum_{j=0}^{k+1} f^{(j)}(-1)D_j(x)$$

with the fundamental polynomials given in (2.12)-(2.15) satisfies for $x \in [-1,1]$

$$|f'(x) - Q_m'(x;f)| = w\left(f^{(r)}; \frac{1}{n}\right) O\left(n^{k-r+\frac{5}{2}}\right).$$

Proof. For $k = 0$ we refer to (1.4), proved by Xie and Zhou [10], and we prove tha case $k \geq 1$. Let $f \in C^r[-1,1]$, then by the theorem of Gopengauz [3] for every $m \geq 4r+5$ there exists a polynomial $p_m(x)$ of degree at most m such that for $j = 0, \ldots, r$

$$\left|f^{(j)}(x) - p_m^{(j)}(x)\right| \leq M_{r,j} \left(\frac{\sqrt{1-x^2}}{m}\right)^{r-j} w\left(f^{(r)}; \frac{\sqrt{1-x^2}}{m}\right),$$

where $w\left(f^{(r)};.\right)$ denotes the modulus of continuity of the function $f^{(r)}(x)$ and the constants $M_{r,j}$ depend only on r and j. Moreover,

$$f^{(j)}(\pm 1) = p_m^{(j)}(\pm 1) \quad (j = 0, \ldots, r).$$

By the uniqueness of the interpolational polynomials $Q_m(x;f)$ it is clear that $Q_m(x;p_m) = p_m(x)$. Hence for $x \in [-1,1]$

$$|f'(x) - Q_m'(x;f)| \leq |f'(x) - p_m'(x)| + |Q_m'(x;p_m) - Q_m'(x;f)| \leq$$

$$\leq |f'(x) - p_m'(x)| + \sum_{j=1}^{n-1} |f(x_j) - p_m(x_j)||A_j'(x)| + \sum_{j=1}^{n} |f'(x_j^*) - p_m'(x_j^*)||B_j'(x)| \leq$$

$$\leq M_{r,0} w\left(f^{(r)}; \frac{1}{n}\right) \frac{1}{n^r} \sum_{j=1}^{n-1} (1-x_j^2)|A_j'(x)| +$$

$$+ M_{r,1} w\left(f^{(r)}; \frac{1}{n}\right) \frac{1}{n^{r-1}} \left(1 + \sum_{j=1}^{n} |B_j'(x)|\right).$$

Now applying the estimates (3.1) and (3.4) we have

$$|f'(x) - Q_m'(x; f)| \leq$$

$$\leq O(1) \frac{1}{n^r} w\left(f^{(r)}; \frac{1}{n}\right) n^{k+\frac{5}{2}} + O(1) \frac{1}{n^{r-1}} w\left(f^{(r)}; \frac{1}{n}\right) \left(1 + n^{k+\frac{3}{2}}\right) =$$

$$= O(1) n^{k-r+\frac{5}{2}} w\left(f^{(r)}; \frac{1}{n}\right),$$

which is the statement of the theorem.

As a corollary of (1.5) and Theorem 1 we can state the following convergence theorem.

Theorem 2. *Let $k \geq 0$ be a fixed integer, $m = 2n + 2k + 1$, $n \geq k + 4$, and let the knots $\{x_i^*\}_{i=1}^n$ and $\{x_i^*\}_{i=1}^n$ be the roots of the ultraspherical polynomials $P_{n-1}^{(k+1)}(x)$ and $P_n^{(k)}(x)$, respectively. If $f \in C^{k+2}[-1,1]$, $f^{(k+2)} \in \mathrm{Lip}\, \alpha$, $\alpha > \frac{1}{2}$, then $Q_m(x; f)$ and $Q_m'(x; f)$ uniformly converge to $f(x)$ and $f'(x)$, respectively, on $[-1,1]$ as $n \to \infty$.*

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